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The Scalar Curvature of a Projectively Invariant Metric Defined by the Kernel Function

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ABSTRACT

Considering a projectively invariant metric τ defined by the kernel function on a strongly convex bounded domain $\Omega \subset \mathbb{R}^n$, we study the asymptotic expansion of the scalar curvature with respect to the distance function, and use the Fubini-Pick invariant to describe the second term in the expansion. This asymptotic expansion implies that if $n \geq 3$ and (Ω, τ) has constant scalar curvature, then the convex domain is projectively equivalent to a ball.

Keywords: *Scalar curvature, Fubini-Pick invariant, kernel function.*

AMS Subject Classification (2020): *Primary: 52A20; Secondary: 53A15; 34E05.*

1. Introduction

Let Ω be a strongly convex bounded domain with smooth boundary in $\mathbb{R}^n$, and Ω^* be the dual of Ω defined by $\Omega^* = int\{\xi \in \mathbb{R}_n | 1 + \langle x, \xi \rangle \geq 0,\ for\ x \in \Omega\}$. Sasaki [3] defined the characteristic function χ and the kernel function κ of Ω as follows

$$\chi(x) = \int_{\Omega^*} n!(1 + \langle \xi, x \rangle)^{-n-1} \mathrm{d}\xi, \tag{1.1}$$

$$\kappa(x) = \int_{\Omega^*} (2n+1)!(1 + \langle \xi, x \rangle)^{-2n-2} \cdot \chi_{\Omega^*}(\xi)^{-1} \mathrm{d}\xi. \tag{1.2}$$

Next he defined two metrics

$$\omega = -\chi^{\frac{1}{n+1}} \cdot \mathrm{dd}(\chi^{\frac{-1}{n+1}}), \qquad and \qquad \tau = -\kappa^{\frac{1}{2n+2}} \cdot \mathrm{dd}(\kappa^{\frac{-1}{2n+2}}), \tag{1.3}$$

and proved they are complete Riemannian metrics and invariant under projective transformations. Also he showed metrics ω and τ coincide with the Blaschke metric on a hyperbolic affine hypersphere when Ω is projectively homogeneous.

In the paper [4], Sasaki-Yagi first gave the boundary behaviors of derivatives of the functions χ and κ, then showed that the sectional curvatures of both metric ω and τ tend to -1 on the boundary $\partial\Omega$. Wu [5] studied the asymptotic expansion of the scalar curvature H of (Ω, ω) and obtained

$$H(x) = -n(n-1) + 2^{-\frac{2}{n+1}} J(y) \cdot dist(x, \partial\Omega) + O(dist(x, \partial\Omega)^2), \tag{1.4}$$

where J is the Fubini-Pick invariant of the boundary $\partial\Omega$. By the theorem of Maschke-Pick-Berwald [1]: Every locally strongly convex hypersurface with vanishing Fubini-Pick invariant must be a hyperquadric. Hence

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the estimate (1.4) implies that if $n \geq 3$ and the scalar curvature of (Ω, ω) is a constant, then Ω is projectively equivalent to a ball. When $n = 2$, the Fubini-Pick invariant of the boundary curve is zero, Wu [5] found that the third term of the asymptotic expansion is also zero. Sasaki [3] also defined the $p-$th characteristic function χ_p and its metric ω_p, Wu [6] also considered the asymptotic expansion of the derivatives of χ_p, and proved that the sectional curvature of (Ω, ω_p) tend to -1 on the boundary $\partial\Omega$.

In this paper, we study the asymptotic expansion of the scalar curvature of (Ω, τ) with respect to the distance function, and use the Fubini-Pick invariant to describe the second term in the expansion. We obtain

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a strongly convex bounded domain with smooth boundary, and R be the scalar curvature of (Ω, τ). For x near $\partial\Omega$, choose $y(x)$ so that $dist(x, y) = dist(x, \partial\Omega)$, then*

$$R(x) = -n(n-1) + 2^{-\frac{2}{n+1}} J(y) \cdot dist(x, \partial\Omega) + O(dist(x, \partial\Omega)^2), \tag{1.5}$$

where J is the Fubini-Pick invariant of the boundary $\partial\Omega$.

By the result of Maschke-Pick-Berwald, we have

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a strongly convex bounded domain with smooth boundary. If the scalar curvature of (Ω, τ) is a constant, then Ω is projectively equivalent to a ball.*

2. Boundary behaviors of the derivatives of $\kappa(x)$

In this section we need the calculations in [4]. Let Ω be a strongly convex bounded domain with smooth boundary which contains the origin, and let Ω^* be the dual of Ω. The star mapping defined as

$$x^* = -grad\chi(x) \cdot ((n+1)\chi(x) + \langle grad\chi(x), x\rangle)^{-1} \tag{2.1}$$

is a diffeomorphism of Ω onto Ω^*, and can be smoothly extended to $\partial\Omega$.

