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The Generalized Hermite-Hadamard-Fejer Type Inequalities for Generalized Fractional Integral Operators

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Abstract

In [11], Sarikaya and Ertuğral described a new left-sided and right-sided generalized fractional integrals. The goal of this study obtain the new Hermite-Hadamard-Fejér inequality for the this generalized fractional integrals. New results obtained from the this study are supported from previous works.

Keywords: *Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, Riemann-Liouville fractional integral, convex function.*
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1. Introduction

The subject of fractional calculus (integrals and derivatives) has gained considerable importance. The importance of this subject, whose popularity has increased over the past decades, is that it has various applications in many branches such as mathematics and engineering. It has extensions and generalizations in more variants as well as applications for various problems involving special functions of mathematical science. This issue it is still extensively studied by many authors, see for example ([3], [7],[9],[10],[12],[14]). One of the important applications of fractional integrals is the Hermite-Hadamard inequality, see Ref ([3], [6],[10],[11]). The classical Hermite-Hadamard inequality is as follows:

$$h\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b h(x)\,dx\leq\frac{h(a)+h(b)}{2} \tag{1.1}$$

where $h:I\to\mathbb{R}$ is a convex mapping defined on the interval $I\subseteq\mathbb{R}$ and $a,b\in I$, with $a<b$.
Within years Hermite-Hadamard type inequalities and fractional integrals became a very attractive subject to mathematicians due to widespread use, and many different forms of Hermite-Hadamard inequalities involving fractional (i.e., noninteger) operators were introduced for different classes of functions; see ([3], [9],[10]).
The best-known inequalities about the integral mean of a convex function are the Hermite Hadamard inequalities or their weighted versions, called the Hermite-Hadamard-Fejér inequalities (see, (1.1) and (1.2) in the references ([1],[2],[4],[5], [7],[8],[12]-[18]). In [5], Fejer obtained a downward inequality for the Hermite-Hadamard inequality (1.1) in weighted generalization as the following:

Theorem 1.1. *Let $h:[a,b]\to\mathbb{R}$ be a convex function. So the following inequality*

$$h\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\leq\frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx\leq\frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx \tag{1.2}$$

is valid when $g:[a,b]\to\mathbb{R}$ is non-negative, symmetric to $(a+b)/2$ (i.e. $g(x)=g(a+b-x)$), and integrable.

In [11], Sarikaya and Ertuğral described a new left-sided and right-sided generalized fractional integrals as the following:-

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Definition 1.2. *Let $h\in L[a,b]$. The Generalized Fractional integrals ${}_{a^+}I_\varphi h$ and ${}_{b^-}I_\varphi h$ with $a\geq 0$ are defined by*

$$ {}_{a^+}I_\varphi h(x)=\int_a^x \frac{\varphi(x-t)}{x-t}h(t)\,dt,\ x>a \tag{1.3}$$

$$ {}_{b^-}I_\varphi h(x)=\int_x^b \frac{\varphi(t-x)}{t-x}h(t)\,dt,\ x<b, \tag{1.4}$$

where $\varphi:[0,\infty)\to[0,\infty)$ a function which satisfies the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t}dt<\infty. \tag{1.5}$$

$$\frac{1}{A_1}\leq\frac{\varphi(s)}{\varphi(r)}\leq A_1 \text{ for } \frac{1}{2}\leq\frac{s}{r}\leq 2 \tag{1.6}$$

$$\frac{\varphi(r)}{r^2}\leq A_2\frac{\varphi(s)}{s^2} \text{ for } s\leq r \tag{1.7}$$

$$\left|\frac{\varphi(r)}{r^2}-\frac{\varphi(s)}{s^2}\right|\leq A_3|r-s|\frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2}\leq\frac{s}{r}\leq 2 \tag{1.8}$$

where $A_1,A_2,A_3>0$ are independent of $r,s>0$. If $\varphi(r)r^\alpha$ is increasing for some $\alpha\geq 0$ and $\frac{\varphi(r)}{r^\beta}$ is decreasing for some $\beta\geq 0$, then φ satisfies (1.5-1.8), see [11].

