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Research Article

Some New Generalized Inequalities with regard to Two-Dimensional Preinvex Functions

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Abstract
In this study, we investigated the general convexity of functions which is named preinvexity. Firstly, we generalized Hermite-Hadamard type integral inequality for two-dimensional preinvex functions. Then, we obtained a generalization of Ostrowski type integral inequality for two-dimensional preinvex functions. Besides, we derived some new generalized inequalities related to these functions.

Keywords: Preinvexity, Hermite-Hadamard type inequalities, Ostrowski inequality, Two-dimensional preinvex functions.

İki-Boyutlu Preinveks Fonksiyonlara İlişkin Bazı Yeni Genelleştirilmiş Eşitsizlikler

Öz
Bu çalışmada, preinvekslik olarak adlandırılan fonksiyonların genel konveksliğini inceledik. İlk olarak, iki-boyutlu preinveks fonksiyonlar için Hermite-Hadamard tipi integral eşitsizliğini genelleştirdik. Daha sonra, iki-boyutlu preinveks fonksiyonlar için Ostrowski tipi eşitsizliğinin bir genelleştirmesini elde ettik. Ayrıca bu fonksiyonlarla ilgili bazı yeni genelleştirilmiş eşitsizlikleri sağladık.

Anahtar Kelimeler: Preinvekslik, Hermite-Hadamard tipi eşitsizlikler, Ostrowski eşitsizliği, İki-boyutlu preinveks fonksiyonlar.

1. Introduction

The famous Hermite-Hadamard type integral inequality ($\mathcal{HTI}$) for convex functions is as follows [4]:

$$f\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(\omega)d\omega \leq \frac{f(\alpha)+f(\beta)}{2}.$$

It should be known that preinvexity indicates a generalization of convexity. In other words, the function $f: I \subset \mathbb{R} \to \mathbb{R}$ is known preinvex with respect to ϕ, if the following inequality holds:

$$f\big(\omega+\curlyvee\phi(\upsilon,\omega)\big) \leq (1-\curlyvee)f(\omega)+\curlyvee f(\upsilon)$$

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for all $\omega, \upsilon \in I$ and $\curlyvee \in [0,1]$. In this place, the invex bifunction $\phi: I \times I \rightarrow \mathbb{R}^n$ fulfills following Condition C [11]

$$\phi(\upsilon, \upsilon + \curlyvee\phi(\omega, \upsilon)) = -\curlyvee\phi(\omega, \upsilon);$$
$$\phi(\omega, \upsilon + \curlyvee\phi(\omega, \upsilon)) = (1 - \curlyvee)\phi(\omega, \upsilon)$$

for all $\omega, \upsilon \in I$ and $\curlyvee \in [0,1]$. [13] obtained $\mathcal{HTI}$ for the $f: [\mu, \mu + \phi(\delta, \mu)] \rightarrow (0, \infty)$ as follows:

$$f\left(\frac{2\mu + \phi(\delta, \mu)}{2}\right) \leq \frac{1}{\phi(\delta, \mu)} \int_{\mu}^{\mu+\phi(\delta,\mu)} f(\omega)d\omega \leq \frac{f(\mu) + f(\delta)}{2}$$

for all $\mu, \delta \in I^{\circ}$ with $\mu < \mu + \phi(\delta, \mu)$. He expressed that the sub-intervals of the preinvex function are also preinvex.

The classical $\mathcal{HTI}$ for every convex function on $[\alpha, \beta]$ was generalized by [1]. He proved also some sharps Ostrowski's type integral inequality ($\mathcal{OTI}$) for every positive convex function on $[\alpha, \beta]$. [3] got $\mathcal{HTI}$ for convex functions on the coordinates. Some generalizations of $\mathcal{HTI}$ and $\mathcal{OTI}$ on $[\alpha, \beta] \times [\xi, \vartheta]$ were proved by [12].

