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A Logarithmic Finite Difference Method for Numerical Solutions of the Generalized Huxley Equation

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Abstract. In this paper, numerical solutions of generalized Huxley equation are obtained by using a new scheme: Implicit logarithmic finite difference method (I-LFDM). The efficiency of the presented method is illustrated by a numerical example for different cases of parameters which confirm that obtained results are in good agreement with the exact solutions and numerical solutions obtained by some other methods in literature. The method is analyzed by von-Neumann stability analysis method and it is displayed that the method is unconditionally stable.

1. INTRODUCTION

Nonlinear partial differential equations are often used to model most of the problems in various fields such as physics, chemistry, biology, mathematics and engineering. One of these nonlinear partial differential equations is generalized Huxley equation.
The generalized Huxley equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \beta u\left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right), \qquad a < x < b, \quad t > 0 \tag{1}$$

with initial condition

$$u\left(x, 0\right) = f(x), \qquad a < x < b$$

and boundary conditions

$$u(a,t) = g_1\left(t\right), \quad u(b,t) = g_2\left(t\right), \quad t > 0$$

describes the propagation of a nerve impulse in nerve fibers and the movement of the wall in liquid crystals. Where $f(x)$, $g_1(t)$ and $g_2(t)$ are known functions, $\delta, \beta \geq 0$and $\gamma \in (0.1)$are given parameters.
Various numerical methods have been used to solve the equation (1) numerically by many researchers. Hashim et. al. [9] applied the Adomian decomposition method to solve the equation numerically. Variational iteration method (VIM) has been used to obtain the numerical solutions of the equation by Batiha et. al. [2]. Hashemi et. al. [8] used the homotopy perturbation method (HPM) and then Hemida and Mohamed [10] used the homotopy analysis method (HAM) for obtaining the numerical solutions of the

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equation. Inan [12, 13] used the explicit exponential finite difference method and implicit exponential finite difference method (I-EFDM) to solve the equation.
In this study, we present the implicit logarithmic finite difference method to obtain the numerical solutions of the generalized Huxley equation. Logarithmic finite difference methods have been used to solve various equations in literature. İsmail and Al-Basyoni [14] used the closed logarithmic finite difference method to solve the Troesch problem numerically. Srivastava et al. [16] used the closed logarithmic finite difference method to solve two-dimensional Burgers equation systems. The one-dimensional coupled Burgers equation was solved by Srivastava et al. [15] using the closed logarithmic finite difference method. Aljaboori [1] used the Crank-Nicolson logarithmic finite difference method to solve the combined Burgers equation numerically. El-Azab et al. [7] obtained numerical solutions of the Korteweg de Vries Burger (KdVB) equation using the open logarithmic finite difference method. Celikten et. al. [3] used the explicit logarithmic finite difference schemes to solve the Burgers equation. Modified Burgers equation as solved by Celikten [4] using the explicit logarithmic finite difference schemes. Celikten [5] obtained the numerical solutions of Burgers equation by using implicit and fully implicit logarithmic finite difference methods. Celikten and Sürek [6] used the explicit logarithmic finite difference method to solve the generalized Burgers-Fisher equation numerically.

2. MATERIALS AND METHODS

2.1. IMPLICIT LOGARITHMIC FINITE DIFFERENCE METHOD

We demonstrate the finite difference approximation of $u(x,t)$at the node point (x_i,t_n) by u_i^n in which $x_i = ih(i = 0,1,\ldots,N)$, $t_n = t_0 + nk(n = 0,1,2,\ldots)$, $h = \frac{b-a}{N}$ is the node size in x direction and $k$is the time step.
We reorganize Equation (1) to acquire

$$\frac{\partial u}{\partial t} = \beta u\left(1-u^{\delta}\right)\left(u^{\delta}-\gamma\right)+\frac{\partial^2 u}{\partial x^2}. \tag{2}$$

Multiplying equation (2) by e^u, we acquire the following equation:

$$\frac{\partial e^u}{\partial t} = e^u\left(\beta u\left(1-u^{\delta}\right)\left(u^{\delta}-\gamma\right)+\frac{\partial^2 u}{\partial x^2}\right) \tag{3}$$

using the finite difference approximations for derivatives in Equation (3) the following implicit logarithmic finite difference scheme is acquired
I-EFDM

$$u_i^{n+1} = u_i^n + \ln\left\{1 + k\left[\beta u_i^n\left(1-\left(u_i^n\right)^{\delta}\right)\left(\left(u_i^n\right)^{\delta}-\gamma\right)+\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{h^2}\right]\right\} \tag{4}$$