Furthermore, they noticed that these generalized fractional integrals may contain some types of fractional integrals such as Riemann-Liouville fractional integral, $k-$Riemann-Liouville fractional integral, Katugampola fractional integral, conformable fractional integral, etc., with some special choices.

In [11], for this new generalized fractional integral defined by Sarikaya and Ertuğral, let's recall the basic lemma of the Hermite-Hadamard inequality given by them.

Theorem 1.3. *Given a positive function $h:[a,b]\to\mathbb{R}$ mapping on (a,b) with $a<b$ and $h\in L[a,b]$. If h is a convex function on $[a,b]$, then we get the following expression*

$$h\left(\frac{a+b}{2}\right)\leq\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi h(b)+{}_{b^-}I_\varphi h(a)\right]\leq h(a)+h(b), \tag{1.9}$$

where $\Lambda(1)=\int_0^1\frac{\varphi((b-a)t)}{t}dt$.

Meanwhile, the following basic lemma, which was used to obtain some Hermite-Hadamard inequalities, was obtained by Sarıkaya et. al in [11].

Lemma 1.4. *Given a differentiable $h:[a,b]\to\mathbb{R}$ mapping on (a,b) with $a<b$. If $h'\in L[a,b]$, then the below expression fot the generalized fractional integrals holds:*

$$\begin{aligned} &\frac{h(a)+h(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi h(b)+{}_{b^-}I_\varphi h(a)\right] \\ =&\frac{b-a}{2\Lambda(1)}\int_0^1[\Lambda(1-t)-\Lambda(t)]h'(ta+(1-t)b)\,dt, \end{aligned}\tag{1.10}$$

where $\Lambda(1)=\int_0^1\frac{\varphi((b-a)t)}{t}dt$.

Theorem 1.5. *Given a differentiable $h:[a,b]\to\mathbb{R}$ mapping on (a,b) with $a<b$. If $|h'|$ is a convex function on $[a,b]$, then, then the below expression fot the generalized fractional integrals holds:*

$$\begin{aligned} &\left|\frac{h(a)+h(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi h(b)+{}_{b^-}I_\varphi h(a)\right]\right| \\ \leq&\frac{b-a}{2\Lambda(1)}\left(\int_0^1 t|\Lambda(1-t)-\Lambda(t)|\,dt\right)\left[|h'(a)|+|h'(b)|\right], \end{aligned}\tag{1.11}$$

where $\Lambda(1)=\int_0^1\frac{\varphi((b-a)t)}{t}dt$.

On the other hand, let's remind the following inequality that we used in our main results.

Lemma 1.6. *([15, 17]) For $0<\alpha\leq 1$ and $0\leq a<b$, we have*

$$|a^\alpha-b^\alpha|\leq(b-a)^\alpha$$

Firstly the purpose of this paper is to give the Hermite-Hadamard-Fejer inequality using a general class of this generalized fractional integral operator, which is the predominant generalization of the Hermite-Hadamard inequality (1.9). We will then obtain new inequalities for generalized fractional integrals related to the right hand side of Hermite-Hadamard-Fejér type integral inequality.

2. MAIN RESULTS

Before we get our next result in this section, we need the following lemma:

Lemma 2.1. *If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$ with $a<b$, then*

$$_{a^+}I_\varphi g(b) ={}_{b^-}I_\varphi g(a)=\frac{1}{2}\left[_{a^+}I_\varphi g(b)+{}_{b^-}I_\varphi g(a)\right].$$

Proof. Since g is symmetric to $(a+b)/2$, we have $g(a+b-x)=g(x)$, for all $x\in[a,b]$. Hence, in the following integral setting $x=tb+(1-t)a$ and $dx=(b-a)dt$ gives

$$\begin{aligned} {}_{a^+}I_\varphi g(b) &= \int_a^b \frac{\varphi((b-x))}{(b-x)} g(x)\,dx \\ &= \int_a^b \frac{\varphi(x-a)}{(x-a)} g(a+b-x)\,dx \\ &= \int_a^b \frac{\varphi(x-a)}{(x-a)} g(x)\,dx ={}_{b^-}I_\varphi g(a). \end{aligned}$$