For a multi-index $\alpha' = (\alpha_1, \alpha_2, ..., \alpha_{n-1})$ with $|\alpha'| = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$, we use the notations

$$\xi' = (\xi_1, \xi_2, ..., \xi_{n-1}), \quad (\xi')^{\alpha'} = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_{n-1}^{\alpha_{n-1}}.$$

For a fixed point $y \in \partial\Omega$, we choose coordinates $(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ such that $y = (0, 0, ..., 0, y_n)$ and $dist(ky, \partial\Omega) = dist(ky, y)$ for $k \in (0, 1)$ sufficiently near 1. For simplicity, we assume $y_n = 1$. We choose coordinate such that the boundary $\partial\Omega$ around y is written as

$$x_n = 1 - \tfrac{1}{4}\sum_{i=1}^{n-1}(x_i)^2 + \tfrac{1}{6}\sum a_{ijk}x_ix_jx_k + \sum_{|\alpha'|>3} a_{\alpha'}(x')^{\alpha'} + O(|x'|^{2N_0+2}), \tag{2.2}$$

where $x' = (x_1, x_2, ..., x_{n-1})$ and N_0 is a sufficiently large integer. Let y^* denote the image of y by the star mapping, then the boundary of $\partial\Omega^*$ around y^* is written as

$$\xi_n = -1 + \sum_{i=1}^{n-1}(\xi_i)^2 + \tfrac{1}{6}\sum b_{ijk}\xi_i\xi_j\xi_k + \sum_{|\alpha'|>3} b_{\alpha'}(\xi')^{\alpha'} + O(|\xi'|^{2N_0+2}). \tag{2.3}$$

Set $x = ky$ and $1 + \xi_n = t$, then $1 + \langle\xi, x\rangle = kt + 1 - k$. The derivative of κ is given by

$$\begin{aligned}\kappa_\beta(x) : &= \frac{\partial^{|\beta|}(\kappa)}{\partial x^\beta} \\ &= (-1)^{|\beta|}(2n + |\beta| + 1)! \int_0^b (kt + 1 - k)^{-2n-2-|\beta|}(t-1)^{\beta_n}B(t)\mathrm{d}t,\end{aligned} \tag{2.4}$$

where

$$B(t) = \int_{\Omega^* \cap \{1+\xi_n = t\}} (\xi')^{\beta'}(\chi_{\Omega^*}(\xi))^{-1}\mathrm{d}\xi', \qquad b = \max_{\Omega^*}\{1 + \xi_n\}. \tag{2.5}$$

Choose ξ with $l(\xi) := 1 + \xi_n$ sufficiently small so that $d(\xi, \partial\Omega^*)$ is attained by a unique point $\varsigma \in \partial\Omega^*$. Then $\varsigma = (\varsigma', \varsigma_n)$ satisfies

$$\varsigma_i - \xi_i = (t - l(\varsigma'))\tfrac{\partial l}{\partial \xi_i}(\varsigma'). \tag{2.6}$$

Hence there exist the following approximate identities

$$d(\xi, \varsigma) = |t - l(\varsigma')|(1 + \sum_{|\alpha'|>1} a^1_\alpha (\varsigma')^{\alpha'} + O(|\varsigma'|^{2N_1})), \tag{2.7}$$

$$\mathrm{d}\xi' = \mathrm{d}(\varsigma')(1 + \sum_{|\alpha'|, m\geq 1} a^2_{\alpha,m}(\varsigma')^{\alpha'}|t - l(\varsigma')|^m + O(|\varsigma'|^{2N_1}) + O(|t - l(\varsigma')|^{N_1})), \tag{2.8}$$

$$(\xi')^{\beta'} = (\varsigma')^{\beta'}(1 + \sum_{m\geq 1} a^3_m |t - l(\varsigma')|^m + O(|t - l(\varsigma')|^{N_1})), \tag{2.9}$$

where N_1 is a sufficiently large integer greater than the following integer N_2.

By the formula (2.3), the Gauss curvature of $\partial\Omega^*$ at ς has the expansion

$$\gamma(\varsigma) = 2^{n-1} + \sum_{|\alpha'|\geq 1} a^4_\alpha (\varsigma')^{\alpha'} + O(|\varsigma'|^{2N_1}). \tag{2.10}$$

The characteristic function χ_{Ω^*} has the asymptotic expansion (see [3])

$$\chi_{\Omega^*}(\xi) = c_0(0)2^{\frac{n-1}{2}}\gamma(\varsigma)^{\frac{1}{2}} d(\xi, \varsigma)^{-\frac{n+1}{2}} \cdot (1 + \sum_{m\geq 1} a^5_m d(\xi, \varsigma)^m + O(d(\xi, \varsigma)^{N_2})), \tag{2.11}$$

where $N_2 = [\frac{n}{2}]$, and $c_0(0) = \pi^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})$.

It follows that

$$\begin{aligned}(\xi')^{\beta'}(\chi_{\Omega^*})^{-1}\mathrm{d}\xi' \;&=\; \tfrac{1}{2^{n-1}c_0(0)}(\varsigma')^{\beta'}|t - l(\varsigma')|^{\frac{n+1}{2}}(1 + \sum_{|\alpha'|+m\geq 1} a^6_{\alpha,m}(\varsigma')^{\alpha'}|t - l(\varsigma')|^m \\ &+\; O(|\varsigma'|^{2N_2}) + O(|t - l(\varsigma')|^{N_2}))\mathrm{d}\varsigma'. \end{aligned} \tag{2.12}$$

Put

$$P_{q,m} = \sum_{|\alpha'|=q} a^6_{\alpha,m}(\varsigma')^{\alpha'}, \quad P_{0,0} = 1. \tag{2.13}$$

Then the estimate of $B(t)$ is obtained by computing the integral

$$\tfrac{1}{2^{n-1}c_0(0)} \int_{\Omega^*\cap\{0\leq l(\varsigma')\leq t\}} (\varsigma')^{\beta'} P_{q,m}(\varsigma')|t - l(\varsigma')|^{m+\frac{n+1}{2}}\mathrm{d}\varsigma'. \tag{2.14}$$

Relative to the polar coordinates $\varsigma_i = rf_i$, we have

$$l(\varsigma') = r^2(1 + \sum_{p\geq 1}\epsilon_p r^p + O(r^{2N_0})), \tag{2.15}$$

where N_0 is a sufficiently large integer greater than N_2.

