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Gap Between Operator Norm and Spectral Radius for the Square of Antidiagonal Block Operator Matrices

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Abstract
In this work, the gap between operator norm and spectral radius for the square of antidiagonal block operator matrices in the direct sum of Banach spaces has been investigated, and also the gap between operator norm and numerical radius for the square of same matrices in the direct sum of Hilbert spaces has been studied.

Keywords: Antidiagonal operator matrix, Numerical radius, Operator norm, Spectral radius
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1. Introduction

As it is known from the mathematical literature that one of the fundamental problems of the spectral theory of linear operators is to obtain the spectrum set, numerical range set and calculate spectral and numerical radii of a given operator. In many cases, serious theoretical and technical difficulties are encountered in finding the spectrum set and numerical range of non-selfadjoint linear bounded operators. Note that there is one formula for the calculation of the spectral radius $r(A)$ of the linear bounded operator in any Banach space $r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ [1]. On the other hand, it is also known that $r(A) \leq w(A) \leq \|A\|$ and $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$ for $A \in L(H)$.

In addition for the linear normal bounded operator A in Hilbert space we have the following relations $r(A) = w(A) = \|A\|$.

It is beneficial to recall that for the spectrum set $\sigma(A)$ and numerical range $W(A)$ of any linear bounded operator A the following spectral inclusion holds $\sigma(A) \subset \overline{W(A)}$ (See [1, 2] for more information).

In [3] some spectral radius inequalities for 2×2 block operator matrix, sum, product, and commutators of two linear bounded Hilbert space operators have been examined. In [4] some estimates for numerical and spectral radii of the Frobenius companion matrix has been obtained.

Some upper and lower bounds for the numerical radius in Hilbert space operators have been obtained in [5].

In [6] some estimates for spectral and numerical radii have been obtained for the product, sum, commutator, anticommutator of two Hilbert spaces operators.

In [7] several numerical radius inequalities for $n\times n$ block operator matrices in the direct sum of Hilbert spaces have been proved. The numerical radius inequalities for $n\times n$ accretive matrices have also been given in [8].

Several new norms and numerical radius inequalities for 2×2 block operator matrices have been researched in [9].

Recently, several new $\mathbb{A}$-numerical radius inequalities for many type $n\times n$ block operator matrices have been offered in [9] in the direct sum of Hilbert spaces.

Subadditivity of the spectral radius of commutative two operators in Banach spaces has been investigated in [10]. By the same author the subadditivity and submultiplicativity properties of local spectral radius of bounded positive operators have been researched in Banach spaces [11]. The same properties of local spectral radius in partially ordered Banach spaces have been established in [12]. In Banach space ordered by a normal and generating core, several inequalities for the spectral radius of a positive commutator of positive operators have been surveyed in [13].

The numerical range and numerical radius of some Volterra integral operator in Hilbert Lebesgue spaces at finite interval have been considered in [14, 15].

Demuth's open problem in 2015 had a great impact on the emergence and shaping of the subject examined in this article (see [16]).

This paper is organized as fallows: In section 2 one important result will be contrived. The evaluations of gaps between operator norm with spectral and numerical radii will be given in Section 3 and Section 4, respectively. And also, throughout this paper, we will use the notations as fallows:

$$(\cdot,\cdot)_H := (\cdot,\cdot),\ \|\cdot\|_H := \|\cdot\|,\ (\cdot,\cdot)_{H_m} := (\cdot,\cdot)_m,\ \|\cdot\|_{H_m} := \|\cdot\|_m,\ 1 \le m \le n.$$

2. Auxiliary important result

It will be proved the following elementary result.

Theorem 2.1. *For each $n \in \mathbb{N}$ and numbers $a_1, a_2, ..., a_n \in \mathbb{R}$ and $b_1, b_2, ..., b_n \in \mathbb{R}$*

$$\min_{1\le m\le n}(a_m - b_m) \le \max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m \le \max_{1\le m\le n}(a_m - b_m)$$

are true.