Thus, the desired result of the lemma is obtained. □

Remark 2.2. *Under the hypotheses of Lemma 3, if we choose $\varphi(x)=\frac{x^\alpha}{\Gamma(\alpha)}$, then we get the expression of Lemma 3 in [7].*

Theorem 2.3. *Given a $h:[a,b]\to\mathbb{R}$ convex function and $h\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then the below expression for the generalized fractional integrals holds:*

$$\begin{aligned} & h\left(\frac{a+b}{2}\right)\left[_{a^+}I_\varphi g(b)+{}_{b^-}I_\varphi g(a)\right] \\ \le\ & \left[_{a^+}I_\varphi(hg)(b)+{}_{b^-}I_\varphi(hg)(a)\right] \\ \le\ & \frac{h(a)+h(b)}{2}\left[_{a^+}I_\varphi g(b)+{}_{b^-}I_\varphi g(a)\right]. \end{aligned} \tag{2.1}$$

Proof. Since h is a convex function on $[a,b]$, we have for all $t\in[0,1]$

$$\begin{aligned} h\left(\frac{a+b}{2}\right) &= h\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right) \\ &\le \frac{h(ta+(1-t)b)+h(tb+(1-t)a)}{2}. \end{aligned} \tag{2.2}$$

Multiplying both sides of (2.2) by $2\frac{\varphi((b-a)t)}{t}g(tb+(1-t)a)$ then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} & 2h\left(\frac{a+b}{2}\right)\int_0^1 \frac{\varphi((b-a)t)}{t} g(tb+(1-t)a)\,dt \\ \le\ & \int_0^1 \frac{\varphi((b-a)t)}{t}\left[h(ta+(1-t)b)+h(tb+(1-t)a)\right] g(tb+(1-t)a)\,dt \\ =\ & \int_0^1 \frac{\varphi((b-a)t)}{t} h(ta+(1-t)b)\, g(tb+(1-t)a)\,dt \\ & +\int_0^1 \frac{\varphi((b-a)t)}{t} h(tb+(1-t)a)\, g(tb+(1-t)a)\,dt. \end{aligned}$$

Choosing $x=tb+(1-t)a$, we get

$$\begin{aligned} & 2h\left(\frac{a+b}{2}\right)\int_a^b \frac{\varphi(x-a)}{(x-a)} g(x)\,dx \\ \le\ & \int_a^b \frac{\varphi(x-a)}{(x-a)} h(a+b-x) g(x)\,dx+\int_0^1 \frac{\varphi(x-a)}{(x-a)} h(x)\, g(x)\,dx \\ =\ & \int_a^b \frac{\varphi(b-x)}{(b-x)} h(x)\, g(a+b-x)dx+\int_0^1 \frac{\varphi(x-a)}{(x-a)} h(x)\, g(x)\,dx \\ =\ & \int_a^b \frac{\varphi(b-x)}{(b-x)} h(x)\, g(x)\,dx+\int_0^1 \frac{\varphi(x-a)}{(x-a)} h(x)\, g(x)\,dx. \end{aligned}$$

Therefore, by Lemma 2.1 we have

$$h\left(\frac{a+b}{2}\right)\left[_{a^+}I_\varphi g(b)+{}_{b^-}I_\varphi g(a)\right]\le\left[_{a^+}I_\varphi(hg)(b)+{}_{b^-}I_\varphi(hg)(a)\right]$$

and the first inequality is proved.
For the proof of the second inequality in (2.1) we first note that if h is a convex function, then, for all $t\in[0,1]$, it yields

$$h(ta+(1-t)b)+h(tb+(1-t)a)\leq h(a)+h(b). \tag{2.3}$$

Then multiplying both sides of (2.3) by $\frac{\varphi((b-a)t)}{t}g(tb+(1-t)a)$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1\frac{\varphi((b-a)t)}{t}h(ta+(1-t)b)g(tb+(1-t)a)dt\\&+\int_0^1\frac{\varphi((b-a)t)}{t}h(tb+(1-t)a)g(tb+(1-t)a)dt\\\leq\;&[h(a)+h(b)]\int_0^1\frac{\varphi((b-a)t)}{t}g(tb+(1-t)a)dt\end{aligned}$$

i.e.