Further set $l(\varsigma') = tu^2$, then there exist v_p such that

$$r = \sqrt{t}u(1 + \sum_{p\geq 1} v_p(\sqrt{t}u)^p + O((\sqrt{t}u)^{N_0})), \tag{2.16}$$

Define μ_p by

$$r^{q+n-2+|\beta'|}\mathrm{d}r = (\sqrt{t}u)^{q+n-2+|\beta'|}\sqrt{t}\cdot(1+\sum_{p\geq 1}(p+1)\mu_p(\sqrt{t}u)^p + O((\sqrt{t}u)^{N_0}))\mathrm{d}u. \tag{2.17}$$

Hence

$$\begin{aligned} B(t) &= \tfrac{1}{2^n c_0(0)}\sum_{k,m}\int a_{k,m}B(\tfrac{n+3+2m}{2},\tfrac{n-1+k+|\beta'|}{2})t^{\frac{2n+k+2m+|\beta'|}{2}}(f')^{\beta'}\mathrm{d}S + O(t^{N_2}) \\ &= \sum_{i\geq 0} t^{\frac{2n+i+|\beta'|}{2}}\int \lambda_i (f')^{\beta'}\mathrm{d}S + O(t^{N_2}), \end{aligned} \tag{2.18}$$

where

$$a_{k,m} = \sum_{p+q=k}(p+1)\mu_p\cdot P_{q,m}(f'),\quad a_{0,0}=1, \tag{2.19}$$

and

$$\lambda_i = \tfrac{1}{2^n c_0(0)}\sum_{k+2m=i} a_{k,m}B(\tfrac{n+3+2m}{2},\tfrac{n-1+k+|\beta'|}{2}). \tag{2.20}$$

In the following paper, we denote the distance function $dist(x,\partial\Omega)$ by d. Combining (2.4) and (2.18), Sasaki-Yagi [4] obtained

$$\kappa_\beta = d^{-\frac{2n+|\beta'|+2\beta_n+2}{2}}(\sum_{k=0}^{2N_2-1}\tilde{c}_k(\beta)\cdot d^{\frac{k}{2}} + O(d^{N_2})), \tag{2.21}$$

here

$$\begin{aligned} \tilde{c}_{2p+q}(\beta) &= (-1)^{|\beta'|}\Gamma(\tfrac{2n+|\beta'|+2+2p+q}{2})\Gamma(\tfrac{2n+|\beta'|+2+2\beta_n-2p-q}{2})(\Gamma(\tfrac{2n+|\beta'|+2-2p-q}{2}))^{-1} \\ &\cdot \sum_{j=0}^{p}\Gamma(\tfrac{2n+|\beta'|+2-2j-q}{2})(\Gamma(p-j+1))^{-1}\int\lambda_{2j+q}(f')^{\beta'}\mathrm{d}S, \end{aligned} \tag{2.22}$$

where q takes the value 0 or 1.

Because of the integral formula

$$\int\prod_{i=1}^{n-1}(f_i')^{\beta_i}\mathrm{d}S = \begin{cases} \frac{(n+|\beta'|-1)}{\Gamma(\frac{n+|\beta'|+1}{2})}\prod_{i=1}^{n-1}\Gamma(\frac{\beta_i+1}{2}), & when\ \beta_i, 1\leq i\leq\ n-1\ are\ even, \\ 0, & otherwise, \end{cases} \tag{2.23}$$

Sasaki-Yagi divided into three cases

(case a): If every $\beta_i,\ (i=1,\cdots n-1)$ is even, then $c_{2k+1}(\beta)=0$ for $k\geq 0$,

(case b): If at least one of $\beta_i,(\ i=1,\cdots n-1)$ is odd and $|\beta'|$ is odd, then $c_{2k}(\beta)=0$ for $k\geq 0$,

(case c): If at least one of $\beta_i,\ (i=1,\cdots n-1)$ is odd and $|\beta'|$ is even, then $c_0(\beta)=0,\quad c_{2k+1}(\beta)=0$ for $k\geq 0$.

According to the parity of the index β defined above, they obtained the following expansions

Theorem 2.1.

$$(case\ a)\quad \kappa_\beta = d^{-\frac{2n+|\beta'|+2+2\beta_n}{2}}(\sum_{i=0}^{N_2-1}\tilde{c}_{2i}(\beta)\cdot d^i + O(d^{N_2})),$$

$$(case\ b)\quad \kappa_\beta = d^{-\frac{2n+|\beta'|+1+2\beta_n}{2}}(\sum_{i=0}^{N_2-1}\tilde{c}_{2i+1}(\beta)\cdot d^i + O(d^{N_2-\frac{1}{2}})),$$

$$(case\ c)\quad \kappa_\beta = d^{-\frac{2n+|\beta'|+2\beta_n}{2}}(\sum_{i=0}^{N_2-2}\tilde{c}_{2i+2}(\beta)\cdot d^i + O(d^{N_2-1})).$$

Put

$$\psi = -\kappa^{-\frac{1}{n+1}}, \tag{2.24}$$

then the metric τ defined in (1.3) is given by

$$\tau = -\frac{1}{\sqrt{-\psi}}\mathrm{dd}(\sqrt{-\psi}) := \sum h_{ij}\mathrm{d}x_i\mathrm{d}x_j, \tag{2.25}$$

where

$$h_{ij} = \frac{1}{2}\left(-\frac{\psi_{ij}}{\psi} + \frac{\psi_i\psi_j}{2\psi^2}\right) = \frac{1}{2(n+1)}\frac{\kappa_{ij}}{\kappa} - \frac{2n+3}{4(n+1)^2}\frac{\kappa_i}{\kappa}\frac{\kappa_j}{\kappa}. \tag{2.26}$$

Here and later, $\psi_i, \psi_{ij}, \psi_{ijk}$ denote the usual derivatives, and the matrix (h^{ij}) denotes the inverse of the matrix (h_{ij}). Then the boundary estimates of (h_{ij}) and (h^{ij}) follow from in [4].

