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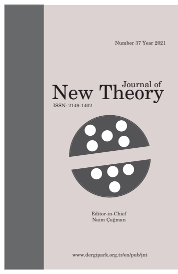

An Examination on to Find 5^{th} Order Bézier Curve in E^3

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Abstract — In this study, we have examined how to find any 5^{th} order Bézier curve with its known first, second and third derivatives, which are the 4^{th} order, the cubic and the quadratic Bézier curves, respectively, based on the control points of given the derivatives. Also we give an example to find the 5^{th} order Bézier curve with the given derivatives.

Keywords — *Bézier curves, derivatives of Bézier curve, cubic Bézier curves*

Mathematics Subject Classification (2020) — 53A04, 53A05

1. Introduction

As a tool of motion controller the Bézier curves are the most preferred ones in computer graphics for animation purposes. For example, Adobe Flash and Synfig are the applications of animation in which the Bézier curves are often integrated. Users sketched the desired path, and the application creates required frames for an object moving along the path. By being aware of the current importance of the Bézier curves, we have been motivated by the following studies. First Bézier-curves with curvature and torsion continuity has been examined in [1]. Also in [2] Bézier curves and surfaces has been studied in deep. In [3], Bézier curves are outlined deeply for Computer-Aided Geometric Design. In [4], Frenet apparatus of the cubic Bézier curves has been examined in E^3. We have already examine the cubic Bézier curves and involutes in [4] and [5], respectively. Before, 5^{th} order Bézier curve and its first, second, and third derivatives based on the control points are examined in [6, 7]. Futher involute, Bertrand mate and Mannheim partner of a cubic Bézier curve based on the control points with matrix form has been examined with Frenet apparatus in [8, 9]. In [10], the Bertrand pairs have also been examined by B-Spline curves. Last but not least, the Bézier curves have been associated to the alternative frame in [11].

2. How to Find 5^{th} Order Bézier Curve

In this study we have motivated by the following questions:

"How to find a 5^{th} order Bézier curve if we know the first, second and third derivatives?"

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Generally, Bézier curves can be defined by $n+1$ control points $P_0, P_1, ..., P_n$ with the parametrization

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} [P_i]$$

Definition 2.1. The 5^{th} order Bézier Curve has the following equation

$$\alpha(t) = \sum_{I=0}^{5} \binom{5}{I} t^I (1-t)^{5-I} [P_I] \qquad t \in [0,1]$$

and matrix representation

$$\alpha(t) = \begin{bmatrix} t^5 & t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}$$

with control points P_0, P_1, P_2, P_3, P_4, and P_5.

Theorem 2.2. The 5^{th} order Bézier curve with given the first derivative and the initial point P_0, has the following control points

$$\begin{aligned} P_1 &= P_0 + \frac{Q_0}{5} \\ P_2 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} \\ P_3 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} + \frac{Q_2}{5} \\ P_4 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} + \frac{Q_2}{5} + \frac{Q_3}{5} \\ P_5 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} + \frac{Q_2}{5} + \frac{Q_3}{5} + \frac{Q_4}{5} \end{aligned}$$

Proof. If the first derivative of 5^{th} order Bézier curve is given,

$$\alpha'(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B\prime] \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

where $[B\prime]$ is the coefficient matrix of the 4^{th} order Bézier curve which is the derivative of the 5^{th} order Bézier curve, then the control points $Q_0, Q_1, ..., Q_4$ are given as in the following way,

$$\begin{aligned} Q_0 &= 5(P_1 - P_0) \\ Q_1 &= 5(P_2 - P_1) \\ Q_2 &= 5(P_3 - P_2) \\ Q_1 &= 5(P_4 - P_3) \\ Q_4 &= 5(P_5 - P_4) \end{aligned}$$

Let the 5^{th} order Bézier curve pass through from a given the initial point P_0.If we take each P_i and replace, we get all the control points based on the Q_i, $0 \leq i \leq 4$.

$$\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{5} \\
P_2 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} \\
P_3 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} + \frac{Q_2}{5} \\
P_4 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} + \frac{Q_2}{5} + \frac{Q_3}{5} \\
P_5 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} + \frac{Q_2}{5} + \frac{Q_3}{5} + \frac{Q_4}{5}
\end{aligned}$$

Hence, this complete rhe proof. □

Corollary 2.3. The derivative of n^{th} order Bézier curve can not has the origin $(0, 0, 0)$ as a control point.

