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Asymptotic Frame Fields of Rational Bézier Curves [1]

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Abstract

Bézier curves are a type of curves used in computer aided design and related fields. The curves can be defined with the help of De Casteljau algorithm, which is one of the most basic elements of curve and surface design, and Bernstein polynomials, which facilitate theoretical developments. A rational Bézier curve may be evaluated by applying the de Casteljau algorithm to both numerator and denominator and finally dividing through. The curves are defined by suitable control points and corresponding scalar weights. In this work, we constitute the asymptotic orthonormal frame field of a spacelike quadratic rational Bézier curve at all points on 2 and 3-dimensional lightlike cones which are degenerate surfaces in Minkowski 3 and 4-spaces. We get the formulas of curvatures for a spacelike quadratic rational Bézier curve 2 and 3-dimensional lightlike cones.

Keywords: *Asymptotic frame field, Rational bézier curve, Lightlike cone*

Rasyonel Bézier Eğrilerinin Asimptotik Çatı Alanları

Öz

Bézier eğrileri, bilgisayar destekli tasarım ve bununla ilişkili alanlarda kullanılan bir eğri türüdür. Bu eğriler eğri ve yüzey tasarımının en temel unsurlarından biri olan De Casteljau algoritması ve teorik gelişmeleri kolaylaştıran Bernstein polinomları yardımıyla tanımlanabilir. Rasyonel bir Bézier eğrisi, hem paya hem de paydaya de Casteljau algoritması uygulanarak ve son olarak bölünerek değerlendirilebilir. Bu eğriler, uygun kontrol noktaları ve karşılık gelen skaler ağırlıklarla tanımlanır. Bu çalışmada, Minkowski 3 ve 4-uzaylarında dejenere yüzeyler olan 2 ve 3-boyutlu lightlike konilerinde bir spacelike kuadratik rasyonel Bézier eğrisinin bütün noktalarında asimptotik ortonormal çatı alanını oluşturduk. Spacelike kuadratik rasyonel Bézier eğrisi 2 ve 3-boyutlu lightlike koniler için eğrilik formüllerini elde ettik.

Anahtar Kelimeler: *Asimptotik çatı alanı, Rasyonel bézier curve, Lightlike koni*

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I. INTRODUCTION

Any space curve is studied by assigning at each point a specific frame in differential geometry. In the 3-dimensional spaces a frame field is a set of three unit vector fields and the ratio of these vectors along the curve is usually expressed with regard to the vectors themselves by the famous Frenet formulas [7]. The curvature and the torsion functions of arc length in the Frenet formulas has attention because a curve in 3-dimenasional spaces is completely determined (up to Euclidean motions) by these functions. Although Frenet frame fields are one of the most common moving frames, alternative frame fields may be preferred as they will be more useful at times. For example, different moving frame fields may be preferred, which are more convenient when studying a curve on a surface, or working with lightlike curves in non-Euclidean spaces, or studying curves on a degenerate surface like a lightlike cone.

Among the most studied non-Euclidean spaces in differential geometry is the Minkowski space equipped with the metric which is a non-degenerate, symmetric and bilinear form. A differentiable curve is called spacelike, timelike or lightlike if its velocity vector is spacelike, timelike or lightlike, respectively. A surface in Minkowski 3-space R_1^3 (or a hypersurface in Minkowski 4-space R_1^4) is called non-degenerate or degenerate if induced metric on its tangent plane is non-degenerate or degenerate, respectively [4]. To study a curve on 2 and 3-dimensional lightlike cones which are respectively degenerate surface in R_1^3 and degenerate hypersurface in R_1^4, it is better to choose a more suitable moving Frame, called the asymptotic orthonormal frame curve field (or cone Frenet frame) of the curve (see [2,3]).

Bézier curves, developed in 1960 by French engineer Pierre Bézier for use in the design of the car bodies, are parametric curve types that have been used in many disciplines recently. Bézier curves are commonly used in industrial design such as automobiles, ships, airframe and product design [1, 5]. Although Bézier curves can represent a wide variety of curves, the conic sections cannot be represented in the Bézier form. In order to be able to include conic sections in the Bézier form, rational quadratic Bézier curves are defined. A rational Bézier curve is defined by control points and corresponding scalar weights. To investigate the geometry of Bézier curves, moving frame fields and curvatures of those curves are studied by different spaces or surfaces [8,9,13].