Nowadays, [15] obtained some generalized inequalities for two-dimensional harmonically convex functions and [20] verified some generalized inequalities about two-dimensional φ-convex functions. There are also many studies on preinvex functions ($\mathcal{PF}$) and $\mathcal{HTI}$ about them (see [2], [5], [6], [7], [8], [14], [16], [17], [18], [21]). Based on the above mentioned information, using by [1] and [12] methods, we verified some important generalized inequalities for $\mathcal{PF}$ on real number line and on the coordinates, respectively.

2. Preliminaries

In this section firstly, two-dimensional $\mathcal{PF}$ were introduced and $\mathcal{HTI}$ for these functions was presented. Lately, some generalized inequalities for two-dimensional $\mathcal{PF}$ were verified.

2.1. Two-Dimensional $\mathcal{PF}$ and Related $\mathcal{HTI}$

Let us give definitions about invexity and preinvexity of functions on the coordinates.

Definition 2.1 ([10])**.** Let the continuous functions $\phi : \Delta = T \times S \rightarrow \mathbb{R}^n$ be invex on the sets $\Delta \subseteq \mathbb{R}^n \times \mathbb{R}^n$, respectively. Then Δ is called invex with respect to ϕ, if

$$(\mu + \curlyvee\phi(\omega, \mu), \delta + \curlyvee\phi(\upsilon, \delta)) \in \Delta$$

for all $(\omega, \upsilon), (\mu, \delta) \in \Delta$, $\curlyvee \in [0,1]$.

Definition 2.2 ([19])**.** Let Δ be called invex with respect to ϕ. A function $F: \Delta \rightarrow \mathbb{R}$ is called preinvex with respect to ϕ on Δ, if the following inequality holds

$$F\big(\omega_1+\gamma\phi(\omega_2,\omega_1),\upsilon_1+\gamma\phi(\upsilon_2,\upsilon_1)\big)\le(1-\gamma)F(\omega_1,\upsilon_1)+\gamma F(\omega_2,\upsilon_2)$$

for all (ω_1,υ_1) , $(\omega_2,\upsilon_2)\in\Delta$ and $\gamma\in[0,1]$. If the above inequality is reversed, then F is called be a pre-concave with respect to ϕ on Δ.

Definition 2.3 ([9])**.** A function $F:\Delta\to\mathbb{R}$ is said to be a preinvex with respect to ϕ on the coordinates, if the following partial mappings are preinvex with respect to ϕ, respectively,

$$F_\upsilon:T\ \to\mathbb{R},\ F_\upsilon(\mu):=F(\mu,\upsilon);\ F_\omega:S\ \to\mathbb{R},\ F_\omega(\delta):=F(\omega,\delta)$$

for all $\omega\in T$ and $\upsilon\in S$.

Let us consider $\mathcal{HTI}$ for related functions. Assume that

$\theta:=[\mu_1,\mu_1+\phi(\mu_2,\mu_1)]\times[\delta_1,\delta_1+\phi(\delta_2,\delta_1)]\subseteq\mathbb{R}_+^2$, for all $\phi(\mu_2,\mu_1)>0,\phi(\delta_2,\delta_1)>0$, $\mu_1<\mu_2,\delta_1<\delta_2$.

Theorem 2.4 ([3])**.** Let $F:\theta\to\mathbb{R}_+$ be a preinvex function with respect to ϕ on the coordinates. Then the following inequality holds

$$\begin{aligned}
&F\left(\mu_1+\frac{1}{2}\phi(\mu_2,\mu_1),\delta_1+\frac{1}{2}\phi(\delta_2,\delta_1)\right)\\
&\le\frac{1}{2\phi(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}F\left(\omega,\delta_1+\frac{1}{2}\phi(\delta_2,\delta_1)\right)d\omega\\
&+\frac{1}{2\phi(\delta_2,\delta_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}F\left(\mu_1+\frac{1}{2}\phi(\mu_2,\mu_1),\upsilon\right)d\upsilon\\
&\le\frac{1}{\phi(\mu_2,\mu_1)\phi(\delta_2,\delta_1)}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}F(\omega,\upsilon)\,d\upsilon d\omega\\
&\le\frac{1}{4\phi(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\big(F(\omega,\delta_1)+F(\omega,\delta_2)\big)d\omega\\
&+\frac{1}{4\phi(\delta_2,\delta_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}\big(F(\mu_1,\upsilon)+F(\mu_2,\upsilon)\big)d\upsilon\\
&\le\frac{F(\mu_1,\delta_1)+F(\mu_1,\delta_2)+F(\mu_2,\delta_1)+F(\mu_2,\delta_2)}{4}.
\end{aligned}$$