Proof. Let first derivative of n^{th} order Bézier curve has the origin $Q_0 = (0, 0, 0)$,

$$\begin{aligned}
Q_i &= n\,(P_{i+1} - P_i) = (0, 0, 0) \\
P_{i+1} &= P_i
\end{aligned}$$

Hence, Bézier curve has $n - 1$ control points hence derivative of n^{th} order Bézier curve cannot has the origin $Q_i(0, 0, 0)$ as a control point. □

Corollary 2.4. If the first derivative of 5^{th} order Bézier curve with given control points Q_i, $0 < i < 4$, is given and 5^{th} order Bézier curve has initial point $P_0 = (0, 0, 0)$,has the following control points

$$P_i = \frac{Q_0 + \cdots + Q_{i-1}}{5}, \ 1 \leq i \leq 5$$

Proof. Since $P_i = P_0 + \frac{Q_0+Q_1+Q_2+\ldots+Q_{i-1}}{n}$, $1 \leq i \leq 5$ and $P_0 = (0, 0, 0)$, it is clear. □

Theorem 2.5. The 5^{th} order Bézier curve with given the end point P_5 and the first derivative, has the following control points as in the following ways

$$\begin{aligned}
P_4 &= P_5 - \frac{Q_4}{5} \\
P_3 &= P_5 - \frac{Q_4}{5} - \frac{Q_3}{5} \\
P_2 &= P_5 - \frac{Q4}{5} - \frac{Q_3}{5} - \frac{Q_2}{5} \\
P_1 &= P_5 - \frac{Q_4}{5} - \frac{Q_3}{5} - \frac{Q_2}{5} - \frac{Q_1}{5} \\
P_0 &= P_5 - \frac{Q_4}{5} - \frac{Q_3}{5} - \frac{Q_2}{5} - \frac{Q_1}{5} - \frac{Q_0}{5}
\end{aligned}$$

Proof. Let the first derivative of 5^{th} order Bézier curve with control points Q_0, Q_1, Q_2, Q_3, Q_4 be given as,

$$\alpha'(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B\prime] \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

where $[B\prime]$ is the coefficient matrix of the 4^{th} order Bézier curve which is the derivative of the 5^{th} order Bézier curve. Hence, the control points Q_0, Q_1, Q_2, Q_3, Q_4 are

$$\begin{aligned}
Q_0 &= 5(P_1 - P_0) \\
Q_1 &= 5(P_2 - P_1) \\
Q_2 &= 5(P_3 - P_2) \\
Q_1 &= 5(P_4 - P_3) \\
Q_4 &= 5(P_5 - P_4)
\end{aligned}$$

If the 5^{th} order Bézier curve passing through the end point P_5, then

$$\begin{aligned}
P_4 &= P_5 - \frac{Q_4}{5} \\
P_3 &= P_4 - \frac{Q_3}{5} \\
P_2 &= P_3 - \frac{Q_2}{5} \\
P_1 &= P_0 - \frac{Q_1}{5} \\
P_0 &= P_1 - \frac{Q_0}{5}
\end{aligned}$$

and we get all the control points P_i based on the Q_0, Q_1, Q_2, Q_3, Q_4

$$\begin{aligned}
P_4 &= P_5 - \frac{Q_4}{5} \\
P_3 &= P_5 - \frac{Q_4}{5} - \frac{Q_3}{5} \\
P_2 &= P_5 - \frac{Q4}{5} - \frac{Q_3}{5} - \frac{Q_2}{5} \\
P_1 &= P_5 - \frac{Q_4}{5} - \frac{Q_3}{5} - \frac{Q_2}{5} - \frac{Q_1}{5} \\
P_0 &= P_5 - \frac{Q_4}{5} - \frac{Q_3}{5} - \frac{Q_2}{5} - \frac{Q_1}{5} - \frac{Q_0}{5}
\end{aligned}$$