Darboux frame and geodesic curvature of a quadratic rational Bézier curve at the end points on two-dimensional sphere have studied in [13]. The frame field for quadratic rational Bézier curves on 2-dimensional sphere have been generalized to hyperquadrics in Minkowski 3-space and Darboux frame and geodesic curvature of a non-null quadratic rational Bézier curve at the end points on 2-dimensional de Sitter space S_1^2 and 2-dimensional anti de Sitter space H_0^2 are presented [11, 12]. However, this issue has not been studied so far in 2-dimensional lightlike cone which includes all lightlike curves in Minkowski 3-space R_1^3.

We deal with geometry of rational Bézier curves on 2-dimensional lightlike cone which are degenerate surface in R_1^3. We obtain the elements of the frame field of a rational Bézier curve on 2-dimensional lightlike cone Q^2 in Minkowski 3-space. Thus, we derive the asymptotic frame field and cone curvature function of a spacelike quadratic rational Bézier curve at all points and the formulas of for the quadratic rational Bézier curve on 2-dimensional lightlike cone Q^2. Moreover, we generalize the results on 3-dimensional lightlike cone which is degenerate hypersurface in Minkowski 4-space R_1^4.

II. PRELIMINARIES

In this section, we give some explanatory materials including short information about cone Frenet frame fields of spacelike curves in a 2- dimensional lightlike cone Q^2 in Minkowski 3-space R_1^3 and 3-dimensional lightlike cone Q^2 in Minkowski 4-space R_1^4.

Minkowski 3-space R_1^3 and Minkowski 4-space R_1^4 are the metric spaces endowed with the symmetric, bilinear and non-degenerate metric $<,>$. A tangent vector x in R_1^n, n=3,4, is called spacelike (resp. timelike, lightlike) if it satisfies $<x,x>> 0$ or $x = 0$ (resp. $<x,x>< 0$, $<x,x>= 0$ and $x \neq 0$). A curve α in R_1^n is said spacelike (resp. timelike, lightlike) if its velocity vector is spacelike (resp. timelike, lightlike). A surface in R_1^3 is a degenerate if induced metric on its tangent plane is degenerate. 2-dimensional lightlike cone Q^2 which is a degenerate surface in R_1^3 is the set of all null vectors of R_1^3 given by

$$Q^2 = Q_1^2(0) = \left\{x \in \mathbb{R}_1^3 \middle| <x,x>= 0\right\} - \{(0,0,0)\}.$$

All curves in a 2-dimensional lightlike cone Q^2 are spacelike [2, 3, 6]. For a spacelike curve $\gamma: I \to Q^2 \subset R_1^3$ with arc length parameter s in 2-dimensional lightlike cone Q^2 in Minkowski 3-space R_1^3, we have the spacelike unit tangent vector field $T(s) = \gamma'(s)$ and the unit normal vector field $N(s) = -\gamma''(s) - \frac{1}{2} < \gamma''(s), \gamma''(s) > \gamma(s)$ satisfying following conditions

$$<\gamma(s), N(s)>= 1,\ <N(s), N(s)>=<T(s), N(s)>= 0.$$

The frame field $\{\boldsymbol{\gamma}, \boldsymbol{T}, \boldsymbol{N}\}$ is called an asymptotic orthonormal frame field along the curve γ in Q^2. The derivative equations of the asymptotic orthonormal frame field are given by

$$\begin{aligned} \gamma'(s) &= T(s), \\ T'(s) &= \kappa(s)\gamma(s) - N(s), \\ N'(s) &= -\kappa(s)T(s), \end{aligned} \tag{1}$$

where $\kappa(s) = -\frac{1}{2} < \gamma''(s), \gamma''(s) >$ is the cone curvature function [2, 10].