3. Main Theorem and Proof

3.1. Generalized Inequalities for $\mathcal{PF}$ on the Real Line

Let us obtain the generalized $\mathcal{HJI}$ for $\mathcal{PF}$ on the real line:

Theorem 3.1. Let $f: [\mu, \mu + \phi(\delta, \mu)] \to \mathbb{R}_+$ be a preinvex function and $f \in [\mu, \mu + \phi(\delta, \mu)]$. Then the inequality holds

$$\frac{\phi(\delta,\mu)}{n}\sum_{k=1}^{n} f\left(\frac{2\omega_{k-1}+\phi(\omega_k,\omega_{k-1})}{2}\right) \leq \int_{\mu}^{\mu+\phi(\delta,\mu)} f(\omega)d\omega$$
$$\leq \frac{\phi(\delta,\mu)}{2n}\left[f(\mu)+2\sum_{k=1}^{n-1} f(\omega_k)+f(\delta)\right], \tag{1}$$

where $\omega_k = \mu + k\frac{\phi(\delta,\mu)}{n}$, $k = 0,1,2, \ldots, n;\ \mu < \delta, n \in N$.

Proof. Because of preinvexity of f on each sub-interval $[\omega_{k-1}, \omega_{k-1} + \phi(\omega_k, \omega_{k-1})] \subseteq [\mu, \mu + \phi(\delta, \mu)]$, $k = 1,2, \ldots, n$, we can write

$$f(\omega_{k-1} + \Upsilon\phi(\omega_k, \omega_{k-1})) \leq \Upsilon f(\omega_k) + (1-\Upsilon)\, f(\omega_{k-1}) \tag{2}$$

for all $\Upsilon \in [0,1]$

Integrating (2) with respect to Υ on $[0,1]$, then

$$\int_0^1 f(\omega_{k-1}+\Upsilon\phi(\omega_k,\omega_{k-1}))d\Upsilon \leq \frac{f(\omega_{k-1})+f(\omega_k)}{2}. \tag{3}$$

Changing of variable $\omega = \omega_{k-1} + \Upsilon\phi(\omega_k, \omega_{k-1})$ in (3), we find

$$\int_{\omega_{k-1}}^{\omega_{k-1}+\phi(\omega_k,\omega_{k-1})} f(\omega)d\omega \leq \frac{\phi(\omega_k,\omega_{k-1})}{2}\big(f(\omega_{k-1})+f(\omega_k)\big). \tag{4}$$

Taking the sum over k from 1 to n on (4), we have

$$\sum_{k=1}^{n}\int_{\omega_{k-1}}^{\omega_{k-1}+\phi(\omega_k,\omega_{k-1})} f(\omega)d\omega = \int_{\mu}^{\mu+\phi(\delta,\mu)} f(\omega)d\omega \leq \sum_{k=1}^{n}\frac{\phi(\omega_k,\omega_{k-1})}{2}\big(f(\omega_{k-1})+f(\omega_k)\big)$$
$$\leq \frac{1}{2}\max_k\{\phi(\omega_k,\omega_{k-1})\}\sum_{k=1}^{n}\big(f(\omega_{k-1})+f(\omega_k)\big)$$
$$= \frac{\phi(\delta,\mu)}{2n}\left[f(\omega_0)+f(\omega_1)+\sum_{k=2}^{n-1}\big(f(\omega_{k-1})+f(\omega_k)\big)+f(\omega_{n-1})+f(\omega_n)\right]$$
$$= \frac{\phi(\delta,\mu)}{2n}\left[f(\mu)+2\sum_{k=1}^{n-1} f(\omega_k)+f(\delta)\right]. \tag{5}$$