Lemma 2.1. *The matrix (h_{ij}) has the form:*

$$\frac{1}{4}\begin{pmatrix} d^{-1} & & & & O(d^{-1}) \\ & \ddots & & O(1) & \vdots \\ O(1) & & \ddots & & \vdots \\ & & & d^{-1} & O(d^{-1}) \\ O(d^{-1}) & \cdots & \cdots & O(d^{-1}) & d^{-2} \end{pmatrix}$$

and (h^{ij}) has the form:

$$4\begin{pmatrix} d & & & & O(d^2) \\ & \ddots & & O(d^2) & \vdots \\ O(d^2) & & \ddots & & \vdots \\ & & & d & O(d^2) \\ O(d)^2) & \cdots & \cdots & O(d^2) & d^2 \end{pmatrix}$$

Take the logarithm on both sides of (2.24) and differentiate with respect to x:

$$\begin{aligned} -(n+1)\frac{\psi_i}{\psi} &= \frac{\kappa_i}{\kappa}, \\ -(n+1)\frac{\psi_{ij}}{\psi} &= \frac{\kappa_{ij}}{\kappa} - \frac{n+2}{n+1}\frac{\kappa_i}{\kappa}\frac{\kappa_j}{\kappa}, \\ -(n+1)\frac{\psi_{ijk}}{\psi} &= \frac{\kappa_{ijk}}{\kappa} - \frac{n+2}{n+1}\left(\frac{\kappa_{ij}}{\kappa}\frac{\kappa_k}{\kappa} + \frac{\kappa_{jk}}{\kappa}\frac{\kappa_i}{\kappa} + \frac{\kappa_{ik}}{\kappa}\frac{\kappa_j}{\kappa}\right) + \frac{(n+2)(2n+3)}{(n+1)^2}\frac{\kappa_i}{\kappa}\frac{\kappa_j}{\kappa}\frac{\kappa_k}{\kappa}. \end{aligned} \tag{2.27}$$

The boundary estimates of ψ is given in [4].

Lemma 2.2. *Assume $n \geq 2$, the derivatives of ψ up to third order have finite continuous values on $\partial\Omega$ except $\psi_{nn}, \psi_{iin}, \psi_{inn}, 1 \leq i < n$ that are of order $d^{-\frac{1}{2}}$ at most, and ψ_{nnn} that is of order $d^{-\frac{3}{2}}$ at most. Assume $n \geq 4$, the derivatives of ψ up to third order have finite continuous values on $\partial\Omega$ except ψ_{nnn} that is of order $d^{-\frac{1}{2}}$ at most.*

3. Improve the boundary estimates of ψ

In this section, we improve the boundary estimates of ψ in the case of $n \leq 3$, which is crucial for the proof of Theorem 1.1. In section 2, the authors wrote the the asymptotic expansion of χ_{Ω^*} as in (2.11), but by [4], the following asymptotic expansion also holds.
If $n = 2$, then

$$\chi_{\Omega^*}(\xi) = 2^{\frac{1}{2}}c_0(0)\gamma(\varsigma)^{\frac{1}{2}}d(\xi,\varsigma)^{-\frac{3}{2}}\left(1 + a_1^5 d(\xi,\varsigma) + O(d(\xi,\varsigma)^{\frac{3}{2}})\right), \tag{3.1}$$

if $n = 3$, then

$$\begin{aligned} \chi_{\Omega^*}(\xi) &= 2c_0(0)\gamma(\varsigma)^{\frac{1}{2}}d(\xi,\varsigma)^{-2}\left(1 + \tilde{a}_1^5 d(\xi,\varsigma) + d^2\cdot O(\log d(\xi,\varsigma)\right) \\ &= 2c_0(0)\gamma(\varsigma)^{\frac{1}{2}}d(\xi,\varsigma)^{-2}\left(1 + \tilde{a}_1^5 d(\xi,\varsigma) + O(d(\xi,\varsigma)^{\frac{3}{2}})\right). \end{aligned} \tag{3.2}$$

Thus for $n \leq 3$, we have

$$\chi_{\Omega^*}(\xi) = 2^{\frac{n-1}{2}} c_0(0)\gamma(\varsigma)^{\frac{1}{2}} d(\xi,\varsigma)^{-\frac{n+1}{2}} \big(1 + a_1^5 d(\xi,\varsigma) + O(d(\xi,\varsigma)^{\frac{3}{2}})\big). \tag{3.3}$$

By the computations in section 2, we have the following estimates:
(case a): If every $\beta_i,\ (i = 1, \cdots n-1)$ is even, then

$$\kappa_\beta = d^{-\frac{2n+|\beta'|+2+2\beta_n}{2}} \big(\tilde{c}_0(\beta) + \tilde{c}_2(\beta)\cdot d + O(d^{\frac{3}{2}})\big). \tag{3.4}$$

(case b): If at least one of $\beta_i,(\ i = 1, \cdots n-1)$ is odd and $|\beta'|$ is odd, then

$$\kappa_\beta = d^{-\frac{2n+|\beta'|+1+2\beta_n}{2}} \big(\tilde{c}_1(\beta) + O(d)\big). \tag{3.5}$$