This complete the proof. □

Corollary 2.6. The 5^{th} order Bézier curve with given the end point $P_5 = (0,0,0)$ and the first derivative has the following control points as in the following ways

$$P_{i-1} = -\frac{Q_0 + \cdots + Q_{i-1}}{5}, 1 \leq i \leq 5$$

Theorem 2.7. The 5^{th} order Bézier curve with given any point P_k, $0 < k < n$, and the first derivative has the following control points

$$\begin{aligned}
P_{k+1} &= P_k + \frac{Q_k}{5} \\
P_{k+2} &= P_k + \frac{Q_k}{5} + \frac{Q_{k+1}}{5} \\
&\cdots \\
P_5 &= P_k + \frac{Q_k}{5} + \frac{Q_{k+1}}{5} + \cdots + \frac{Q_4}{5}
\end{aligned}$$

$$P_{k-1} = P_k - \frac{Q_{k-1}}{5}$$
$$P_{k-2} = P_{k-} - \frac{Q_{k-1}}{5} - \frac{Q_{k-2}}{5}$$
$$\ldots$$
$$P_0 = P_k - \frac{Q_{k-1}}{5} - \frac{Q_{k-2}}{5} - \cdots - \frac{Q_0}{5}$$

Theorem 2.8. The 5^{th} order Bézier curve with given the initial point P_0,the initial point Q_0 of the first derivative and the control points R_0, R_1, R_2, R_3 of the second derivative , has the following control points as in the following way

$$P_1 = P_0 + \frac{Q_0}{5}$$
$$P_2 = P_0 + 2\frac{Q_0}{5} + \frac{R_0}{20}$$
$$P_3 = P_0 + 3\frac{Q_0}{5} + 2\frac{R_0}{20} + \frac{R_1}{20}$$
$$P_4 = P_0 + 4\frac{Q_0}{5} + 3\frac{R_0}{20} + 2\frac{R_1}{20} + \frac{R_2}{20}$$
$$P_5 = P_0 + 5\frac{Q_0}{5} + \frac{4R_0}{20} + \frac{3R_1}{20} + \frac{2R_2}{20} + \frac{R_3}{20}$$

PROOF. The second derivative of 5^{th} order Bézier curve is a cubic Bézier curve with control points R_0, R_1, R_2, R_3. It has the following matrix representation

$$\alpha''(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B''] \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

$$\alpha''(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B''] \begin{bmatrix} (n-1)(Q_1 - Q_0) \\ (n-1)(Q_2 - Q_1) \\ (n-1)(Q_3 - Q_2) \\ (n-1)(Q_4 - Q_3) \end{bmatrix}$$

where $[B'']$ is the coefficient matrix of the *cubic* Bézier curve which is the second derivative of the 5^{th} order Bézier curve. Where control points R_0, R_1, R_2, R_3, and Q_0 are given, we can easily find the Q_1, Q_2, Q_3, Q_4.

$$Q_1 = Q_0 + \frac{R_0}{4}$$
$$Q_2 = Q_0 + \frac{R_0}{4} + \frac{R_1}{4}$$
$$Q_3 = Q_0 + \frac{R_0}{4} + \frac{R_1}{4} + \frac{R_2}{4}$$
$$Q_4 = Q_0 + \frac{R_0}{4} + \frac{R_1}{4} + \frac{R_2}{4} + \frac{R_3}{4}$$

Also if the initial control point P_0 is given we can find easly control points of 5^{th} order Bézier curve

$$
\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{5} \\
P_2 &= P_0 + \frac{Q_0}{5} + \frac{Q_0 + \frac{R_0}{4}}{5} \\
P_3 &= P_0 + \frac{Q_0}{5} + \frac{Q_0 + \frac{R_0}{4}}{5} + \frac{Q_0 + \frac{R_0}{4} + \frac{R_1}{4}}{5} \\
P_4 &= P_0 + \frac{Q_0}{5} + \frac{Q_0 + \frac{R_0}{4}}{5} + \frac{Q_0 + \frac{R_0}{4} + \frac{R_1}{4}}{5} + \frac{Q_0 + \frac{R_0}{4} + \frac{R_1}{4} + \frac{R_2}{4}}{5} \\
P_5 &= P_0 + \frac{Q_0}{5} + \frac{Q_0 + \frac{R_0}{4}}{5} + \frac{Q_0 + \frac{R_0}{4} + \frac{R_1}{4}}{5} + \frac{Q_0 + \frac{R_0}{4} + \frac{R_1}{4} + \frac{R_2}{4}}{5} + \frac{Q_0 + \frac{R_0}{4} + \frac{R_1}{4} + \frac{R_2}{4} + \frac{R_3}{4}}{5}
\end{aligned}
$$