Then we can write

$$f\left(\frac{2\omega_{k-1}+\phi(\omega_k,\omega_{k-1})}{2}\right)=f\left(\frac{\omega_{k-1}+\curlyvee\phi(\omega_k,\omega_{k-1})}{2}+\frac{\omega_{k-1}+\curlyvee\phi(\omega_k,\omega_{k-1})}{2}\right)$$
$$\leq\frac{1}{2}[f(\omega_{k-1}+\curlyvee\phi(\omega_k,\omega_{k-1}))+f(\omega_{k-1}+\curlyvee\phi(\omega_k,\omega_{k-1}))]. \quad (6)$$

Applying on (6) by using similar way in (2)-(4), we get

$$\frac{\phi(\delta,\mu)}{n}\sum_{k=1}^{n}f\left(\frac{2\omega_{k-1}+\phi(\omega_k,\omega_{k-1})}{2}\right)\leq\int_{\mu}^{\mu+\phi(\delta,\mu)}f(\omega)d\omega. \quad (7)$$

From (5) and (7), we have (1).

Remark 3.2. If $n = 1$ in Theorem 3.1, the inequalities in [13] are obtained.

Let us now obtain a generalized $\mathcal{OTI}$ for $\mathcal{PF}$ on the real:

Theorem 3.3. Let be $f:[\mu,\mu+\phi(\delta,\mu)]\to\mathbb{R}_+$ be a preinvex function. Then the following inequality holds

$$\int_{\mu}^{\mu+\phi(\delta,\mu)}f(\omega)d\omega-\phi(\delta,\mu)f(\upsilon)\leq\frac{\phi(\delta,\mu)}{2n}\left[f(\mu)+2\sum_{k=1}^{n-1}f(\omega_k)+f(\delta)\right] \quad (8)$$

*f*or all $\upsilon\in[\mu,\mu+\phi(\delta,\mu)]$, $\omega_k=\mu+k\frac{\phi(\delta,\mu)}{n}$, $k=0,1,2,\ldots,n;$ $\mu_1<\mu_2$, $n\in N$.

Proof. Let $\upsilon\in w[\omega_{k-1},\omega_{k-1}+\phi(\omega_k,\omega_{k-1})]$, $k=1,2,\ldots,n$. Owing to preinvexity of f on $[\omega_{k-1},\omega_{k-1}+\phi(\upsilon,\omega_{k-1})]$, $k=1,2,\ldots,n$, then

$$\int_{\omega_{k-1}}^{\omega_{k-1}+\phi(\upsilon,\omega_{k-1})}f(\omega)d\omega\leq\frac{\phi(\upsilon,\omega_{k-1})}{2}\big(f(\omega_{k-1})+f(\upsilon)\big). \quad (9)$$

Using preinvexity of f on $[\upsilon,\upsilon+\phi(\omega_k,\upsilon)]$, $k=1,2,\ldots,n$, we have

$$\int_{\upsilon}^{\upsilon+\phi(\omega_k,\upsilon)}f(\omega)d\omega\leq\frac{\phi(\omega_k,\upsilon)}{2}\big(f(\upsilon)+f(\omega_k)\big). \quad (10)$$

Adding the inequalities (9) and (10), we have

$$\int_{\omega_{k-1}}^{\omega_{k-1}+\phi(\upsilon,\omega_{k-1})} f(\omega)d\omega + \int_{\upsilon}^{\upsilon+\phi(\omega_k,\upsilon)} f(\omega)d\omega = \int_{\omega_{k-1}}^{\omega_{k-1}+\phi(\omega_k,\omega_{k-1})} f(\omega)d\omega$$
$$\leq \frac{\phi(\upsilon,\omega_{k-1})}{2}\big(f(\omega_{k-1})+f(\upsilon)\big) + \frac{\phi(\omega_k,\upsilon)}{2}\big(f(\upsilon)+f(\omega_k)\big)$$
$$\leq \frac{\phi(\omega_k,\omega_{k-1})}{2}\{f(\omega_{k-1})+f(\omega_k)\} + \frac{\phi(\delta,\mu)}{n}f(\upsilon). \tag{11}$$

Taking the sum over k from 1 to n on (11), we obtain (8).