Proof. In proof, we will use by mathematical induction method.
For $n = 2$, it is clear that

$$\begin{aligned}
\max\{a_1,a_2\} - \max\{b_1,b_2\} &= \frac{1}{2}[(a_1+a_2+|a_1-a_2|) - (b_1+b_2+|b_1-b_2|)] \\
&= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) + (|a_1-a_2| - |b_1-b_2|)] \\
&= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - (|b_1-b_2| - |a_1-a_2|)] \\
&\ge \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - ||b_1-b_2| - |a_1-a_2||] \\
&\ge \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - |(b_1-b_2) - (a_1-a_2)|] \\
&= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - |(a_1-b_1) - (a_2-b_2)|] \\
&= \min\{a_1-b_1, a_2-b_2\}.
\end{aligned}$$

Now assume that

$$\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m \ge \min_{1\le m\le n-1}(a_m - b_m)$$

for any $n \in \mathbb{N}$, $n > 2$. Then one can easily have that

$$\begin{aligned}
\max\{a_1,...,a_n\} - \max\{b_1,...,b_n\} &= \max\{\max_{1\le m\le n-1} a_m, a_n\} - \max\{\max_{1\le m\le n-1} b_m, b_n\} \\
&\ge \min\{\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m, a_n - b_n\} \\
&\ge \min\{\min_{1\le m\le n-1}(a_m - b_m), a_n - b_n\} \\
&= \min_{1\le m\le n}(a_m - b_m).
\end{aligned}$$

From this and by mathematical induction method, for any $n \in \mathbb{N}$

$$\min_{1\le m\le n}(a_m - b_m) \le \max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m$$

holds. Similarly, for $n = 2$ by simple calculations we again have that

$$\begin{aligned}\max\{a_1,a_2\} - \max\{b_1,b_2\} &= \frac{1}{2}[(a_1+a_2+|a_1-a_2|) - (b_1+b_2+|b_1-b_2|)] \\ &= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) + (|a_1-a_2| - |b_1-b_2|)] \\ &= \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - (|a_1-a_2| - |b_1-b_2|)] \\ &\le \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - ||a_1-a_2| - |b_1-b_2||] \\ &\le \frac{1}{2}[((a_1-b_1)+(a_2-b_2)) - |(a_1-b_1) - (a_2-b_2)|] \\ &= \max\{a_1-b_1, a_2-b_2\}.\end{aligned}$$

Now assume that for $n \in \mathbb{N}$, $n > 2$

$$\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m \le \max_{1\le m\le n-1} (a_m - b_m).$$

From this assumption one can have that

$$\begin{aligned}\max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m &= \max\{\max_{1\le m\le n-1} a_m, a_n\} - \max\{\max_{1\le m\le n-1} b_m, b_n\} \\ &\le \max\{\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m, a_n - b_n\} \\ &\le \max\{\min_{1\le m\le n-1} (a_m - b_m), a_n - b_n\} \\ &= \max_{1\le m\le n} (a_m - b_m).\end{aligned}$$

Consequently, by mathematical induction method it is obtained that,

$$\max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m \le \max_{1\le m\le n} (a_m - b_m)$$

holds for any $n \in \mathbb{N}$. This completes the proof of theorem. □

3. Gap between operator norm and spectral radius for the square of antidiagonal block operator matrices

Remember that the traditional direct sum of Banach spaces $\mathfrak{X}_m, 1 \le m \le n$ in the sense of $\ell_p, 1 \le p < \infty$ and the direct sum of linear densely defined closed operator A_m in $\mathfrak{X}_m, 1 \le m \le n$ are defined as

$$\mathfrak{X} = \left(\bigoplus_{m=1}^{n} \mathfrak{X}_m\right)_p = \left\{x = (x_1,\dots,x_n) : x_m \in \mathfrak{X}_m,\ 1 \le m \le n,\ \|x\|_p = \left(\sum_{m=1}^{n} \|x_m\|_{\mathfrak{X}_m}^p\right)^{1/p} < +\infty\right\}$$