Hence, we have the proof. □

Theorem 2.9. The 5^{th} order Bézier curve with given the initial point P_0, the initial point Q_0 of the first derivative, the initial point R_0 of the second derivative and the control points S_0, S_1, S_2 of the third derivative has the following control points as in the following ways

$$
\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{5} \\
P_2 &= P_0 + 2\frac{Q_0}{5} + \frac{R_0}{20} \\
P_3 &= P_0 + 3\frac{Q_0}{5} + 3\frac{R_0}{20} + \frac{S_0}{60} \\
P_4 &= P_0 + 4\frac{Q_0}{5} + 6\frac{R_0}{20} + 3\frac{S_0}{60} + \frac{S_1}{60} \\
P_5 &= P_0 + 5\frac{Q_0}{5} + 10\frac{R_0}{20} + 6\frac{S_0}{60} + 3\frac{S_1}{60} + \frac{S_2}{20}
\end{aligned}
$$

where P_0, Q_0, R_0, and S_0, S_1, S_2 must be given.

PROOF. The third derivative of 5^{th} order Bézier curve is a quadratic Bézier curve with control points S_0, S_1, S_2 . Also it has the following matrix representation

$$
\alpha'''(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B\ ''' \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix}
$$

$$
\alpha'''(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B\ ''' \end{bmatrix} \begin{bmatrix} (n-2)(R_1 - R_0) \\ (n-2)(R_2 - R_1) \\ (n-2)(R_3 - R_2) \end{bmatrix}
$$

Where $[B\ ''']$ is the coefficient matrix of the quadratic Bézier curve which is the second derivative of the 5^{th} order Bézier curve. Since te control points S_0, S_1, S_2, and R_0 are given, by solving the following system

$$
\begin{aligned}
R_1 &= R_0 + \frac{S_0}{3} \\
R_2 &= R_0 + \frac{S_0}{3} + \frac{S_1}{3} \\
R_3 &= R_0 + \frac{S_0}{3} + \frac{S_1}{3} + \frac{S_2}{3}
\end{aligned}
$$

We can easily find the R_1, R_2, R_3. Also if the initial control point Q_0 of first derivative is given we can find easly Q_i control points of 5^{th} order Bézier curve

$$\begin{aligned}
Q_1 &= Q_0 + \frac{R_0}{4} \\
Q_2 &= Q_0 + 2\frac{R_0}{4} + \frac{S_0}{12} \\
Q_3 &= Q_0 + 3\frac{R_0}{4} + 2\frac{S_0}{12} + \frac{S_1}{12} \\
Q_4 &= Q_0 + 4\frac{R_0}{4} + 3\frac{S_0}{12} + 2\frac{S_1}{12} + \frac{S_2}{12}
\end{aligned}$$

Hence,

$$\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{5} \\
P_2 &= P_0 + 2\frac{Q_0}{5} + \frac{R_0}{20} \\
P_3 &= P_0 + 3\frac{Q_0}{5} + 2\frac{R_0}{20} + \frac{R_1}{20} \\
&= P_0 + 3\frac{Q_0}{5} + 2\frac{R_0}{20} + \frac{R_0 + \frac{S_0}{3}}{20} \\
&= P_0 + 3\frac{Q_0}{5} + 3\frac{R_0}{20} + \frac{S_0}{60} \\
P_4 &= P_0 + 4\frac{Q_0}{5} + 3\frac{R_0}{20} + 2\frac{R_1}{20} + \frac{R_2}{20} \\
&= P_0 + 4\frac{Q_0}{5} + 3\frac{R_0}{20} + 2\frac{R_0 + \frac{S_0}{3}}{20} + \frac{R_0 + \frac{S_0}{3} + \frac{S_1}{3}}{20} \\
&= P_0 + 4\frac{Q_0}{5} + 6\frac{R_0}{20} + 3\frac{S_0}{60} + \frac{S_1}{60} \\
P_5 &= P_0 + 5\frac{Q_0}{5} + 4\frac{R_0}{20} + 3\frac{R_1}{20} + 2\frac{R_2}{20} + \frac{R_3}{20} \\
&= P_0 + 5\frac{Q_0}{5} + 4\frac{R_0}{20} + 3\frac{R_0 + \frac{S_0}{3}}{20} + 2\frac{R_0 + \frac{S_0}{3} + \frac{S_1}{3}}{20} + \frac{R_0 + \frac{S_0}{3} + \frac{S_1}{3} + \frac{S_2}{3}}{20} \\
&= P_0 + 5\frac{Q_0}{5} + 10\frac{R_0}{20} + 6\frac{S_0}{60} + 3\frac{S_1}{60} + \frac{S_2}{20}
\end{aligned}$$