3-dimensional lightlike cone Q^3 which is a degenerate hypersurface in R_1^4 is the set of all null vectors of R_1^4 given by

$$Q^3 = \left\{x \in \mathbb{R}_1^4 \middle| <x,x>= 0\right\} - \{(0,0,0,0)\}.$$

All curves in a 3-dimensional lightlike cone Q^3 are spacelike [3]. For a spacelike curve $\gamma: I \to Q^3 \subset R_1^4$ with arc length parameter s in 3-dimensional lightlike cone Q^3 in Minkowski 4-space R_1^4, we have the spacelike unit tangent vector field $T(s) = \gamma'(s)$, the unit normal vector field $N(s) = -\gamma''(s) - \frac{1}{2} < \gamma''(s), \gamma''(s) > \gamma(s)$ and the binormal vector field $Q(s) = \frac{1}{\tau(s)}\{\gamma'''(s) + < \gamma''(s), \gamma''(s) > \gamma'(s) + < \gamma'''(s), \gamma''(s) > \gamma(s)\}$ satisfying following conditions

$$\begin{aligned} &<\gamma(s), \gamma(s)>= <N(s), N(s)>=<\gamma(s), T(s)>=<\gamma(s), Q(s)>= 0, \\ &<T(s), N(s)>=<N(s), Q(s)>=<T(s), Q(s)>= 0, \\ &<\gamma(s), N(s)>=<T(s), T(s)>=<Q(s), Q(s)>= 1, \end{aligned}$$

where $\kappa(s) = -\frac{1}{2} < \gamma''(s), \gamma''(s) >$ and $\tau = \sqrt{<\gamma'''(s), \gamma'''(s)> - <\gamma''(s), \gamma''(s)>^2}$ is the cone torsion (or second cone curvature). The frame field $\{\boldsymbol{\gamma}, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{Q}\}$ is the asymptotic orthonormal frame field along the curve γ in Q^3. The derivative equations of the asymptotic orthonormal frame field are given by

$$\begin{aligned} \gamma'(s) &= T(s), \\ T'(s) &= \kappa(s)\gamma(s) - N(s), \\ N'(s) &= -\kappa(s)T(s) - \tau(s)Q(s), \\ Q'(s) &= \tau(s)\gamma(s) \end{aligned} \tag{2}$$

[3].

A rational Bézier curve of degree n with control points $b_0, b_1, \ldots, b_n$ and corresponding scalar weights ω_i, $0 \leq i \leq n$, is defined to be

$$P(t) = \frac{\sum_{i=0}^{n} \omega_i b_i B_{i,n}(t)}{\sum_{i=0}^{n} \omega_i B_{i,n}(t)}, \quad t \in [0,1]$$

where

$$B_{i,n}(t) = \begin{cases} \dfrac{n!}{(n-i)!\, i!}, & if\ 0 \le i \le n \\ 0, & otherwise \end{cases}$$

are called the Bernstein polynomials, with the understanding that if $\omega_i = 0$, then $\omega_i b_i$ is to be replaced by b_i. It is assumed that all the weights are non-zero. Rational Bézier curves are called quadratics for $n = 2$ [1, 5]. If $\omega_0 = \omega_1 = \cdots = \omega_n$, then the curve becomes an integral Bézier curve.

III. QUADRATIC RATIONAL BEZIER CURVES IN 2-DIMENSIONAL LIGHTLIKE CONE

In this section we give certain geometric properties for a spacelike quadratic Bézier curve in 2-dimensional lightlike cone $Q^2 \subset R_1^3$. Now, we suppose that a spacelike quadratic rational Bézier curve with its arc length s, weights ω_i control points b_i, $0 \le i \le 2$,

$$P(s) = \frac{\sum_{i=0}^{2} \omega_i b_i B_{i,2}(s)}{\sum_{i=0}^{2} \omega_i B_{i,2}(s)}, \tag{3}$$

lies in 2-dimensional lightlike cone Q^2. So we have <P(s),P(s)>=0. The first order derivative of the spacelike quadratic rational Bézier curve is given by the following equation

$$P'(s) = \frac{\sum_{i=0}^{2} a_i B_{i,2}(s)}{\left(\sum_{i=0}^{2} \omega_i B_{i,2}(s)\right)^2}, \tag{4}$$

where $a_0 = 2\omega_0\omega_1\Delta^1 b_0$, $a_1 = \omega_0\omega_2(b_2 - b_0), a_2 = 2\omega_1\omega_2\Delta^1 b_1$ such that $\Delta^1 b_0 = b_1 - b_0$ and $\Delta^1 b_1 = b_2 - b_1$. The second derivative of (3) is given by

$$P''(s) = \frac{\sum_{i=0}^{5} c_i B_{i,5}(s)}{\left(\sum_{i=0}^{2} \omega_i B_{i,2}(s)\right)^4}, \tag{5}$$