3.2. Some Generalized Inequalities for Two-Dimensional $\mathcal{PF}$

Now, let us prove the following generalized $\mathcal{HTI}$ for two-dimensional $\mathcal{PF}$:

Theorem 3.4. Let $F:\theta \to \mathbb{R}_+$ be a preinvex function with respect to ϕ on θ. Then the following inequality holds

$$\frac{\phi(\delta_2,\delta_1)}{2n}\sum_{k=1}^{n}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F\left(\omega,\frac{2\upsilon_{k-1}+\phi(\upsilon_k,\upsilon_{k-1})}{2}\right)d\omega$$
$$+\frac{\phi(\mu_2,\mu_1)}{2n}\sum_{k=1}^{n}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F\left(\frac{2\omega_{k-1}+\phi(\omega_k,\omega_{k-1})}{2},\upsilon\right)d\upsilon \tag{12}$$
$$\leq \frac{1}{\phi(\mu_2,\mu_1)\phi(\delta_2,\delta_1)}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\omega_1,\upsilon_1)d\omega d\upsilon$$
$$\leq \frac{\phi(\delta_2,\delta_1)}{4n}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} [F(\omega,\delta_1)+F(\omega,\delta_2)]d\omega$$
$$+\frac{\phi(\mu_2,\mu_1)}{4n}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} [F(\mu_1,\upsilon)+F(\mu_2,\upsilon)]d\upsilon$$
$$+\frac{\phi(\delta_2,\delta_1)}{2n}\sum_{k=1}^{n-1}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F(\omega,\upsilon_k)d\omega + \frac{\phi(\mu_2,\mu_1)}{2n}\sum_{k=1}^{n-1}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\omega_k,\upsilon)d\upsilon,$$

where $\omega_k = \mu_1 + k\frac{\phi(\mu_2,\mu_1)}{n}$, $\upsilon_k = \delta_1 + k\frac{\phi(\delta_2,\delta_1)}{n}$, $k = 0,1,2,\ldots,n;\ n \in N$.

Proof. By using Theorem 3.1, we have

$$\frac{\phi(\delta_2,\delta_1)}{n}\sum_{k=1}^{n} F\left(\omega,\frac{2\upsilon_{k-1}+\phi(\upsilon_k,\upsilon_{k-1})}{2}\right) \leq \int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\omega,\upsilon)d\upsilon$$

$$\leq \frac{\phi(\delta_2,\delta_1)}{2n}\left[F(\omega,\delta_1)+F(\omega,\delta_2)+2\sum_{k=1}^{n-1}F\,(\omega,\upsilon_k)\right].$$

Consequently,

$$\frac{\phi(\delta_2,\delta_1)}{n}\sum_{k=1}^{n}F\left(\omega,\frac{2\upsilon_{k-1}+\phi(\upsilon_k,\upsilon_{k-1})}{2}\right)\leq\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}F\,(\omega,\upsilon)d\upsilon \qquad (13)$$
$$\leq \frac{\phi(\delta_2,\delta_1)}{2n}\left[F(\omega,\delta_1)+F(\omega,\delta_2)+2\sum_{k=1}^{n-1}F\,(\omega,\upsilon_k)\right].$$

Integrating all sides of (13) over $[\mu_1,\mu_1+\phi(\mu_2,\mu_1)]$, we get

$$\frac{\phi(\delta_2,\delta_1)}{n}\sum_{k=1}^{n}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}F\left(\omega,\frac{2\upsilon_{k-1}+\phi(\upsilon_k,\upsilon_{k-1})}{2}\right)d\omega$$
$$\leq\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}F\,(\omega,\upsilon)d\upsilon\,d\omega \qquad (14)$$
$$\leq\frac{\phi(\delta_2,\delta_1)}{2n}\left[\begin{array}{c}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}F\,(\omega,\delta_1)d\omega+\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}F\,(\omega,\delta_2)d\omega\\ +2\sum_{k=1}^{n-1}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}F\,(\omega,\upsilon_k)d\omega\end{array}\right].$$