This complete the proof. □

2.1. An Example to Find 5^{th} Order Bézier Curve with Given First Derivative

In this section, we will give an example to find 5^{th} order Bézier curves which are defined in E^3. For more detail, see [3].

Example 2.10. Let $\alpha(t) = \big(74t^5 - 210t^4 + 180t^3 - 50t^2 + 5t + 1, -79t^5 + 185t^4 - 130t^3 + 10t^2 + 10t + 1, -63t^5 + 95t^4 - 30t^3 - 5t + 2\big)$ be an example of a 5^{th} order Bézier curve with control points, $P_0 = (1,1,2), P_1 = (2,3,1), P_2 = (-2,6,0), P_3 = (7,-3,-4), P_4 = (5,0,5), P_5 = (0,-3,-1)$.

The control points of the first derivative of 5^{th} order of a Bézier curve are

$$\begin{aligned}
Q_0 &= 5(P_1 - P_0) = 5((2,3,1) - (1,1,2)) = (5,10,-5) \\
Q_1 &= 5(P_2 - P_1) = 5((-2,6,0) - (2,3,1)) = (-20,15,-5) \\
Q_2 &= 5(P_3 - P_2) = 5((7,-3,-4) - (-2,6,0)) = (45,-45,-20) \\
Q_3 &= 5(P_4 - P_3) = 5((5,0,5) - (7,-3,-4)) = (-10,15,45) \\
Q_4 &= 5(P_5 - P_4) = 5((0,-3,-1) - (5,0,5)) = (-25,-15,-30)
\end{aligned}$$

5^{th} order Bézier curve with given the first derivative and the initial point P_0, has the following control points

$$\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{5} \\
P_2 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} \\
P_3 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} + \frac{Q_2}{5} \\
P_4 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} + \frac{Q_2}{5} + \frac{Q_3}{5} \\
P_5 &= P_0 + \frac{Q_0}{5} + \frac{Q_1}{5} + \frac{Q_2}{5} + \frac{Q_3}{5} + \frac{Q_4}{5}
\end{aligned}$$

P_0 and $Q_0, Q_1, \ldots, Q_4$ *are given*

$$\begin{aligned}
P_1 &= (1,1,2) + \frac{(5,10,-5)}{5} = (2,3,1) \\
P_2 &= (1,1,2) + \frac{(5,10,-5)}{5} + \frac{(-20,15,-5)}{5} = (-2,6,0) \\
P_3 &= (1,1,2) + \frac{(5,10,-5)}{5} + \frac{(-20,15,-5)}{5} + \frac{(45,-45,-20)}{5} = (7,-3,-4) \\
P_4 &= (1,1,2) + \frac{(5,10,-5)}{5} + \frac{(-20,15,-5)}{5} + \frac{(45,-45,-20)}{5} + \frac{(-10,15,45)}{5} = (5,0,5) \\
P_5 &= (1,1,2) + \frac{(5,10,-5)}{5} + \frac{(-20,15,-5)}{5} + \frac{(45,-45,-20) + (-10,15,45) + (-25,-15,-30)}{5} \\
&= (0,-3,-1)
\end{aligned}$$