$$\left[{}_{a^+}I_\varphi(hg)(b)+{}_{b^-}I_\varphi(hg)(a)\right]\leq\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_\varphi g(b)+{}_{b^-}I_\varphi g(a)\right].$$

The proof is completed. □

Remark 2.4. *Under the hypotheses of Theorem 4, if we choose $\varphi(x)=\frac{x^\alpha}{\Gamma(\alpha)}$, then we get the expression of Theorem 4 in [7].*

Remark 2.5. *Under the hypotheses of Theorem 4, if we choose $\varphi(x)=x$, then the inequality (2.1) becomes the inequality (1.2).*

Remark 2.6. *Under the hypotheses of Theorem 4, if we choose $\varphi(x)=\frac{x^\alpha}{\Gamma(\alpha)}$, $g(x)=1$, then the inequality (2.1) becomes as follow which is found in [10];*

$$h\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha h(b)+J_{b^-}^\alpha h(a)\right]\leq\frac{h(a)+h(b)}{2},$$

with $\alpha>0$.

Corollary 2.7. *In Theorem 4, if we take $\varphi(x)=\frac{x^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the inequality (2.1) becomes as follow;*

$$\begin{aligned}&h\left(\frac{a+b}{2}\right)\left[I_{a^+,k}^\alpha g(b)+I_{b^-,k}^\alpha g(a)\right]\\\leq\;&\left[I_{a^+,k}^\alpha(hg)(b)+I_{b^-,k}^\alpha(hg)(a)\right]\\\leq\;&\frac{h(a)+h(b)}{2}\left[I_{a^+,k}^\alpha g(b)+I_{b^-,k}^\alpha g(a)\right].\end{aligned}$$

Lemma 2.8. *Given a $h:[a,b]\to\mathbb{R}$ differentiable mapping on (a,b) with $a<b$ and $h'\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, then the below expression for the generalized fractional integrals holds:*

$$\begin{aligned}&\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_\varphi g(b)+{}_{b^-}I_\varphi g(a)\right]-\left[{}_{a^+}I_\varphi(hg)(b)+{}_{b^-}I_\varphi(hg)(a)\right]\\=\;&\int_a^b\left[\int_a^t\frac{\varphi(b-s)}{(b-s)}g(s)ds-\int_t^b\frac{\varphi(s-a)}{(s-a)}g(s)ds\right]h'(t)dt.\end{aligned}\tag{2.4}$$

Proof. It suffices to note that, we get the following expression

$$\begin{aligned}I\;=\;&\int_a^b\left[\int_a^t\frac{\varphi(b-s)}{(b-s)}g(s)ds-\int_t^b\frac{\varphi(b-a)}{(b-a)}g(s)ds\right]h'(t)dt\\=\;&\int_a^b\left(\int_a^t\frac{\varphi(b-s)}{(b-s)}g(s)ds\right)h'(t)dt+\int_a^b\left(-\int_t^b\frac{\varphi(s-a)}{(s-a)}g(s)ds\right)h'(t)dt\\=\;&I_1+I_2.\end{aligned}$$

