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Stability analysis and periodictly properties of a class of rational difference equations

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ABSTRACT

The goal of this study is to investigate the global, local, and boundedness of the recursive sequence

$$T_{\eta+1} = r + \frac{p_1 T_{\eta-l_1}}{T_{\eta-m_1}} + \frac{q_1 T_{\eta-m_1}}{T_{\eta-l_1}} + \frac{p_2 T_{\eta-l_2}}{T_{\eta-m2}} + \frac{q_2 T_{\eta-m_2}}{T_{\eta-l_2}} + \ldots + \frac{p_s T_{\eta-l_s}}{T_{\eta-m_s}} + \frac{q_s T_{\eta-m_s}}{T_{\eta-l_s}},$$

where the initial values $T_{-l_1,}, T_{-l_{12}}, \ldots T_{-l_s,}$, T_{-m_1}, $T_{-m_2}$and T_{-m_s} are arbitrary positive real numbers. It also investigates periodic solutions for special case of above equations.

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1. Introduction

The main proposal of this paper gets the behavior of the solutions such as local Stability, global stability and boundedness character of the following difference equation

$$T_{\eta+1} = r + \frac{p_1 T_{\eta-l_1}}{T_{\eta-m_1}} + \frac{q_1 T_{\eta-m_1}}{T_{\eta-l_1}} + \frac{p_2 T_{\eta-l_2}}{T_{\eta-m2}} + \frac{q_2 T_{\eta-m_2}}{T_{\eta-l_2}} + \ldots + \frac{p_s T_{\eta-l_s}}{T_{\eta-m_s}} + \frac{q_s T_{\eta-m_s}}{T_{\eta-l_s}}, \quad \eta \geq 0, \tag{1}$$

where $l_1, l_2, \ldots, l_s, m_1, m_2, \ldots, m_s, s$, are positive constants and the initial values $T_{-l_1,}, T_{-l_{12}}, \ldots T_{-l_s,}$, T_{-m_1}, T_{-m_2} and T_{-m_s} are arbitrary positive real numbers. In adition, numerical results are provided to confirm theorems. Let $L = \max\{l_1, l_2, \ldots, l_s, m_1, m_2, \ldots, m_s\}$.

Let us introduce some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$g : I^{k+1} \longrightarrow I,$$

be a continuously differentiable function. Then for every set of initial values $T_{-k}, T_{-k+1}, \ldots, T_0 \in I$, the difference equation

$$T_{\eta+1} = g(T_\eta, T_{\eta-1}, \ldots, T_{\eta-k}), \quad \eta = 0, 1, \ldots, \tag{2}$$

has a unique solution $\{T_\eta\}_{\eta=-k}^{\infty}$ [13].

A point $\overline{T} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{T} = g(\overline{T}, \overline{T}, \ldots, \overline{T}).$$

That is, $T_\eta = \overline{T}$ for $\eta \geq 0$, is a solution of Eq. (2), or equivalently, $\overline{T}$ is a fixed point of g.

(Stability)

(i) The equilibrium point $\overline{T}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $T_{-k}, T_{-k+1}, \ldots, T_{-1}, T_0 \in I$ with

$$\left|T_{-k} - \overline{T}\right| + \left|T_{-k+1} - \overline{T}\right| + \ldots + \left|T_0 - \overline{T}\right| < \delta,$$

we have

$$\left|T_\eta - \overline{T}\right| < \epsilon \quad \text{for all} \quad \eta \geq -k.$$

(ii) The equilibrium point $\overline{T}$ of Eq.(2) is locally asymptotically stable if $\overline{T}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $T_{-k}, T_{-k+1}, ..., T_{-1}, T_0 \in I$ with

$$\left|T_{-k} - \overline{T}\right| + \left|T_{-k+1} - \overline{T}\right| + ... + \left|T_0 - \overline{T}\right| < \gamma,$$

we have

$$\lim_{\eta \to \infty} T_\eta = \overline{T}.$$

(iii) The equilibrium point $\overline{T}$ of Eq.(2) is global attractor if for all $T_{-k}, T_{-k+1}, ..., T_{-1}, T_0 \in I$, we have

$$\lim_{\eta \to \infty} T_\eta = \overline{T}.$$

(iv) The equilibrium point $\overline{T}$ of Eq.(2) is globally asymptotically stable if $\overline{T}$ is locally stable, and $\overline{T}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{T}$ of Eq.(2) is unstable if $\overline{T}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{T}$ is the linear difference equation

$$y_{\eta+1} = \sum_{i=0}^{k} \frac{\partial g(\overline{T}, \overline{T}, ..., \overline{T})}{\partial T_{\eta-i}} y_{\eta-i}. \qquad (3)$$