Here we need not consider the (case c). In the following estimates, the first $n-1$ components of $\beta = (\beta', \beta_n)$ and the last component β_n play different roles. By (2.22), we have

$$\tilde{c}_i(\beta', \beta_n + 1) = \tfrac{2n+|\beta'|+2\beta_n+2-i}{2} \tilde{c}_i(\beta', \beta_n), \tag{3.6}$$

$$\tilde{c}_0(\beta_1, ..., \beta_i + 2, ..., \beta_{n-1}, \beta_n) = \tfrac{(2n+|\beta'|+2\beta_n+2)(\beta_i+1)}{4} \tilde{c}_0(\beta_1, ..., \beta_i, ..., \beta_{n-1}, \beta_n). \tag{3.7}$$

We need not to make any distinction among the first $n-1$ components, hence use the abbreviation $(p;q)$ for denoting $\beta = (p, 0...0, q)$. By (3.4)-(3.5), we have

$$\begin{aligned}
\kappa &= d^{-(n+1)}(\tilde{c}_0(0) + \tilde{c}_2(0)\cdot d + O(d^{\frac{3}{2}})),\\
\kappa_1 &= d^{-(n+1)}(\tilde{c}_1(1;0) + O(d)),\\
\kappa_n &= d^{-(n+2)}(\tilde{c}_0(0;1) + \tilde{c}_2(0;1)\cdot d + O(d^{\frac{3}{2}})),\\
\kappa_{11} &= d^{-(n+2)}(\tilde{c}_0(2;0) + \tilde{c}_2(2;0)\cdot d + O(d^{\frac{3}{2}})),\\
\kappa_{1n} &= d^{-(n+2)}(\tilde{c}_1(1;1) + O(d)),\\
\kappa_{nn} &= d^{-(n+3)}(\tilde{c}_0(0;2) + \tilde{c}_2(0;2)\cdot d + O(d^{\frac{3}{2}})),\\
\kappa_{1nn} &= d^{-(n+3)}(\tilde{c}_1(1;2) + O(d)),\\
\kappa_{11n} &= d^{-(n+3)}(\tilde{c}_0(2;1) + \tilde{c}_2(2;1)\cdot d + O(d^{\frac{3}{2}})),\\
\kappa_{nnn} &= d^{-(n+4)}(\tilde{c}_0(0;3) + \tilde{c}_2(0;3)\cdot d + O(d^{\frac{3}{2}})).
\end{aligned}$$

Put

$$c_1 = \tfrac{\tilde{c}_1(1;0)}{\tilde{c}_0(0)}, \quad c_2 = \tfrac{\tilde{c}_2(0)}{\tilde{c}_0(0)}. \tag{3.8}$$

Hence by (3.6)-(3.8), we have

$$\begin{aligned}
\tfrac{\kappa_1}{\kappa} &= c_1 + O(d),\\
\tfrac{\kappa_n}{\kappa} &= d^{-1}\big(\tfrac{\tilde{c}_0(0;1)}{\tilde{c}_0(0)} + (\tfrac{\tilde{c}_2(0;1)}{\tilde{c}_0(0)} - c_2\tfrac{\tilde{c}_0(0;1)}{\tilde{c}_0(0)})\cdot d + O(d^{\frac{3}{2}})\big),\\
&= (n+1)d^{-1} - c_2 + O(d^{\frac{1}{2}}),\\
\tfrac{\kappa_{11}}{\kappa} &= \tfrac{n+1}{2}d^{-1} + O(1),\\
\tfrac{\kappa_{1n}}{\kappa} &= (n+1)c_1 d^{-1} + O(1),\\
\tfrac{\kappa_{nn}}{\kappa} &= d^{-2}\big(\tfrac{\tilde{c}_0(0;2)}{\tilde{c}_0(0)} + (\tfrac{\tilde{c}_2(0;2)}{\tilde{c}_0(0)} - c_2\tfrac{\tilde{c}_0(0;2)}{\tilde{c}_0(0)})\cdot d + O(d^{\frac{3}{2}})\big),\\
&= (n+2)(n+1)d^{-2} - 2(n+1)c_2 d^{-1} + O(d^{-\frac{1}{2}}),\\
\tfrac{\kappa_{11n}}{\kappa} &= \tfrac{(n+2)(n+1)}{2}d^{-2} + O(d^{-1}),\\
\tfrac{\kappa_{1nn}}{\kappa} &= (n+2)(n+1)c_1 d^{-2} + O(d^{-1}),\\
\tfrac{\kappa_{nnn}}{\kappa} &= d^{-3}\big(\tfrac{\tilde{c}_0(0;3)}{\tilde{c}_0(0)} + (\tfrac{\tilde{c}_2(0;3)}{\tilde{c}_0(0)} - c_2\tfrac{\tilde{c}_0(0;3)}{\tilde{c}_0(0)})\cdot d + O(d^{\frac{3}{2}}\big).\\
&= (n+3)(n+2)(n+1)d^{-3} - 3(n+2)(n+1)c_2 d^{-2} + O(d^{-\frac{3}{2}}).
\end{aligned}$$

By (2.27) we know

$$\begin{aligned}
-(n+1)\tfrac{\psi_{11n}}{\psi} &= \tfrac{\kappa_{11n}}{\kappa} - \tfrac{n+2}{n+1}\left(\tfrac{\kappa_{11}}{\kappa}\tfrac{\kappa_n}{\kappa} + 2\tfrac{\kappa_{1n}}{\kappa}\tfrac{\kappa_1}{\kappa}\right) + \tfrac{(n+2)(2n+3)}{(n+1)^2}\tfrac{\kappa_1}{\kappa}\tfrac{\kappa_1}{\kappa}\tfrac{\kappa_n}{\kappa},\\
-(n+1)\tfrac{\psi_{1nn}}{\psi} &= \tfrac{\kappa_{1nn}}{\kappa} - \tfrac{n+2}{n+1}\left(\tfrac{\kappa_{nn}}{\kappa}\tfrac{\kappa_1}{\kappa} + 2\tfrac{\kappa_{1n}}{\kappa}\tfrac{\kappa_n}{\kappa}\right) + \tfrac{(n+2)(2n+3)}{(n+1)^2}\tfrac{\kappa_1}{\kappa}\tfrac{\kappa_n}{\kappa}\tfrac{\kappa_n}{\kappa},\\
-(n+1)\tfrac{\psi_{nnn}}{\psi} &= \tfrac{\kappa_{nnn}}{\kappa} - 3\tfrac{n+2}{n+1}\tfrac{\kappa_{nn}}{\kappa}\tfrac{\kappa_n}{\kappa} + \tfrac{(n+2)(2n+3)}{(n+1)^2}\tfrac{\kappa_n}{\kappa}\tfrac{\kappa_n}{\kappa}\tfrac{\kappa_n}{\kappa}.
\end{aligned}$$