By integration by parts and Lemma 2.1, we get

$$\begin{aligned}I_1\;=\;&\left(\int_a^t\frac{\varphi(b-s)}{(b-s)}g(s)ds\right)h(t)\Big|_a^b-\int_a^b\frac{\varphi(b-t)}{(b-t)}g(t)h(t)dt\\=\;&\left(\int_a^b\frac{\varphi(b-s)}{(b-s)}g(s)ds\right)h(b)-\int_a^b\frac{\varphi(b-t)}{(b-t)}(hg)(t)dt\\=\;&\left[h(b)_{a^+}I_\varphi g(b)-{}_{a^+}I_\varphi(hg)(b)\right]\\=\;&\left[\frac{h(b)}{2}\left[{}_{a^+}I_\varphi g(b)+{}_{b^-}I_\varphi g(a)\right]-{}_{a^+}I_\varphi(hg)(b)\right]\end{aligned}$$

and similarly

$$\begin{aligned} I_2 &= \left(-\int_t^b \frac{\varphi(s-a)}{(s-a)} g(s)ds\right) h(t)\Big|_a^b - \int_a^b \frac{\varphi(t-a)}{(t-a)} g(t) h(t) dt \\ &= \left(\int_a^b \frac{\varphi(s-a)}{(s-a)} g(s)ds\right) h(a) - \int_a^b \frac{\varphi(t-a)}{(t-a)} (hg)(t) dt \\ &= \left[\frac{h(a)}{2}\left[{}_{a^+}I_\varphi g(b) + {}_{b^-}I_\varphi g(a)\right] - {}_{b^-}I_\varphi (hg)(a)\right]. \end{aligned}$$

Thus, we can write

$$\begin{aligned} I &= I_1 + I_2 \\ &= \left\{\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_\varphi g(b) + {}_{b^-}I_\varphi g(a)\right] - \left[{}_{a^+}I_\varphi (hg)(b) + {}_{b^-}I_\varphi (hg)(a)\right]\right\}. \end{aligned}$$

Thus we obtain (2.4) which completes the proof. □

Remark 2.9. *Under the hypotheses of Lemma 4, if we choose* $\varphi(x) = \frac{x^\alpha}{\Gamma(\alpha)}$, *then we get the expression of Lemma 4 in [7].*

Remark 2.10. *Under the hypotheses of Lemma 4, if we choose* $\varphi(x) = x$, *then we get the following;*

$$\begin{aligned} &(h(a)+h(b))\int_a^b g(t)dt - 2\int_a^b h(t) g(t) dt \\ &= \int_a^b \left[\int_a^t \frac{\varphi(b-s)}{(b-s)} g(s)ds - \int_t^b \frac{\varphi(s-a)}{(s-a)} g(s)ds\right] h'(t) dt. \end{aligned}$$

Moreover this result is same Lemma 2.6 in [5]

Remark 2.11. *Under the hypotheses of Lemma 4, if we choose* $\varphi(x) = \frac{x^\alpha}{\Gamma(\alpha)}$, $g(x) = 1$, *then inequality (2.4) becomes as follow which is found in [10];*

$$\left(\frac{h(a)+h(b)}{2}\right) - \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] = \frac{b-a}{2}\int_0^1 \left[(1-t)^\alpha - t^\alpha\right] h'(ta+(1-t)b) dt.$$

Corollary 2.12. *In Lemma 4, if we take* $\varphi(x) = \frac{x^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then the inequality (2.4) becomes as follow;*

$$\begin{aligned} &\left(\frac{h(a)+h(b)}{2}\right)\left[I_{a^+,k}^\alpha g(b) + I_{b^-,k}^\alpha g(a)\right] - \left[I_{a^+,k}^\alpha (hg)(b) + I_{b^-,k}^\alpha (hg)(a)\right] \\ &= \int_a^b \left[\int_a^t \frac{(b-s)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} g(s)ds - \int_t^b \frac{(s-a)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} g(s)ds\right] h'(t) dt. \end{aligned} \tag{2.5}$$