where $1 \le i \le N-1$.
Equation (4) is a system of nonlinear difference equations. We assume this nonlinear system of equations in the form

$$G(W) = 0 \tag{5}$$

where $G = [g_1, g_2, \ldots, g_{N-1}]^T$ and $W = \left[u_1^{n+1}, u_2^{n+1}, \ldots, u_{N-1}^{n+1}\right]^T$. The nonlinear Equation (5) is linearized using Newton's iterative approach, which yields the following iteration:
1) Determine $W^{(0)}$, a first guess.
2) For $m = 0,1,2,3\ldots$up to convergency do:
Resolve $J\left(W^{(m)}\right)\delta^{(m)} = -G\left(W^{(m)}\right)$;
Adjust $W^{(m+1)} = W^{(m)} + \delta^{(m)}$ where $J(W^{(m)})$ the Jacobian matrix which is appraised analytically. The initial estimate is based on the solution from the previous time step. The Newton iteration is halted at every time step when $\left\|G(W^{(m)})\right\| \le 10^{-5}$.

2.2. LOCAL TRUNCATION ERROR AND CONSISTENCY

In order to analyze the local truncation errors of the numerical scheme (4), the nonlinear term of the scheme has been linearized by replacing the quantity $\left(u_i^n\right)^\delta$ by local constant $\tilde{U}$. Hence the numerical scheme (4), convert into

$$u_i^{n+1} = u_i^n + \ln\left\{1 + k\left[\beta u_i^n\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}\right]\right\} \tag{6}$$

Since the scheme (6) is logarithmic, the examination will be improved by expanding the logarithmic term of the scheme into a Taylor's series. Hilal et al. [11] applied the same procedure to calculate the local truncation error of exponential finite difference schemes and examine their stability. If the scheme's logarithmic term is expanded to a Taylor series and the first term is used, the scheme can be expressed as:

$$u_i^{n+1} = u_i^n + k\beta u_i^n\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) + k\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}\right] \tag{7}$$

Expansion of the terms u_i^{n+1}, u_{i+1}^{n+1} and u_{i-1}^{n+1}about the point $(x_i, t_n)$by Taylor' s series and substitution into

$$T_i^n = u_i^{n+1} - u_i^n - k\beta u_i^n\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) - k\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}\right]$$

leads to

$$T_i^n = \left[\frac{\partial u}{\partial t} - \beta u\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right) - \frac{\partial^2 u}{\partial x^2}\right]_i^n + \frac{k}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + \ldots$$

Therefore the principal part of the local truncation error is as follows:

$$\frac{k}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n$$

Hence the local truncation error is $T_i^n = O(k) + O\left(h^2\right)$

Since $\lim\limits_{h,k\to 0}\left[O(k) + O\left(h^2\right)\right] = 0$ presented scheme is consistent. And the scheme is first order in time and second order in space.

2.3. STABILITY ANALYSIS

We will utilize the von Neumann stability analysis to analyze the scheme's stability, where the growth factor of a characteristic Fourier mode is specified as follows:

$$u_i^n = \varepsilon^n e^{I\phi ih},\ I = \sqrt{-1}. \tag{8}$$

von Neumann stability analysis is used to analyze the stability of finite difference schemes applied to linear partial differential equations. So we will investigate the stability of linear form of the scheme. By substituting the (8) equality into the (7) linear form of the scheme, we get the growth factors as follows:

$$\varepsilon = \frac{1 + k\beta\left(1-\tilde{U}\right)\left(\tilde{U}-\gamma\right)}{1 + \frac{2k}{h^2}\sin^2\frac{\phi h}{2}}.$$

Stability condition in von-Neumann method is $|\varepsilon| \le 1$

$|\varepsilon| \le 1$since $\beta \ge 0$ and $\gamma \in (0.1)$.Therefore I-LFDM generalized Huxley equation is unconditionally stable.

3. NUMERICAL RESULTS AND DISCUSSION

Implicit logarithmic finite difference method is used to acquire the numerical solutions of the generalized Huxley equation. To demonstrate the correctness of results L_2 and L_∞error norms:

$$L_2 = \|U - u_N\|_2 = \sqrt{h\sum_{j=0}^{N}\left|U_j - (u_N)_j\right|^2},$$

$$L_\infty = \|U - u_N\|_\infty = \max_j \left|U_j - (u_N)_j\right|$$

are used, in which U and u indicate the exact and computed numerical solutions, respectively. In all numerical computations we took as $h = 0.01$ and $k = 0.0001$.