where

$$c_0 = 2\omega_0^3\omega_2\,(b_2 - b_0) + (4\omega_0^3\omega_1 + 8\omega_0^2\omega_1^2)\,\Delta^1 b_0,$$

$$c_1 = \frac{1}{5}(6\omega_0^3\omega_2\,(b_2 - b_0) + (8\omega_0^2\omega_1^2 - 16\omega_0\omega_1^3 - 8\omega_0^2\omega_1\omega_2)\Delta^1 b_0 + (4\omega_0^2\omega_1\omega_2)\Delta^1 b_1),$$

$$c_2 = \frac{1}{10}\Big((8\omega_0\omega_1^2\omega_2 + 8\omega_0^2\omega_1\omega_2)\Delta^1 b_1 + (16\omega_0^2\omega_1\omega_2 - 8\omega_0\omega_1^2\omega_2 - 4\omega_0^2\omega_2^2)(b_2 - b_0) - 24\omega_0\omega_1^2\omega_2\,\Delta^1 b_0\Big),$$

$$c_3 = \frac{1}{10}(24\omega_0\omega_1^2\omega_2\,\Delta^1 b_1 - (8\omega_0\omega_1^2\omega_2 + 8\omega_0\,\omega_1\omega_2^2)\Delta^1 b_0 + (-16\omega_0\,\omega_1\omega_2^2 + 8\omega_0\omega_1^2\omega_2 + 4\omega_0^2\omega_2^2)(b_2 - b_0)),$$

$$c_4 = \frac{1}{5}\Big((-8\omega_1^2\omega_2^2 + 16\omega_1^3\omega_2 + 8\omega_0\,\omega_1\omega_2^2)\Delta^1 b_1 - 6\omega_0\,\omega_2^3(b_2 - b_0) - 4\omega_0\,\omega_1\omega_2^2\Delta^1 b_0\Big)$$

and

$$c_5 = (-4\omega_1\omega_2^3 + 8\omega_1^2\omega_2^2)\Delta^1 b_1 - 2\omega_0\omega_2^3(b_2 - b_0)$$

[12].

In the following theorem, we give the asymptotic orthonormal frame field of a spacelike quadratic rational Bézier curve in 2-dimensional lightlike cone $Q^2 \subset R_1^3$ for all $t \in R$.

Theorem 1. Let P(s) be a spacelike quadratic rational Bézier curve parametrized by arc length s in 2-dimensional lightlike cone Q^2 in Minkowski 3-space R_1^3. Then the asymptotic orthonormal frame field $\{\boldsymbol{P}, \boldsymbol{T}, \boldsymbol{N}\}$ and cone curvature function along P(s) are given by

$$\mathrm{P}(s) = \frac{\sum_{i=0}^{2}\omega_i b_i B_{i,2}(s)}{\sum_{i=0}^{2}\omega_i B_{i,2}(s)},$$

$$T(s) = \frac{\sum_{i=0}^{2} a_i B_{i,2}(s)}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^2},$$

$$N(s) = -\frac{1}{2}\left(\frac{2\sum_{i=0}^{5}\sum_{k=0}^{2} c_i\omega_k B_{i,5}(s)\, B_{k,2}(s)\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^4 + \sum_{i,j=0}^{5}\sum_{k=0}^{2}\omega_k b_k B_{i,5}(s) B_{j,5}(s)\, B_{k,2}(s) < c_i, c_j >}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^9}\right) \quad (6)$$

and

$$\kappa(s) = -\frac{1}{2}\frac{\sum_{i,j=0}^{5} B_{i,5}(s) B_{j,5}(s) < c_i, c_j >}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^4\left(\sum_{i=0}^{2}\omega_j B_{j,2}(s)\right)^4}. \quad (7)$$

Proof. Assume that P(s) is a spacelike quadratic rational Bézier curve parametrized by arc length s defined by (3) in 2-dimensional lightlike cone Q^2 in Minkowski 3-space R_1^3. The formulation of P(s) and T(s) can be clearly see from (3) and (4). Taking into consideration (5), we have

$$\begin{aligned}
\kappa(s) &= -\frac{1}{2} < P''(s), P''(s) > \\
&= -\frac{1}{2} < \frac{\sum_{i=0}^{5} c_i B_{i,5}(s)}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^4}, \frac{\sum_{j=0}^{5} c_j B_{j,5}(s)}{\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^4} > \\
&= -\frac{1}{2}\frac{\sum_{i,j=0}^{5} B_{i,5}(s) B_{j,5}(s) < c_i, c_j >}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^4\left(\sum_{i=0}^{2}\omega_j B_{j,2}(s)\right)^4}.
\end{aligned} \quad (8)$$