Theorem 2.13. *Let* $h : I \subset \mathbb{R} \to \mathbb{R}$ *be a diferentiable mapping on* I° *and* $h' \in L[a,b]$ *with* $a < b$, *and also* $|h'|$ *is convex on* $[a,b]$ *and* $g : [a,b] \to \mathbb{R}$ *is continuous and symetric with respect to* $\frac{a+b}{2}$. *The following holds for the generalized fractional integrals;*

$$\begin{aligned} &\left|\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_\varphi g(b) + {}_{b^-}I_\varphi g(a)\right] - \left[{}_{a^+}I_\varphi (hg)(b) + {}_{b^-}I_\varphi (hg)(a)\right]\right| \\ &\le \frac{\|g\|_\infty}{(b-a)}\left[|h'(a)| + |h'(b)|\right] M_2 \end{aligned} \tag{2.6}$$

where

$$M_2 := \int_a^{\frac{a+b}{2}} \left[\int_{a+b-t}^t \left|\frac{\varphi(b-s)}{(b-s)}\right| ds\right] dt + \int_{\frac{a+b}{2}}^b \left[\int_{a+b-t}^t \left|\frac{\varphi(b-s)}{(b-s)}\right| ds\right] dt.$$

Proof. From Lemma 2.8, we have

$$\begin{aligned} &\left|\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_\varphi g(b) + {}_{b^-}I_\varphi g(a)\right] - \left[{}_{a^+}I_\varphi (hg)(b) + {}_{b^-}I_\varphi (hg)(a)\right]\right| \\ &\le \int_a^b \left|\int_a^t \frac{\varphi(b-s)}{(b-s)} g(s)ds - \int_t^b \frac{\varphi(s-a)}{(s-a)} g(s)ds\right| |h'(t)| dt. \end{aligned} \tag{2.7}$$

Since $|h'|$ is convex on $[a,b]$, we know that for $t\in[a,b]$

$$\left|h'(t)\right| = \left|h'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right| \leq \frac{b-t}{b-a}\left|h'(a)\right| + \frac{t-a}{b-a}\left|h'(b)\right| \tag{2.8}$$

and since $g:[a,b]\to\mathbb{R}$ is symmetric to $(a+b)/2$ we write

$$\int_t^b \frac{\varphi(s-a)}{(s-a)}g(s)ds = \int_a^{a+b-t}\frac{\varphi(b-s)}{(b-s)}g(a+b-s)ds = \int_a^{a+b-t}\frac{\varphi(b-s)}{(b-s)}g(s)ds$$

then we have

$$\begin{aligned} & \left|\int_a^t \frac{\varphi(b-s)}{(b-s)}g(s)ds - \int_t^b \frac{\varphi(b-s)}{(b-s)}g(s)ds\right| \\ = & \left|\int_t^{a+b-t}\frac{\varphi(b-s)}{(b-s)}g(s)ds\right| \\ \leq & \begin{cases} \int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}g(s)\right|ds & , t\in\left[a,\frac{a+b}{2}\right] \\ \int_{a+b-t}^t \left|\frac{\varphi(b-s)}{(b-s)}g(s)\right|ds & , t\in\left[\frac{a+b}{2},b\right] \end{cases}. \end{aligned} \tag{2.9}$$

A combination of (2.7), (2.8) and (2.9), we get

$$\begin{aligned} & \left|\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_\varphi g(b) + {}_{b^-}I_\varphi g(a)\right] - \left[{}_{a^+}I_\varphi(hg)(b) + {}_{b^-}I_\varphi(hg)(a)\right]\right| \\ \leq & \int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}g(s)\right|ds\right)\left(\frac{b-t}{b-a}\left|h'(a)\right| + \frac{t-a}{b-a}\left|h'(b)\right|\right)dt \\ & + \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t \left|\frac{\varphi(b-s)}{(b-s)}g(s)\right|ds\right)\left(\frac{b-t}{b-a}\left|h'(a)\right| + \frac{t-a}{b-a}\left|h'(b)\right|\right)dt \\ \leq & \frac{ess\sup_{t\in[a,b]}|g(x)|}{(b-a)}\left[\left|h'(a)\right| + \left|h'(b)\right|\right]M_2. \end{aligned}$$

We obtain the desired result. This completes the proof. □

Remark 2.14. *Under the hypotheses of Theorem 5, if we choose* $\varphi(x) = \frac{x^\alpha}{\Gamma(\alpha)}$*, then we get the expression of Theorem 5 in [7].*