and

$$\begin{aligned}A &= \bigoplus_{m=1}^{n} A_m,\ A : D(A) \subset \mathfrak{X} \to \mathfrak{X}, \\ D(A) &= \{x = (x_1,\dots,x_n) \in \mathfrak{X} : x_m \in D(A_m),\ 1 \le m \le n,\ Ax = (A_1x_1, A_2x_2, \dots, A_nx_n) \in \mathfrak{X}\},\end{aligned}$$

respectively [17].

Definition 3.1. *[16] For any operator $A \in L(\mathscr{X})$ in any Banach space $\mathscr{X}$, $gap(A)$ denotes as fallows*

$$gap(A) = \|A\| - r(A).$$

Theorem 3.2. *For any $1 \le m \le n$ $\mathfrak{X}_m$ be a Banach space and $A_m \in L(\mathfrak{X}_m)$, $\mathfrak{X} = (\bigoplus_{m=1}^{n} \mathfrak{X}_m)_p$ and*

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & A_1 \\ 0 & 0 & \cdots & A_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & 0 & \cdots & 0 & 0 \end{pmatrix} : \mathfrak{X} \to \mathfrak{X}.$$

Then for the operator $A \in L(\mathfrak{X})$

$$\min_{1\le m\le n} gap(A_m A_{n-m+1}) \le gap(A^2) \le \max_{1\le m\le n} gap(A_m A_{n-m+1}).$$

Proof. It is clear that

$$A^2 = \begin{pmatrix} A_1A_n & & & \\ & A_2A_{n-1} & & 0 \\ 0 & & \ddots & \\ & & & A_nA_1 \end{pmatrix} : \mathfrak{X} \to \mathfrak{X}.$$

Then from this we obtain

$$\|A^2\| = \max_{1\le m\le n} \|A_m A_{n-m+1}\| \text{ [18]}$$

and

$$\sigma(A^2) = \bigcup_{m=1}^{n} \sigma(A_m A_{n-m+1}) \text{ [19]}.$$

In this case

$$r(A^2) = \max_{1\le m\le n} r(A_m A_{n-m+1}).$$

Consequently by Theorem 2.1 it implies that

$$\begin{aligned}
\min_{1\le m\le n} gap(A_m A_{n-m+1}) &\le gap(A^2) \\
&= \|A^2\| - r(A^2) \\
&= \max_{1\le m\le n} \|A_m A_{n-m+1}\| - \max_{1\le m\le n} r(A_m A_{n-m+1}) \\
&\le \max_{1\le m\le n} gap(A_m A_{n-m+1}).
\end{aligned}$$

□

Corollary 3.3. *It is known that by spectral mapping theorem $r(A^2) = r^2(A)$ [3]. Then from the inequality*

$$\|A^2\| - r(A^2) \le \|A\|^2 - r^2(A)$$

and last theorem it implies that

$$\frac{\min\limits_{1\le m\le n} gap(A_m A_{n-m+1})}{\|A\| + r(A)} \le gap(A)$$

for $A \ne 0$. From this, additionally for $A \ne 0$ we have

$$\frac{1}{2\|A\|} \min_{1\le m\le n} gap(A_m A_{n-m+1}) \le gap(A).$$

Example 3.4. *Let us*

$$\mathfrak{X}_1 = \mathfrak{X}_2 = L_2(0,1),\ \mathfrak{X} = (\mathfrak{X}_1 \oplus \mathfrak{X}_2)_2,\ A_1, A_2 : L_2(0,1) \to L_2(0,1),$$

$$A_1 f_1 = \alpha_1 V f_1 = \alpha_1 \int_0^t f_1(s)ds,\ f_1 \in L_2(0,1),\ \alpha_1 > 0,$$

$$A_2 f_2 = \alpha_2 V f_2 = \alpha_2 \int_0^t f_2(s)ds,\ f_2 \in L_2(0,1),\ \alpha_2 > 0,$$

$$A : \mathfrak{X} \to \mathfrak{X},\ \begin{pmatrix} 0 & \alpha_1 V \\ \alpha_2 V & 0 \end{pmatrix}.$$