By using (3), (5) and (8), we obtain

$$\begin{aligned}
N(s) &= -P''(s) - \frac{1}{2} < P''(s), P''(s) > P(s) \\
&= -\frac{\sum_{i=0}^{5} c_i B_{i,5}(s)}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^4} - \frac{1}{2}\left(\frac{\sum_{i,j=0}^{5}\sum_{k=0}^{2}\omega_k b_k B_{i,5}(s)\, B_{j,5}(s)\; B_{k,2}(s) < c_i, c_j >}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^4\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^4\sum_{k=0}^{2}\omega_k B_{k,2}(s)}\right) \\
&= -\frac{1}{2}\left(\frac{2\sum_{i=0}^{5}\sum_{k=0}^{2} c_i\omega_k B_{i,5}(s)\, B_{k,2}(s)\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^4 + \sum_{i,j=0}^{5}\sum_{k=0}^{2}\omega_k b_k B_{i,5}(s) B_{j,5}(s)\, B_{k,2}(s) < c_i, c_j >}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^9}\right).
\end{aligned}$$

The following results give the asymptotic frame and the cone curvature for a spacelike quadratic rational Bézier curve at the points P(0) = b_0 and P(1) = b_2, respectively. The proofs are clear.

Corollary 1. The asymptotic orthonormal frame field $\{\boldsymbol{P},\boldsymbol{T},\boldsymbol{N}\}$ and cone curvature function κ of the spacelike quadratic rational Bézier curve $\mathrm{P(s)}$ parametrized by arclength s and defined by (3) at $\mathrm{s}=0$ in 2-dimensional lightlike cone Q^2 in R_1^3 are defined by

$$\mathrm{P}(0)=b_0,\quad T(0)=2\frac{\omega_1}{\omega_0}\Delta^1 b_0,\ N(0)=-\frac{1}{2}\left(\frac{2c_0{\omega_0}^4+\|c_0\|^2 b_0}{{\omega_0}^8}\right)$$

and

$$\kappa(0)=-\frac{1}{2}\frac{\|c_0\|^2}{{\omega_0}^8}.$$

Corollary 2. The asymptotic orthonormal frame field $\{\boldsymbol{P},\boldsymbol{T},\boldsymbol{N}\}$ and cone curvature function κ of the spacelike quadratic rational Bézier curve $\mathrm{P(s)}$ parametrized by arclength s and defined by (3) at $\mathrm{s}=1$ in 2-dimensional lightlike cone Q^2 in R_1^3 are defined by

$$\mathrm{P}(1)=b_2,\quad T(1)=2\frac{\omega_1}{\omega_2}\Delta^1 b_1,\ N(1)=-\frac{1}{2}\left(\frac{2c_5{\omega_2}^4+\|c_5\|^2 b_2}{{\omega_2}^8}\right)$$

and

$$\kappa(1)=-\frac{1}{2}\frac{\|c_5\|^2}{{\omega_2}^8}.$$

IV. QUADRATIC RATIONAL BEZIER CURVES IN 3-DIMENSIONAL LIGHTLIKE CONE

In this section we obtain the asymptotic orthonormal frame of a spacelike quadratic Bézier curve in 3-dimensional lightlike cone $Q^3\subset R_1^4$. Now, we suppose that a spacelike quadratic rational Bézier curve with its arc length s, weights ω_i control points b_i, $0\leq i\leq 2$, defined by (3) in $Q^3\subset R_1^4$. So we have <P(s),P(s)>=0. The third order derivative of the spacelike quadratic rational Bézier curve is given by the following equation

$$P'''(s)=\frac{\sum_{i=0}^{12} h_i B_{i,12}(s)}{(\sum_{i=0}^{2}\omega_i B_{i,2}(s))^8}, \tag{9}$$