Remark 2.15. *Under the hypotheses of Theorem 5, if we choose* $\varphi(x) = x$*, then we get the following expression;*

$$\begin{aligned} & \left|(h(a)+h(b))\int_a^b g(t)dt - 2\int_a^b h(t)g(t)dt\right| \\ \leq & \frac{(b-a)}{4}\|g\|_\infty\left[\left|h'(a)\right| + \left|h'(b)\right|\right]. \end{aligned}$$

Remark 2.16. *Under the hypotheses of Theorem 5, if we choose* $\varphi(x) = \frac{x^\alpha}{\Gamma(\alpha)}$, $g(x) = 1$*, then inequality (2.6) becomes as follow which is found in [10];*

$$\begin{aligned} & \left|\left(\frac{h(a)+h(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha h(b) + J_{b^-}^\alpha h(a)\right]\right| \\ \leq & \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[\left|h'(a)\right| + \left|h'(b)\right|\right]. \end{aligned}$$

Corollary 2.17. *Under the hypotheses of Theorem 5, if we choose* $\varphi(x) = \frac{x^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$*, then the inequality (2.6) becomes as follow;*

$$\begin{aligned} & \left|\left(\frac{h(a)+h(b)}{2}\right)\left[I_{a^+,k}^\alpha g(b) + I_{b^-,k}^\alpha g(a)\right] - \left[I_{a^+,k}^\alpha(hg)(b) + I_{b^-,k}^\alpha(hg)(a)\right]\right| \\ \leq & \frac{\|g\|_\infty}{(b-a)}\left[\left|h'(a)\right| + \left|h'(b)\right|\right]\int_{\frac{a+b}{2}}^b\left[\int_{a+b-t}^t \frac{(b-s)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}ds\right]dt. \end{aligned} \tag{2.10}$$

Theorem 2.18. *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a diferentiable mapping on I° and $h' \in L[a,b]$ with $a < b$, and also $|h'|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symetric with respect to $\frac{a+b}{2}$. The following holds for the generalized fractional integrals;*

$$\left|\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_{\varphi}g(b)+{}_{b^-}I_{\varphi}g(a)\right]-\left[{}_{a^+}I_{\varphi}(hg)(b)+{}_{b^-}I_{\varphi}(hg)(a)\right]\right| \tag{2.11}$$
$$\leq \frac{2\|g\|_{\infty}}{(b-a)^{1/q}}\left\{\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right]dt+\int_{\frac{a+b}{2}}^{b}\left[\int_{a+b-t}^{t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right]dt\right\}^{1-1/q}$$
$$\times\left\{\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right]\left((b-t)\left|h'(a)\right|^q+(t-a)\left|h'(b)\right|^q\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^{b}\left[\int_{a+b-t}^{t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right]\left((b-t)\left|h'(a)\right|^q+(t-a)\left|h'(b)\right|^q\right)dt\right\}^{1/q},$$

where $q > 1$.

Proof. Using Lemma 2.8, Hölder's inequality, (2.9) and the convexity of $|f'|^q$, it follows that

$$\left|\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_{\varphi}g(b)+{}_{b^-}I_{\varphi}g(a)\right]-\left[{}_{a^+}I_{\varphi}(hg)(b)+{}_{b^-}I_{\varphi}(hg)(a)\right]\right|$$
$$\leq\left[\int_a^b\left|\int_t^{a+b-t}\frac{\varphi(b-s)}{(b-s)}g(s)ds\right|dt\right]^{1-1/q}\left[\int_a^b\left|\int_t^{a+b-t}\frac{\varphi(b-s)}{(b-s)}g(s)ds\right|\left|h'(t)\right|^q dt\right]^{1/q}$$
$$\leq\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}g(s)\right|ds\right)dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|\frac{\varphi(b-s)}{(b-s)}g(s)\right|ds\right)dt\right]^{1-1/q}$$
$$\times\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}g(s)\right|ds\right)\left|h'(t)\right|^q dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|\frac{\varphi(b-s)}{(b-s)}g(s)\right|ds\right)\left|h'(t)\right|^q dt\right]^{1/q} \tag{2.12}$$
$$\leq\frac{ess\sup_{t\in[a,b]}|g(x)|}{(b-a)^{1/q}}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right)dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right)dt\right]^{1-1/q}$$
$$\times\left\{\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right)\left((b-t)\left|h'(a)\right|^q+(t-a)\left|h'(b)\right|^q\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right)\left((b-t)\left|h'(a)\right|^q+(t-a)\left|h'(b)\right|^q\right)dt\right\}^{1/q}$$