Theorem A [12]: Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$T_{\eta+1} + pT_\eta + qT_{\eta-k} = 0, \quad \eta = 0, 1, ...$$

Theorem A can be easily extended to a general linear equations of the form

$$T_{\eta+k} + p_1 T_{\eta+k-1} + ... + p_k T_\eta = 0, \quad \eta = 0, 1, ..., \qquad (3)$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Then Eq.(4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Consider the following equation

$$T_{\eta+1} = h(T_\eta, T_{\eta-1}, T_{\eta-2}). \qquad (4)$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [13]: Let $[a, b]$ be an interval of real numbers and assume that

$$h : [a, b]^3 \to [a, b],$$

is a continuous function satisfying the following properties :

(a) $h(x, y, z)$ is non-decreasing in x and z in $[a, b]$ for each $y \in [a, b]$, and is non-increasing in $y \in [a, b]$ for each x and z in $[a, b]$;

(b) If $(n, N) \in [a, b] \times [a, b]$ is a solution of the system

$$N = h(N, n, N) \quad a\eta d \quad n = h(n, N, n),$$

then

$$n = N.$$

Then Eq.(4) has a unique equilibrium $\overline{T} \in [a, b]$ and every solution of Eq.(4) converges to $\overline{T}$.

The increasing worldwide attention paid to the study of many characteristics, of behaviors and dynamics of difference equations, such as stability, periodicity, boundedness character, is not a coincidence. The applications of distinction equations have recently been the basice of numerous sciences and that is the cause why the principle of difference equations stays the important thing participant not only in mathematics however also in different sciences that employ its implementations. Many mathematicians find the research on difference equations interesting and fruitful because it supports the analysis and modeling of various phenomena in everyday life [15]. For example, Elsayed [15] discovered a new technique to get second and third periodic solution of the recursive sequence that is given by

$$T_{\eta+1} = a + \frac{bT_\eta}{T_{\eta-1}} + \frac{bT_{\eta-1}}{T_\eta},$$

Chatzarakis et al. in [5] focused on study periodic and boundedness, local and global stability of a class of nonlinear difference equations given by

$$T_{\eta+1} = a + \frac{bT_\eta^2}{(T_\eta + d)\, T_{\eta-1}},$$

The dynmical analysis of the following difference equations

$$T_{\eta+1} = a_\eta + \frac{T_\eta^p}{T_{\eta-1}^p},$$

is examined by Khan and El-Metwally [18].

The global attractivity and local stability of the difference equation

$$T_{\eta+1} = \frac{T_{\eta-1}}{c + dT_{\eta-1}T_{\eta-2}},$$

have investigated by Yang et al. [24].

Khaliq et al. [17] studies the dynamical behavior of solutions of the seventh order difference equation

$$T_{\eta+1} = aT_{\eta-3} + \frac{\alpha T_{\eta-3}T_{\eta-7}}{\beta T_{\eta-3} + \gamma T_{\eta-7}}.$$

Cinar [6] has figuer out how to obtain solution of the difference problem

$$T_{\eta+1} = \frac{aT_{\eta-1}}{1 + bT_{\eta}T_{\eta-1}}.$$

Alayachi et al. [7] studied qualitative behavior and boundedness of the difference equation

$$T_{\eta+1} = aT_{\eta-1} + \frac{\alpha T_{\eta-1}T_{\eta-3}}{\beta T_{\eta-3} + \gamma T_{\eta-5}}.$$

Another associated papers on rational difference equations see [1-25].

2. Behavior of the Solutions of Eq. (1)

In this secion we investigated the behavior of the solution of Eq. (1).