In this case,

$$\|A_1\| = \frac{2\alpha_1}{\pi},\ r(A_1) = 0,\ gap(A_1) = \frac{2\alpha_1}{\pi},$$
$$\|A_2\| = \frac{2\alpha_2}{\pi},\ r(A_2) = 0,\ gap(A_2) = \frac{2\alpha_2}{\pi}$$

and

$$r(A_1A_2) = r(A_2A_1) = \{0\}\ [3].$$

Then by Theorem 3.2

$$\frac{2}{\pi} min\{\alpha_1,\alpha_2\} < gap(A^2) < \frac{2}{\pi} max\{\alpha_1,\alpha_2\},$$

where $A^2 = \begin{pmatrix} \alpha_1\alpha_2V^2 & 0 \\ 0 & \alpha_1\alpha_2V^2 \end{pmatrix}$.

4. Gap between operator norm and numerical radius for the square of antidiagonal block operator matrices

Let us for each $1 \le m \le n < \infty$ H_m be a Hilbert space with inner product $(\cdot,\cdot)_m$ and $H = \bigoplus_{m=1}^{n} H_m, 1 \le m \le n$ be direct sum of Hilbert spaces $H_m, 1 \le m \le n$ with inner product

$$(x,y) = \sum_{m=1}^{n} (x_m,y_m)_m,\ x = (x_1,...,x_m),\ y = (y_1,...,y_m) \in H.$$

In this section it will be investigated gap between operator norm and numerical radius of block antidiagonal operator matrix in the form

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & A_1 \\ 0 & 0 & \cdots & A_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & 0 & \cdots & 0 & 0 \end{pmatrix} : H \to H,$$

in case when $A_m \in L(H_{n-m+1}, H_m),\ 1 \le m \le n$. Then $A \in L(H)$.

Definition 4.1. *[2] The numerical range and numerical radius of an operator T in any Hilbert space $(\mathscr{H},(\cdot,\cdot))$ are*

$$W(T) = \{(Tx,x) : x \in \mathscr{H}, \|x\| = 1\},\ w(T) = sup\{|\lambda| : \lambda \in W(T)\},$$

respectively. And also along of this section the nonnegative number $wgap(T) = \|T\| - w(T)$ will be called the numerical gap of the operator T.

Note that $\frac{1}{2}\|T\| \le w(T) \le \|T\|$ and $w(T) \le \|T\| \le 2w(T)$ for any $T \in L(\mathscr{H})$ (see [1] and [2]).

Theorem 4.2. *For the block antidiagonal operator matrix A in Hilbert space H*

$$\min_{1\le m\le n} wgap(A_mA_{n-m+1}) \le wgap(A^2) \le \max_{1\le m\le n} wgap(A_mA_{n-m+1})$$

are true.

Proof. In this case the simple calculations shown that

$$A^2 = \begin{pmatrix} A_1A_n & & & \\ & A_2A_{n-1} & & 0 \\ & 0 & \ddots & \\ & & & A_nA_1 \end{pmatrix} : H \to H.$$

In this case from [20] it is clear that

$$\|A^2\| - w(A^2) = \max_{1\le m\le n} \|A_mA_{n-m+1}\| - \max_{1\le m\le n} w(A_mA_{n-m+1}).$$

□

From this relation and proof of Theorem 2.1 it is clear that

$$\min_{1\leq m\leq n} wgap(A_m A_{n-m+1}) \leq wgap(A^2) \leq \max_{1\leq m\leq n} wgap(A_m A_{n-m+1}).$$

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Author's contributions

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