where

$$h_0=5(c_1-c_0)r_0-c_0z_0,$$

$$h_1=\frac{1}{12}(5(c_1-c_0)r_1+20(c_2-c_1)r_0-c_0z_1-5c_1z_0),$$

$$h_2=\frac{1}{66}(5(c_1-c_0)r_2+20(c_2-c_1)r_1+30(c_3-c_2)r_0-c_0z_2-5c_1z_1-10c_2z_0),$$

$$h_3=\frac{1}{220}(5(c_1-c_0)r_3+20(c_2-c_1)r_2+30(c_3-c_2)r_1+20(c_4-c_3)r_0-c_0z_3-5c_1z_2-10c_2z_1-10c_3z_0),$$

$$h_4=\frac{1}{495}(5(c_1-c_0)r_4+20(c_2-c_1)r_3+30(c_3-c_2)r_2+20(c_4-c_3)r_1+5(c_5-c_4)r_0-c_0z_4-5c_1z_3-10c_2z_2-10c_3z_1-5c_4z_0),$$

$$h_5=\frac{1}{792}(5(c_1-c_0)r_5+20(c_2-c_1)r_4+30(c_3-c_2)r_3+20(c_4-c_3)r_2+5(c_5-c_4)r_1-c_0z_5-5c_1z_4-10c_2z_3-10c_3z_2-5c_4z_1-c_5z_0),$$

$h_6 = \frac{1}{924}(5(c_1 - c_0)r_6 + 20(c_2 - c_1)r_5 + 30(c_3 - c_2)r_4 + 20(c_4 - c_3)r_3 + 5(c_5 - c_4)r_2 - c_0 z_6 - 5c_1 z_5 - 10c_2 z_4 - 10c_3 z_3 - 5c_4 z_2 - c_5 z_1)$,

$h_7 = \frac{1}{792}(5(c_1 - c_0)r_7 + 20(c_2 - c_1)r_6 + 30(c_3 - c_2)r_5 + 20(c_4 - c_3)r_4 + 5(c_5 - c_4)r_3 - c_0 z_7 - 5c_1 z_6 - 10c_2 z_5 - 10c_3 z_4 - 5c_4 z_3 - c_5 z_2)$,

$h_8 = \frac{1}{495}(5(c_1 - c_0)r_8 + 20(c_2 - c_1)r_7 + 30(c_3 - c_2)r_6 + 20(c_4 - c_3)r_5 + 5(c_5 - c_4)r_4 - 5c_1 z_7 - 10c_2 z_6 - 10c_3 z_5 - 5c_4 z_4 - c_5 z_3)$,

$h_9 = \frac{1}{220}(20(c_2 - c_1)r_8 + 30(c_3 - c_2)r_7 + 20(c_4 - c_3)r_6 + 5(c_5 - c_4)r_5 - 10c_2 z_7 - 10c_3 z_6 - 5c_4 z_5 - c_5 z_4)$,

$h_{10} = \frac{1}{66}(30(c_3 - c_2)r_8 + 20(c_4 - c_3)r_7 + 5(c_5 - c_4)r_6 - 10c_3 z_7 - 5c_4 z_6 - c_5 z_5)$,

$h_{11} = \frac{1}{12}(20(c_4 - c_3)r_8 + 5(c_5 - c_4)r_7 - 5c_4 z_7 - c_5 z_6)$,

$h_{12} = 5(c_5 - c_4)r_8 - c_5 z_7$

and

$z_o = 8\omega_0^3(\omega_1 - \omega_0)$, $z_1 = 8\omega_0^2(6\omega_1^2 + \omega_0\omega_2 - 7\omega_0\omega_1)$,

$z_2 = 24\omega_0(4\omega_1^3 + 3\omega_0\omega_1\omega_2 - \omega_0^2\omega_2 - 6\omega_0\omega_1^2)$,

$z_3 = 64\omega_1^4 + 192\omega_0\omega_1^2\omega_2 + 24\omega_0^2\omega_2^2 - 120\omega_0^2\omega_1\omega_2 - 160\omega_0\omega_1^3$,

$z_4 = 160\omega_1^3\omega_2 + 120\omega_0\omega_1\omega_2^2 - 64\omega_1^4 - 192\omega_0\omega_1^2\omega_2 - 24\omega_0^2\omega_2^2$,

$z_5 = 24\omega_2(6\omega_1^2\omega_2 + \omega_0\omega_2^2 - 4\omega_1^3 - 3\omega_0\omega_1\omega_2)$,