The direct computations show that ψ_{11n}, ψ_{1nn} have finite continuous values on $\partial\Omega$, and ψ_{nnn} is that of order $d^{-\frac{1}{2}}$ at most. Based on Lemma 2.2, we have

Lemma 3.1. *Assume $n \geq 2$, the derivatives of ψ up to third order have finite continuous values on $\partial\Omega$ except ψ_{nnn} that is of order $d^{-\frac{1}{2}}$ at most.*

4. The scalar curvature of (Ω, τ)

Now we need compute the coefficients $\tilde{c}_0(0)$ and $\tilde{c}_1(\beta)$ explicitly. By (2.19)-(2.22), we have

$$\tilde{c}_0(0) = \tfrac{1}{2^n c_0(0)}(\Gamma(n+1))^2 B(\tfrac{n-1}{2}, \tfrac{n+3}{2}) \int \mathrm{d}S = \tfrac{(n+1)!}{2^n}. \tag{4.1}$$

We also have

$$\begin{aligned}
\tilde{c}_1(\beta) &= (-1)^{|\beta'|}\Gamma(\tfrac{2n+|\beta'|+3}{2})\Gamma(\tfrac{2n+|\beta'|+1+2\beta_n}{2}) \int \lambda_1 \cdot (f')^{\beta'} \mathrm{d}S\\
&= \tfrac{1}{2^n c_0(0)}(-1)^{|\beta'|}\Gamma(\tfrac{n+3}{2})\Gamma(\tfrac{n+|\beta'|}{2})\Gamma(\tfrac{2n+|\beta'|+1+2\beta_n}{2}) \int a_{1,0} \cdot (f')^{\beta'} \mathrm{d}S. \qquad (4.2)
\end{aligned}$$

Formulas (2.7)-(2.12) give

$$\sum_{|\alpha'|=1} a^6_{\alpha,0}(\varsigma')^{\alpha'} = -\tfrac{1}{2^n}\sum_{|\alpha'|=1} a^4_{\alpha}(\varsigma')^{\alpha'}.$$

By (2.3)-(2.19), we get

$$\begin{aligned}
a_{1,0} &= 2\mu_1 + P_{1,0}(f')\\
&= (n+|\beta'|)v_1 + \sum_{|\alpha'|=1} a^6_{\alpha,0} \cdot (f')^{\alpha'} \qquad (4.3)\\
&= -\tfrac{n+|\beta'|}{12}\sum b_{ijk}f_if_jf_k - \tfrac{1}{2^n}\sum_{|\alpha'|=1} a^4_{\alpha} \cdot (f')^{\alpha'}. \qquad (4.4)
\end{aligned}$$

By (2.3), we get the Gauss curvature of $\partial\Omega^*$ at ς has the expansion

$$\gamma(\varsigma) = 2^{n-1} + 2^{n-2}\sum b_{iik}\varsigma_k + O(|\varsigma'|^2). \tag{4.5}$$

On the other hand, By (2.2)-(2.3) and the definition of star mapping we have the relation (see the section 4 in [3])

$$b_{ijk} = -8a_{ijk}. \tag{4.6}$$

Hence

$$\begin{aligned}
a_{1,0} &= -\tfrac{n+|\beta'|}{12}\sum b_{ijk}f_if_jf_k - \tfrac{1}{4}\sum b_{iik}f_k\\
&= \tfrac{2}{3}(n+|\beta'|)\sum a_{ijk}f_if_jf_k + 2\sum a_{iik}f_k. \qquad (4.7)
\end{aligned}$$

Next we expand the scalar curvature of (Ω, τ) with respect to the distance function, the Riemannian curvature tensor of (Ω, τ) is given by (see [2])

$$R_{ijkl} = -(h_{il}h_{jk} - h_{ik}h_{jl}) - \tfrac{1}{16\psi^2}\sum_{p,q=1}^{n} h^{pq}(\psi_{pil}\psi_{qjk} - \psi_{pik}\psi_{qjl}), \tag{4.8}$$

Hence the scalar curvature R of (Ω, τ) is given by

$$R = -n(n-1) + \tfrac{1}{16\psi^2} \sum h^{il}h^{jk}h^{pq}(\psi_{pik}\psi_{qjl} - \psi_{pil}\psi_{qjk}). \tag{4.9}$$

Lemma 2.1 and Lemma 3.1 give

$$\begin{aligned} R &= -n(n-1) + \tfrac{1}{16}\sum_{i\neq j} h^{ii}h^{jj}\sum_{p<n} h^{pp}((\tfrac{\psi_{pij}}{\psi})^2 - \tfrac{\psi_{pii}}{\psi}\tfrac{\psi_{pjj}}{\psi}) + O(d^2) \\ &= -n(n-1) + d^2\sum_{i\neq j}\sum_{p<n} h^{pp}((\tfrac{\psi_{pij}}{\psi})^2 - \tfrac{\psi_{pii}}{\psi}\tfrac{\psi_{pjj}}{\psi}) + O(d^2) \\ &= -n(n-1) + 4d^3\sum_{i,j,p\neq}((\tfrac{\psi_{pij}}{\psi})^2 - \tfrac{\psi_{pii}}{\psi}\tfrac{\psi_{pjj}}{\psi}) \\ &+ 8d^3\sum_{i\neq j}((\tfrac{\psi_{iij}}{\psi})^2 - \tfrac{\psi_{iii}}{\psi}\tfrac{\psi_{ijj}}{\psi}) + O(d^2), \end{aligned} \tag{4.10}$$

where "$i, j, p \neq$" means these indices are different from each other.