Similarly,

$$\frac{\phi(\mu_2,\mu_1)}{n}\sum_{k=1}^{n}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}F\left(\frac{2\omega_{k-1}+\phi(\omega_k,\omega_{k-1})}{2},\upsilon\right)d\upsilon$$
$$\leq\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}F(\omega,s)d\omega d\upsilon \qquad (15)$$
$$\leq\frac{\phi(\mu_2,\mu_1)}{2n}\left[\begin{array}{c}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}F\,(\mu_1,\upsilon)d\upsilon+\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}F\,(\mu_2,\upsilon)d\upsilon\\ +2\sum_{k=1}^{n-1}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}F\,(\omega_k,\upsilon)d\upsilon\end{array}\right].$$

Adding (14) and (15), we have (12). That completes the proof.

Remark 3.5. If n = 1 in Theorem 3.4, the inequalities in [3] are obtained.

Corollary 3.6. Under the assumptions of Theorem 3.4, we have the following inequalities

$$\begin{aligned}
&\sum_{k=1}^{n} F\left(\frac{2\mu_1+\phi(\mu_2,\mu_1)}{2},\frac{2\upsilon_{k-1}+\phi(\upsilon_k,\upsilon_{k-1})}{2}\right)\\
&+\sum_{k=1}^{n} F\left(\frac{2\omega_{k-1}+\phi(\omega_k,\omega_{k-1})}{2},\frac{2\delta_1+\phi(\delta_2,\delta_1)}{2}\right)\\
&\le \frac{n}{\phi(\delta_2,\delta_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F\left(\frac{2\mu_1+\phi(\mu_2,\mu_1)}{2},\upsilon\right)d\upsilon \\
&+\frac{n}{\phi(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F\left(\omega,\frac{2\delta_1+\phi(\delta_2,\delta_1)}{2}\right)d\omega;
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\frac{n}{\phi(\delta_2,\delta_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} [F(\mu_1,\upsilon)+F(\mu_2,\upsilon)]d\upsilon\\
&+\frac{n}{\phi(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} [F(\omega,\delta_1)+F(\omega,\delta_2)]d\omega\\
&\le F(\mu_1,\delta_1)+F(\mu_1,\delta_2)+F(\mu_2,\delta_1)+F(\mu_2,\delta_2)\\
&+\sum_{k=1}^{n-1}[F(\mu_1,\upsilon_k)+F(\mu_2,\upsilon_k)+F(\omega_k,\delta_1)+F(\omega_k,\delta_2)].
\end{aligned} \tag{17}$$

Let us generalize $\mathcal{OTJ}$ for two-dimensional $\mathcal{PF}$:

Theorem 3.7. Let $F:\theta\to\mathbb{R}_+$ be a preinvex function with respect to ϕ on θ. Then

$$\begin{aligned}
&\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\omega,\upsilon)d\upsilon d\omega\\
&\le \frac{\phi(\mu_2,\mu_1)(n+1)}{4n}\left[\begin{array}{c}\displaystyle\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\mu_1,\upsilon)d\upsilon+\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\mu_2,\upsilon)d\upsilon\\ \displaystyle +\frac{2}{(n+1)}\sum_{k=1}^{n-1}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\omega_k,\upsilon)d\upsilon\end{array}\right]
\end{aligned} \tag{18}$$

$$+\frac{\phi(\delta_2,\delta_1)(n+1)}{4n}\left[\begin{array}{c}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F\,(\omega,\delta_1)d\omega+\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F\,(\omega,\delta_2)d\omega \\ +\frac{2}{(n+1)}\sum_{k=1}^{n-1}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F\,(\omega,\upsilon_k)d\omega\end{array}\right],$$

where ω_k and υ_k are defined as in Theorem 3.4.