The second derivative of 5^{th} order of a Bézier curve as a 3^{rd} order Bézier curve with control points R_0, R_1, R_2, R_3 are given as in the following way

$$\begin{aligned}
R_0 &= 4\left(Q_1 - Q_0\right) = 4\left((-20,15,-5) - (5,10,-5)\right) = (-100,20,0) \\
R_1 &= 4\left(Q_2 - Q_1\right) = 4\left((45,-45,-20) - (-20,15,-5)\right) = (260,-240,-60) \\
R_2 &= 4\left(Q_3 - Q_2\right) = 4\left((-10,15,45) - (45,-45,-20)\right) = (-220,240,260) \\
R_3 &= 4\left(Q_4 - Q_3\right) = 4\left((-25,-15,-30) - (-10,15,45)\right) = (-60,-120,-300)
\end{aligned}$$

The 5^{th} order Bézier curve with given the initial point P_0,the initial point Q_0 of the first derivative and the control points R_0, R_1, R_2, R_3 of the second derivation , has the following control points as in the following ways

$$\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{5} \\
P_2 &= P_0 + 2\frac{Q_0}{5} + \frac{R_0}{20} \\
P_3 &= P_0 + 3\frac{Q_0}{5} + 2\frac{R_0}{20} + \frac{R_1}{20} \\
P_4 &= P_0 + 4\frac{Q_0}{5} + 3\frac{R_0}{20} + 2\frac{R_1}{20} + \frac{R_2}{20} \\
P_5 &= P_0 + 5\frac{Q_0}{5} + \frac{4R_0}{20} + \frac{3R_1}{20} + \frac{2R_2}{20} + \frac{R_3}{20}
\end{aligned}$$

where P_0, Q_0, R_0, R_1, R_3 are given.

$$\begin{aligned}
P_1 &= (1,1,2) + \tfrac{(5,10,-5)}{5} = (2,3,1)\\
P_2 &= (1,1,2) + 2\tfrac{(5,10,-5)}{5} + \tfrac{(-100,20,0)}{20} = (-2,6,0)\\
P_3 &= (1,1,2) + 3\tfrac{(5,10,-5)}{5} + 2\tfrac{(-100,20,0)}{20} + \tfrac{(260,-240,-60)}{20} = (7,-3,-4)\\
P_4 &= (1,1,2) + 4\tfrac{(5,10,-5)}{5} + 3\tfrac{(-100,20,0)}{20} + 2\tfrac{(260,-240,-60)}{20} + \tfrac{(-220,240,260)}{20} = (5,0,5)\\
P_5 &= (1,1,2) + \tfrac{5(5,10,-5)}{5} + \tfrac{4(-100,20,0)}{20} + \tfrac{3(260,-240,-60)}{20} + \tfrac{2(-220,240,260)}{20} + \tfrac{(-60,-120,-300)}{20}\\
&= (0,-3,-1)
\end{aligned}$$

The third derivative of 5^{th} order of a Bézier curve using the control points S_0, S_1, and S_2 of 5^{th} order Bézier Curve are

$$\begin{aligned}
S_0 &= 3\,(R_1 - R_0) = 3\,((260,-240,-60) - (-100,20,0)) = (1080,-780,-180)\\
S_1 &= 3\,(R_2 - R_1) = 3\,((-220,240,260) - (260,-240,-60)) = (-1440,1440,960)\\
S_2 &= 3\,(R_3 - R_2) = 3\,((-60,-120,-300) - (-220,240,260)) = (480,-1080,-1680)
\end{aligned}$$

3. Conclusion

We know that the first derivative of a 5^{th} order Bézier curve is 4^{th} order Bézier curve with 5 control points,in this study we have examined that, when any 4^{th} order Bézier curve with 5 control points is given, how to find 5^{th} order Bézier curve. Also, the second derivative of a 5^{th} order Bézier curve is cubic Bézier curve with 4 control points, we have examined that, when any cubic Bézier curve with 4 control points is given, how to find 5^{th} order Bézier curve. Further the third derivative of a 5^{th} order Bézier curve is a quadratic Bézier curve with 3 control points. We have examined that, when any cubic Bézier curve with three control points is given, how to find 5^{th} order Bézier curve. As a result we can choose any initial or end control points except origin of any order derivatives and we can write which we want exactly the 5^{th} order Bézier curve.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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