$z_6 = 8\omega_2^2(7\omega_1\omega_2 - \omega_0\omega_2 - 6\omega_1^2)$, $z_7 = 8\omega_2^3(\omega_2 - \omega_1)$,

$r_o = \omega_0^4$, $r_1 = 8\omega_0^3\omega_1$, $r_2 = 4\omega_0^2(\omega_0\omega_2 + 6\omega_1^2)$, $r_3 = 8\omega_0\omega_1(3\omega_0\omega_2 + 4\omega_1^2)$,

$r_4 = 6\omega_0^2\omega_2^2 + 48\omega_0\omega_1^2\omega_2 + 16\omega_1^4$, $r_5 = \omega_1\omega_2(24\omega_0\omega_2 + 32\omega_1)$,

$r_6 = 4\omega_2^2(6\omega_1^2 + \omega_0\omega_2)$, $r_7 = 8\omega_1\omega_2^3$, $r_8 = \omega_2^4$.

In the following theorem, we give the asymptotic orthonormal frame field and curvatures of a spacelike quadratic rational Bézier curve in 3-dimensional lightlike cone $Q^3 \subset R_1^4$ for all $t \in R$.

Theorem 2. Let P(s) be a spacelike quadratic rational Bézier curve parametrized by arc length s in 3-dimensional lightlike cone Q^3 in Minkowski 4-space R_1^4. Then the asymptotic orthonormal frame field $\{\boldsymbol{P}, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{Q}\}$, cone curvature and cone torsion along P(s) are respectively given by

$$P(s)=\frac{\sum_{i=0}^{2}\omega_i b_i B_{i,2}(s)}{\sum_{i=0}^{2}\omega_i B_{i,2}(s)},$$

$$T(s)=\frac{\sum_{i=0}^{2}a_i B_{i,2}(s)}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{2}},$$

$$N(s) = -\frac{1}{2}\left(\frac{2\sum_{i=0}^{5}\sum_{k=0}^{2}c_i\omega_k B_{i,5}(s)\ B_{k,2}(s)\ \left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^{4}+\sum_{i,j=0}^{5}\sum_{k=0}^{2}\omega_k b_k B_{i,5}(s)B_{j,5}(s)\ B_{k,2}(s)<c_i,c_j>}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{9}}\right),$$

$$Q(s)=\frac{1}{\tau(s)}\left\{\frac{\sum_{i=0}^{12}h_i B_{i,12}(s)}{\left(\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{8}\right.}+\frac{\left(\sum_{i,j=0}^{5}B_{i,5}(s)B_{j,5}(s)<c_i,c_j>\right)\left(\sum_{k=0}^{2}a_k B_{k,2}(s)\right)}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{10}}+\frac{\left((\sum_{i=0}^{12}\sum_{j=0}^{5}B_{i,12}(s)\,B_{j,5}(s)<h_i,c_j>\right)\left(\sum_{k=0}^{2}w_k b_k B_{k,2}(s)\right)}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{13}}\right\},$$

$$\kappa(s)=-\frac{1}{2}\frac{\sum_{i,j=0}^{5}B_{i,5}(s)B_{j,5}(s)<c_i,c_j>}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{4}\left(\sum_{i=0}^{2}\omega_j B_{j,2}(s)\right)^{4}}$$

and

$$\tau(s)=\frac{1}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{4}}\left(\sum_{i,j=0}^{12}B_{i,12}(s)B_{j,12}(s)<h_i,h_j>-\left(\sum_{i,j=0}^{5}B_{i,5}(s)B_{j,5}(s)<c_i,c_j>\right)^{2}\right)^{\frac{1}{2}}. \qquad (10)$$

Proof. Assume that P(s) is a spacelike quadratic rational Bézier curve parametrized by arc length s defined by (3) in 3-dimensional lightlike cone Q^3 in Minkowski 4-space R_1^4. The formulation of P(s), T(s), $N(s)$ and κ(s) are clearly seen from (3), (4), (6) and (7). Now we focus on the proof of Q(s) and τ(s). From (9), we get

$$<P'''(s),P'''(s)>=\frac{<\sum_{i=0}^{12}h_i B_{i,12}(s),\sum_{j=0}^{12}h_j B_{j,12}(s)>}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{8}\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^{8}}$$

or

$$=\frac{\sum_{i,j=0}^{12}B_{i,12}(s)B_{j,12}(s)<h_i,h_j>}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{8}\left(\sum_{j=0}^{2}\omega_j B_{j,2}(s)\right)^{8}}=\frac{\sum_{i,j=0}^{12}B_{i,12}(s)B_{j,12}(s)<h_i,h_j>}{\left(\sum_{i=0}^{2}\omega_i B_{i,2}(s)\right)^{16}}. \qquad (11)$$

From (8) and (11), we get (10).