Proof. By using Theorem 3.3 for $F_\upsilon(\omega)=F(\omega,\upsilon)$ at $\omega=\mu_2$, we get

$$\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F\,(\omega,\upsilon)d\omega-\phi(\mu_2,\mu_1)F(\mu_2,\upsilon)$$
$$\leq\frac{\phi(\mu_2,\mu_1)}{2n}\left[F(\mu_1,\upsilon)+2\sum_{k=1}^{n-1}F\,(\omega_k,\upsilon)+F(\mu_2,\upsilon)\right]. \tag{19}$$

Integrating all of (19) with respect to υ on $[\delta_1,\delta_1+\phi(\delta_2,\delta_1)]$, we have

$$\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F\,(\omega,\upsilon)d\omega d\upsilon$$
$$\leq\frac{\phi(\mu_2,\mu_1)}{2n}\left[\begin{array}{c}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F\,(\mu_1,\upsilon)d\upsilon+(1+2n)\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F\,(\mu_2,\upsilon)d\upsilon \\ +2\sum_{k=1}^{n-1}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F\,(\omega_k,\upsilon)d\upsilon\end{array}\right]. \tag{20}$$

Similarly, for $F_\upsilon(\omega)=F(\omega,\upsilon)$ at $\omega=\mu_1$,

$$\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F\,(\omega,\upsilon)\,d\omega d\upsilon$$

$$\leq \frac{\phi(\mu_2,\mu_1)}{2n}\left[\begin{array}{c}(1+2n)\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\mu_1,v)dv+\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\mu_2,v)dv \\ +2\sum_{k=1}^{n-1}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\omega_k,v)dv\end{array}\right]. \quad (21)$$

Adding (20) and (21), we can write

$$\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\omega,v)d\omega dv$$

$$\leq \frac{\phi(\mu_2,\mu_1)}{2n}\left[\begin{array}{c}(n+1)\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\mu_1,v)dv+(n+1)\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\mu_2,v)dv \\ +2\sum_{k=1}^{n-1}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(v_k,v)dv\end{array}\right]. \quad (22)$$

Using the similar way as Theorem 3.3 for $F_\omega(v)=F(\omega,v)$ at $v=\delta_1$ and $v=\delta_2$, then

$$\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\omega,v)\, d\omega dv$$

$$\leq \frac{\phi(\delta_2,\delta_1)}{2n}\left[\begin{array}{c}(n+1)\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F(\omega,\delta_1)d\omega+(n+1)\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F(\omega,\delta_2)d\omega \\ +2\sum_{k=1}^{n-1}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)} F(\omega_k,v)dv\end{array}\right]. \quad (23)$$

By adding (22) and (23), the inequality (18) is obtained.

Corollary 3.8. Under the assumptions of Theorem 3.7, for $n=1,2$ we obtain the following inequalities.

$$\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F(\omega,v)\, d\omega dv$$

$$\leq \frac{\phi(\delta_2,\delta_1)}{2}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}[F(\omega,\delta_1)+F(\omega,\delta_2)]d\omega$$
$$+\frac{\phi(\mu_2,\mu_1)}{2}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}[F(\mu_1,\upsilon)+F(\mu_2,\upsilon)]d\upsilon\,;$$

$$\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)} F\,(\omega,\upsilon)\,d\omega d\upsilon$$
$$\leq \frac{\phi(\delta_2,\delta_1)}{8}\int_{\mu_1}^{\mu_1+\phi(\mu_2,\mu_1)}\left[3F(\omega,\delta_1)+2F\left(\omega,\frac{2\delta_1+\phi(\delta_2,\delta_1)}{2}\right)+3F(\omega,\delta_2)\right]d\omega$$
$$+\frac{\phi(\mu_2,\mu_1)}{8}\int_{\delta_1}^{\delta_1+\phi(\delta_2,\delta_1)}\left[3F(\mu_1,\upsilon)+2F\left(\frac{2\mu_1+\phi(\mu_2,\mu_1)}{2},\upsilon\right)+3F(\mu_2,\upsilon)\right]d\upsilon.$$

4. Conclusion

In this paper, we obtained some generalized inequalities for $\mathscr{PF}$ on the real line and on the coordinates. Similar results for various classes of these functions can be verified. Applying some type of inequalities for functions is another promising direction for future research.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

The authors contributed equally.

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