In the following we assume the indices rang from 1 to $n-1$, and for the multi-index $\beta = (0...0, \underset{i}{1}, 0...0, \underset{j}{1}, 0...0)$, we use $\tilde{c}_s(i,j)$ to denote $\tilde{c}_s(\beta)$ in Theorem 2.1; if $\beta = (0...0, \underset{i}{1}, 0...0)$, we use $\tilde{c}_s(i)$ to denote $\tilde{c}_s(\beta)$; if $\beta = (0...0, \underset{i}{1}, 0...0, \underset{j}{1}, 0...0, \underset{k}{1}, 0...0)$, we use $\tilde{c}_s(i,j,k)$ to denote the coefficients $\tilde{c}_s(\beta)$. By Theorem 2.1, we have

$$\begin{aligned} \kappa &= d^{-(n+1)}(\tilde{c}_0(0) + O(d)), \\ \kappa_i &= d^{-(n+1)}(\tilde{c}_1(i) + O(d)), \\ \kappa_{ii} &= d^{-(n+2)}(\tilde{c}_0(i,i) + O(d)), \\ \kappa_{ij} &= d^{-(n+1)}(\tilde{c}_2(i,j) + O(d)), \quad i \neq j, \\ \kappa_{iii} &= d^{-(n+2)}(\tilde{c}_1(i,i,i) + O(d)), \\ \kappa_{iij} &= d^{-(n+2)}(\tilde{c}_1(i,i,j) + O(d)), \quad i \neq j, \\ \kappa_{ijk} &= d^{-(n+2)}(\tilde{c}_1(i,j,k) + O(d)), \quad i,j,k \neq . \end{aligned}$$

It follows that

$$\tfrac{\kappa_i}{\kappa} = \tfrac{\tilde{c}_1(i)}{\tilde{c}_0(0)} + O(d), \quad \tfrac{\kappa_{ii}}{\kappa} = \tfrac{\tilde{c}_0(i,i)}{\tilde{c}_0(0)}d^{-1} + O(1), \quad \tfrac{\kappa_{ij}}{\kappa} = O(1), \quad i \neq j, \tag{4.11}$$

$$\tfrac{\kappa_{iii}}{\kappa} = \tfrac{\tilde{c}_1(i,i,i)}{\tilde{c}_0(0)}d^{-1} + O(1), \quad \tfrac{\kappa_{iij}}{\kappa} = \tfrac{\tilde{c}_1(i,i,j)}{\tilde{c}_0(0)}d^{-1} + O(1), \quad i \neq j, \tag{4.12}$$

$$\tfrac{\kappa_{ijk}}{\kappa} = \tfrac{\tilde{c}_1(i,j,k)}{\tilde{c}_0(0)}d^{-1} + O(1), \quad i,j,k \neq . \tag{4.13}$$

From (2.27), we get

$$-(n+1)\tfrac{\psi_{ijk}}{\psi} = \tfrac{\tilde{c}_1(i,j,k)}{\tilde{c}_0(0)}d^{-1} + O(1), \quad i,j,k \neq, \tag{4.14}$$

$$-(n+1)\tfrac{\psi_{iij}}{\psi} = (\tfrac{\tilde{c}_1(i,i,j)}{\tilde{c}_0(0)} - \tfrac{n+2}{n+1}\tfrac{\tilde{c}_1(j)}{\tilde{c}_0(0)}\tfrac{\tilde{c}_0(i,i)}{\tilde{c}_0(0)})d^{-1} + O(1), \quad i \neq j, \tag{4.15}$$

$$-(n+1)\tfrac{\psi_{iii}}{\psi} = (\tfrac{\tilde{c}_1(i,i,i)}{\tilde{c}_0(0)} - 3\tfrac{n+2}{n+1}\tfrac{\tilde{c}_1(i)}{\tilde{c}_0(0)}\tfrac{\tilde{c}_0(i,i)}{\tilde{c}_0(0)})d^{-1} + O(1). \tag{4.16}$$

We can always choose a coordinate system as in section 2 and such that (see the appendix in [3])

$$\sum_m a_{mmi} = 0, \qquad for\ 1 \leq i \leq n-1. \tag{4.17}$$

From (4.2) and (4.7), we have

$$\tilde{c}_1(\beta) = \tfrac{(-1)^{|\beta'|}}{3}\tfrac{n+|\beta'|}{2^{n-1}c_0(0)}\Gamma(\tfrac{n+3}{2})\Gamma(\tfrac{n+|\beta'|}{2})\Gamma(\tfrac{2n+|\beta'|+1+2\beta_n}{2}) \cdot \sum a_{ijk}\int f_i f_j f_k \cdot (f')^{\beta'}\mathrm{d}S. \tag{4.18}$$