Hence, we obtain the desired result. This completes the proof. □

Remark 2.19. *Under the hypotheses of Theorem 6, if we choose $\varphi(x)=\frac{x^{\alpha}}{\Gamma(\alpha)}$, then we get the expression of Theorem 6 in [7].*

We can express another inequality for $q > 1$ as follows:

Theorem 2.20. *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a diferentiable mapping on I° and $h' \in L[a,b]$ with $a < b$, and also $|h'|^q$, $q > 1$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symetric with respect to $\frac{a+b}{2}$. The following holds for the generalized fractional integrals;*

$$\left|\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_{\varphi}g(b)+{}_{b^-}I_{\varphi}g(a)\right]-\left[{}_{a^+}I_{\varphi}(hg)(b)+{}_{b^-}I_{\varphi}(hg)(a)\right]\right| \tag{2.13}$$
$$\leq\frac{\|g\|_{\infty}}{(b-a)}\left\{\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right]^p dt+\int_{\frac{a+b}{2}}^{b}\left[\int_{a+b-t}^{t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right]^p dt\right\}^{1/p}$$
$$\times\left\{\frac{\left|h'(a)\right|^q+\left|h'(b)\right|^q}{2}\right\}^{1/q}.$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.8, Hölder's inequality, (2.9) and the convexity of $|h'|^q$, it follows that

$$\begin{aligned}
&\left|\left(\frac{h(a)+h(b)}{2}\right)\left[{}_{a^+}I_\varphi g(b)+{}_{b^-}I_\varphi g(a)\right]-\left[{}_{a^+}I_\varphi (hg)(b)+{}_{b^-}I_\varphi (hg)(a)\right]\right| \\
\leq\;& \left[\int_a^b\left|\int_t^{a+b-t}\frac{\varphi(b-s)}{(b-s)}g(s)ds\right|^p dt\right]^{1/p}\left[\int_a^b|h'(t)|^q dt\right]^{1/q} \\
\leq\;& ess\sup_{t\in[a,b]}|g(x)|\left\{\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right]^p dt+\int_{\frac{a+b}{2}}^b\left[\int_{a+b-t}^t\frac{|\varphi(b-s)|}{(b-s)}ds\right]^p dt\right\}^{1/p} \\
&\times\left\{\left(\frac{b-t}{b-a}|h'(a)|^q+\frac{t-a}{b-a}|h'(b)|^q\right)dt\right\}^{1/q} \\
=\;& \frac{\|g\|_\infty}{(b-a)}\left\{\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right]^p dt+\int_{\frac{a+b}{2}}^b\left[\int_{a+b-t}^t\left|\frac{\varphi(b-s)}{(b-s)}\right|ds\right]^p dt\right\}^{1/p} \\
&\times\left\{\frac{|h'(a)|^q+|h'(b)|^q}{2}\right\}^{\frac{1}{q}}.
\end{aligned}$$

□

Remark 2.21. *Under the hypotheses of Theorem 7, if we choose* $\varphi(x)=\dfrac{x^\alpha}{\Gamma(\alpha)}$ *then we get the expression of Theorem 7 in [7].*

Remark 2.22. *Under the hypotheses of Theorem 7, if we choose* $\varphi(x)=x$*, then the inequality (2.13) becomes the inequality found in [17].*

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