By using (3), (4), (5), (8), (9) and (10), we obtain

$$Q(s)=\frac{1}{\tau(s)}\{P'''(s)+<P''(s),P''(s)>P'(s)+<P'''(s),P''(s)>P(s)\}$$

$$= \frac{1}{\tau(s)} \left\{ \frac{\sum_{i=0}^{12} h_i B_{i,12}(s)}{(\sum_{i=0}^{2} \omega_i B_{i,2}(s))^8} + \frac{\left(\sum_{i,j=0}^{5} B_{i,5}(s) B_{j,5}(s) <c_i, c_j>\right)\left(\sum_{k=0}^{2} a_k B_{k,2}(s)\right)}{\left(\sum_{i=0}^{2} \omega_i B_{i,2}(s)\right)^{10}} + \right.$$

$$\left. \frac{\left((\sum_{i=0}^{12} \sum_{j=0}^{5} B_{i,12}(s) B_{j,5}(s) <h_i, c_j>\right)\left(\sum_{k=0}^{2} w_k b_k B_{k,2}(s)\right)}{\left(\sum_{i=0}^{2} \omega_i B_{i,2}(s)\right)^{13}} \right\}.$$

The following results give the asymptotic frame, the cone curvature and the cone torsion for a spacelike quadratic rational Bézier curve at the points P(0) = b_0 and P(1) = b_2, respectively. The proofs are clear.

Corollary 3. The asymptotic orthonormal frame field $\{\boldsymbol{P}, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{Q}\}$, cone curvature function κ and cone torsion τ of the spacelike quadratic rational Bézier curve P(s) parametrized by arc length s and defined by (3) at $s = 0$ in 3-dimensional lightlike cone Q^3 in R_1^4 are defined by

$$\mathrm{P}(0) = b_0, \quad T(0) = 2\frac{\omega_1}{\omega_0}\Delta^1 b_0, \ N(0) = -\frac{1}{2}\left(\frac{2c_0{\omega_0}^4 + \|c_0\|^2 b_0}{{\omega_0}^8}\right),$$

$$Q(0) = \frac{1}{\tau(0)}\left\{\frac{h_0}{{\omega_0}^8} + \frac{\|c_0\|^2 a_0}{{\omega_0}^{10}} + \frac{<h_0, c_0>\omega_0 b_0}{{\omega_0}^{13}}\right\},$$

$$\kappa(0) = -\frac{1}{2}\frac{\|c_0\|^2}{{\omega_0}^8}$$

and

$$\tau(0) = \frac{(\|h_0\|^2 - \|c_0\|^4)^{1/2}}{{\omega_0}^4}.$$

Corollary 4. The asymptotic orthonormal frame field $\{\boldsymbol{P}, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{Q}\}$, cone curvature function κ and cone torsion τ of the spacelike quadratic rational Bézier curve P(s) parametrized by arclength s and defined by (3) at $s = 1$ in 3-dimensional lightlike cone Q^3 in R_1^4 are defined by

$$\mathrm{P}(1) = b_2, \quad T(1) = 2\frac{\omega_1}{\omega_2}\Delta^1 b_1, \ N(1) = -\frac{1}{2}\left(\frac{2c_5{\omega_2}^4 + \|c_5\|^2 b_2}{{\omega_2}^8}\right)$$

$$Q(1) = \frac{1}{\tau(1)}\left\{\frac{h_{12}}{{\omega_2}^8} + \frac{\|c_5\|^2 a_2}{{\omega_2}^{10}} + \frac{<h_{12}, c_5>\omega_2 b_2}{{\omega_2}^{13}}\right\},$$

$$\kappa(1) = -\frac{1}{2}\frac{\|c_5\|^2}{{\omega_2}^8}$$

and

$$\tau(1) = \frac{(\|h_{12}\|^2 - \|c_5\|^4)^{1/2}}{{\omega_2}^4}.$$

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