By the integral formula (2.23), we have

$$\begin{aligned}\sum a_{mnp}\int f_m f_n f_p f_i \mathrm{d}S &= a_{iii}\int f_i^4 \mathrm{d}S + 3\sum_{m\neq i} a_{mmi}\int f_m^2 f_i^2 \mathrm{d}S \\ &= \tfrac{3\omega_0}{n+1}a_{iii} + \tfrac{3\omega_0}{n+1}\sum_{m\neq i} a_{mmi} \\ &= \tfrac{3\omega_0}{n+1}\sum_m a_{mmi}, \end{aligned} \tag{4.19}$$

where $\omega_0 = \pi^{\frac{n-1}{2}}(\Gamma(\frac{n+1}{2}))^{-1}$ is the volume of the unit $n-1$ ball.

$$\sum a_{mnp}\int f_m f_n f_p f_i f_j f_k \mathrm{d}S = 6a_{ijk}\int f_i^2 f_j^2 f_k^2 \mathrm{d}S = \tfrac{6\omega_0}{(n+1)(n+3)}a_{ijk}, \qquad i,j,k\neq . \tag{4.20}$$

$$\begin{aligned}\sum a_{mnp}\int f_m f_n f_p f_i^2 f_j \mathrm{d}S &= a_{jjj}\int f_i^2 f_j^4 \mathrm{d}S + 3\sum_{m\neq j} a_{mmj}\int f_i^2 f_j^2 f_m^2 \mathrm{d}S \\ &= a_{jjj}\int f_i^2 f_j^4 \mathrm{d}S + 3\sum_{m\neq i,j} a_{mmi}\int f_i^2 f_j^2 f_m^2 \mathrm{d}S + 3a_{iij}\int f_i^4 f_j^2 \mathrm{d}S \\ &= \tfrac{3\omega_0}{(n+1)(n+3)}a_{jjj} + \tfrac{3\omega_0}{(n+1)(n+3)}\sum_{m\neq i,j} a_{mmj} + \tfrac{9\omega_0}{(n+1)(n+3)}a_{iij} \\ &= \tfrac{3\omega_0}{(n+1)(n+3)}(\sum_m a_{mmj} + 2a_{iij}), \quad i\neq j. \end{aligned} \tag{4.21}$$

$$\begin{aligned}\sum a_{mnp}\int f_m f_n f_p f_i^3 \mathrm{d}S &= a_{iii}\int f_i^6 \mathrm{d}S + 3\sum_{m\neq i} a_{mmi}\int f_i^4 f_m^2 \mathrm{d}S \\ &= \tfrac{15\omega_0}{(n+1)(n+3)}a_{iii} + \tfrac{9\omega_0}{(n+1)(n+3)}\sum_{m\neq i} a_{mmi} \\ &= \tfrac{3\omega_0}{(n+1)(n+3)}(\sum_m 3a_{mmi} + 2a_{iii}). \end{aligned} \tag{4.22}$$

By (4.17)-(4.22), we have

$$\tilde{c}_1(i) = 0, \tag{4.23}$$

$$\tilde{c}_1(i,j,k) = -\tfrac{\omega_0}{2^{n-2}c_0(0)}n!(\Gamma(\tfrac{n+3}{2}))^2 a_{ijk} = -\tfrac{(n+1)^2}{2^n}n!a_{ijk}, \qquad i,j,k\neq, \tag{4.24}$$

$$\tilde{c}_1(i,i,j) = -\tfrac{\omega_0}{2^{n-2}c_0(0)}n!(\Gamma(\tfrac{n+3}{2}))^2 a_{iij} = -\tfrac{(n+1)^2}{2^n}n!a_{iij}, \qquad i\neq j, \tag{4.25}$$

$$\tilde{c}_1(i,i,i) = -\tfrac{\omega_0}{2^{n-2}c_0(0)}n!(\Gamma(\tfrac{n+3}{2}))^2 a_{iii} = -\tfrac{(n+1)^2}{2^n}n!a_{iii}. \tag{4.26}$$

By (4.10), (4.14)-(4.16) and (4.23)-(4.26), we have

$$\begin{aligned} R &= -n(n-1) + \tfrac{4d}{(n+1)^2}\sum_{i,j,p\neq}((\tfrac{\tilde{c}_1(i,j,p)}{\tilde{c}_0(0)})^2 - \tfrac{\tilde{c}_1(i,i,p)}{\tilde{c}_0(0)}\tfrac{\tilde{c}_1(j,j,p)}{\tilde{c}_0(0)}) \\ &+ \tfrac{8d}{(n+1)^2}\sum_{i\neq j}((\tfrac{\tilde{c}_1(i,i,j)}{\tilde{c}_0(0)})^2 - \tfrac{\tilde{c}_1(i,j,j)}{\tilde{c}_0(0)}\tfrac{\tilde{c}_1(i,i,i)}{\tilde{c}_0(0)}) + O(d^2) \\ &= -n(n-1) + 4d\sum_{i,j,p\neq}(a_{ijp}^2 - a_{iip}a_{jjp}) + 8d\sum_{i\neq j}(a_{iij}^2 - a_{ijj}a_{iii}) + O(d^2) \\ &= -n(n-1) + 4d\sum_{i,j,p}(a_{ijp}^2 - a_{iip}a_{jjp}) + O(d^2) \\ &= -n(n-1) + 4d\sum_{i,j,p}a_{ijp}^2 + O(d^2). \end{aligned}$$

From the formula (2.2), we know that the Fubini-Pick invariant of the hypersurface $\partial\Omega$ at the point y is given by (for details see the appendix in [3])

$$J(y) = 2^{\frac{2n+4}{n+1}} \sum_{i,j,k} a_{ijk}^2. \tag{4.27}$$

It follows that

$$R(x) = -n(n-1) + 2^{-\frac{2}{n+1}} J(y) \cdot d + O(d^2).$$

Hence we have proved Theorem 1.1.

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Competing interests

The authors declare that they have no competing interests.

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