3.1. NUMERICAL EXAMPLE OF GENERALIZED HUXLEY EQUATION

Consider the generalized Huxley equation of the form Equation (1) in domain $0 \le x \le 1,\ t > 0$ with initial condition

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}$$

and boundary conditions

$$u(0,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left\{\sigma\gamma\left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right\}\right]^{\frac{1}{\delta}}, u(1,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left\{\sigma\gamma\left(1+\left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right)\right\}\right]^{\frac{1}{\delta}}.$$

The exact solution of this problem is [17]:

$$u(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left\{\sigma\gamma\left(x+\left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right)\right\}\right]^{\frac{1}{\delta}}$$

where $\rho = \sqrt{4\beta(1+\delta)}$ and $\sigma = \delta\rho/4(1+\delta)$.
The numerical solutions of Generalized Huxley Equation obtained by I-LFDM are compared with the exact solutions and numerical solutions obtained by some other methods [2,8-10,12] in literature in Table 1-3. The comparisons for the case $\delta = 1, \beta = 1$and $\gamma = 0.001$ are shown in Table 1 while the comparisons for the case $\delta = 2, \beta = 1$and $\gamma = 0.001$ are shown in Table 2 and for the case $\delta = 3, \beta = 1$and $\gamma = 0.001$ are shown in Table 3. As can be seen from the tables, numerical solutions obtained by the presented method are quite compatible with exact solutions and numerical solutions obtained by some other methods in the literature. In addition, the numerical solutions obtained by the method presented at time $t = 1$ are better than the numerical solutions obtained by some other methods in the literature. L_2 and L_∞ error norms for the case $\delta = 1, \gamma = 0.01$ and different values of β are given in Table 4. L_2 and L_∞ error norms for the case $\delta = 1,\beta = 1$ and different values of γ are given in Table 5. Table 6 presents L_2 and L_∞ error norms for the case $\beta = 1$, $\gamma = 0.001$ and different values of δ. As it can be seen from the tables, the L_2 and L_∞ error norms acquired by the I-LFDM are quite small in all cases.

4. CONCLUSION

In this study, implicit logarithmic finite difference method is used to obtain the numerical solutions of the generalized Huxley equation. The comparison of the numerical solutions obtained by presented method with the exact solutions and the numerical solutions obtained by previous studies in the literature is given by tables. It is clear from the tables that the numerical solutions obtained by I-LFDM are in good agreement with the exact solutions and better than numerical solutions obtained by some other methods in literature. The presented method is an efficient technique for finding numerical solutions for various kinds of nonlinear problems.

Table 1: Exact and numerical solutions for the case $\delta = 1$, $\beta = 1$and $\gamma = 0.001$.

x	t	Exact	I-LFDM	VIM [2], HPM [8], ADM [9]	HAM [10]	I-EFDM [12]
0.1	0.05	5.000302E-4	5.000199E-4	5.000052E-4	5.000100E-4	5.000125 E-4
	0.1	5.000427E-4	5.000276E-4	4.999927E-4	5.000030E-4	5.000102 E-4
	1	5.002676E-4	5.002451E-4	4.997678E-4	4.998680E-4	5.000064 E-4
0.5	0.05	5.001009E-4	5.000778E-4	5.000759E-4	5.000810E-4	5.000768 E-4
	0.1	5.001134E-4	5.000750E-4	5.000634E-4	5.000730E-4	5.000692 E-4
	1	5.003383E-4	5.002758E-4	4.998385E-4	4.999380E-4	5.000572 E-4
0.9	0.05	5.001716E-4	5.001613E-4	5.001466E-4	5.001520E-4	5.001540 E-4
	0.1	5.001841E-4	5.001691E-4	5.001341E-4	5.001440E-4	5.001516 E-4
	1	5.004090E-4	5.003865E-4	4.999092E-4	5.000090E-4	5.001479 E-4

Table 2: Exact and numerical solutions for the case $\delta = 2$, $\beta = 1$and $\gamma = 0.001$.

x	t	Exact	I-LFDM	VIM [2]	HPM [8], ADM [9]	HAM [10]	I-EFDM [12]
0.1	0.05	2.236188E-2	2.236142E-2	2.236077E-2	2.236077E-2	2.236100E-2	2.236110E-2
	0.1	2.236244E-2	2.236177E-2	2.236021E-2	2.236021E-2	2.236070E-2	2.236099E-2
	1	2.237250E-2	2.237149E-2	2.235015E-2	2.235015E-2	2.223546E-2	2.236082E-2
0.5	0.05	2.236447E-2	2.236343E-2	2.236335E-2	2.236335E-2	2.236360E-2	2.236339E-2
	0.1	2.236503E-2	2.236331E-2	2.236279E-2	2.236279E-2	2.236320E-2	2.236305E-2
	1	2.237508E-2	2.237229E-2	2.235273E-2	2.235273E-2	2.235720E-2	2.236251E-2
0.9	0.05	2.236705E-2	2.236659E-2	2.236593E-2	2.236593E-2	2.236620E-2	2.236114E-2
	0.1	2.236761E-2	2.236693E-2	2.236537E-2	2.236537E-2	2.236580E-2	2.236615E-2
	1	2.237766E-2	2.237665E-2	2.235532E-2	2.235531E-2	2.235980E-2	2.236599E-2

Table 3: Exact and numerical solutions for the case $\delta = 3$, $\beta = 1$and $\gamma = 0.001$.

x	t	Exact	I-LFDM	VIM [2]	HPM [8], ADM [9]	HAM [10]	I-EFDM [12]
0.1	0.05	7.937402E-2	7.937239E-2	7.937005E-2	7.937005E-2	7.937080E-2	7.937122E-2
	0.1	7.937601E-2	7.937361E-2	7.936807E-2	7.936807E-2	7.936970E-2	7.937084E-2
	1	7.941169E-2	7.940812E-2	7.933236E-2	7.933234E-2	7.934820E-2	7.937025E-2
0.5	0.05	7.938196E-2	7.937829E-2	7.937799E-2	7.937799E-2	7.937880E-2	7.937814E-2
	0.1	7.938394E-2	7.937784E-2	7.937601E-2	7.937601E-2	7.937760E-2	7.937692E-2
	1	7.941962E-2	7.940971E-2	7.934031E-2	7.934029E-2	7.935620E-2	7.937501E-2
0.9	0.05	7.938989E-2	7.938826E-2	7.938592E-2	7.938592E-2	7.938670E-2	7.938709E-2
	0.1	7.939187E-2	7.938948E-2	7.938394E-2	7.938394E-2	7.938550E-2	7.938671E-2
	1	7.942755E-2	7.942398E-2	7.934825E-2	7.934823E-2	7.936410E-2	7.938612E-2

Table 4: L_2 and L_∞ error norms for the case $\delta = 1$ and $\gamma = 0.01$.

t	L_2			L_∞		
	$\beta = 1$	$\beta = 10$	$\beta = 100$	$\beta = 1$	$\beta = 10$	$\beta = 100$
0.01	4.390519E-7	4.390371E-6	4.388844E-5	4.974336E-7	4.974196E-6	4.972792E-5
0.1	2.851070E-6	2.850957E-5	2.847622E-4	3.825356E-6	3.825217E-5	3.821034E-4
1	4.541154E-6	4.531847E-5	3.835394E-4	6.218224E-6	6.205651E-5	5.263925E-4
10	4.529978E-6	3.594694E-5	1.047479E-7	6.202945E-6	4.923570E-5	1.442949E-7

Table 5: L_2 and L_∞ error norms for the case $\delta = 1$ and $\beta = 1$.

t	L_2			L_∞		
	$\gamma = 0.01$	$\gamma = 0.001$	$\gamma = 0.0001$	$\gamma = 0.01$	$\gamma = 0.001$	$\gamma = 0.0001$
0.01	4.390519E-7	4.410392E-9	4.412417E-11	4.974336E-7	4.996848E-9	7.049136E-12
0.1	2.851070E-6	2.863973E-8	2.865266E-10	3.825356E-6	3.842667E-8	5.420848E-11
1	4.541154E-6	4.561835E-8	4.563772E-10	6.218224E-6	6.246538E-8	8.811778E-11
10	4.529978E-6	4.561961E-8	4.563810E-10	6.202945E-6	6.246721E-8	8.811849E-11

Table 6: L_2 and L_∞ error norms for the case $\beta = 1$ and $\gamma = 0.001$.

t	L_2			L_∞		
	$\delta = 1$	$\delta = 2$	$\delta = 4$	$\delta = 1$	$\delta = 2$	$\delta = 4$
0.01	4.410392E-9	1.972431E-7	1.318968E-6	4.996848E-9	2.234709E-7	1.494356E-6
0.1	2.863973E-8	1.280826E-6	8.564763E-6	3.842667E-8	1.718520E-6	1.149162E-5
1	4.561835E-8	2.039346E-6	1.362625E-5	6.246538E-8	2.792493E-6	1.865864E-5
10	4.561961E-8	2.030148E-6	1.344094E-5	6.246721E-8	2.779902E-6	1.840493E-5

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