2.1. Local Stability

In this subsection we investigate the local stability character of the solutions of Eq.(1).

Theorem 1 Assume that $2\left[|p_1 - q_1| + |p_2 - q_2| + ... + |p_s - q_s|\right] < r + p_1 + q_1 + p_2 + q_2 + ... + p_s + q_s$,then the equiliribum point $\bar{T} = r + p_1 + q_1 + p_2 + q_2 + ... + p_s + q_s$, of Eq.(1) is Locally asymptotically stable.

proof: The equilibrium point of Eq. (1) is given by

$$\bar{T} = r + p_1 + q_1 + p_2 + q_2 + ... + p_s + q_s. \qquad (5)$$

Define a function $g : (0, \infty) \to (0, \infty)$ as

$$g(x_1, y_1, x_2, y_2, ..., x_s, y_s) = r + \frac{p_1 x_1}{y_1} + \frac{q_1 y_1}{x_1} + \frac{p_2 x_2}{y_2} + \frac{q_2 y_2}{x_2} + ... + \frac{p_s x_s}{y_s} + \frac{q_s y_s}{x_s}.$$

Hence we obtain,

$$\begin{aligned}
\frac{\partial g}{\partial x_1}(x_1, y_1, x_2, y_2, ..., x_s, y_s) &= \frac{p_1}{y_1} - \frac{q_1 y_1}{x_1^2},\\
\frac{\partial g}{\partial y_1}(x_1, y_1, x_2, y_2, ..., x_s, y_s) &= -\frac{p_1 x_1}{y_1^2} + \frac{q_1}{x_1},\\
\frac{\partial g}{\partial x_2}(x_1, y_1, x_2, y_2, ..., x_s, y_s) &= \frac{p_2}{y_2} - \frac{q_2 y_2}{x_2^2},\\
\frac{\partial g}{\partial y_2}(x_1, y_1, x_2, y_2, ..., x_s, y_s) &= -\frac{p_2 x_2}{y_2^2} + \frac{q_2}{x_2}, ...,\\
&, ...,\\
\frac{\partial g}{\partial x_s}(x_1, y_1, x_2, y_2, ..., x_s, y_s) &= \frac{p_s}{y_s} - \frac{q_s y_s}{x_s^2},\\
\frac{\partial g}{\partial y_s}(x_1, y_1, x_2, y_2, ..., x_s, y_s) &= -\frac{p_s x_s}{y_s^2} + \frac{q_s}{x_s}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial g}{\partial x_1}(\bar{T}, \bar{T}, \bar{T}, \bar{T}, ..., \bar{T}, \bar{T}) &= \frac{p_1}{\bar{T}} - \frac{q_1}{\bar{T}} = -a_1,\\
\frac{\partial g}{\partial y_1}(\bar{T}, \bar{T}, \bar{T}, \bar{T}, ..., \bar{T}, \bar{T}) &= -\frac{p_1}{\bar{T}} + \frac{q_1}{\bar{T}} = -b_1,\\
\frac{\partial g}{\partial x_2}(\bar{T}, \bar{T}, \bar{T}, \bar{T}, ..., \bar{T}, \bar{T}) &= \frac{p_2}{\bar{T}} - \frac{q_2}{\bar{T}} = a_2,\\
\frac{\partial g}{\partial y_2}(\bar{T}, \bar{T}, \bar{T}, \bar{T}, ..., \bar{T}, \bar{T}) &= -\frac{p_2}{\bar{T}} + \frac{q_2}{\bar{T}} = -b_2, ...,\\
&, ...,\\
\frac{\partial g}{\partial x_s}(\bar{T}, \bar{T}, \bar{T}, \bar{T}, ..., \bar{T}, \bar{T}) &= \frac{p_s}{\bar{T}} - \frac{q_s}{\bar{T}} = -a_s,\\
\frac{\partial g}{\partial y_s}(\bar{T}, \bar{T}, \bar{T}, \bar{T}, ..., \bar{T}, \bar{T}) &= -\frac{p_s}{\bar{T}} + \frac{q_s}{\bar{T}} = -b_s.
\end{aligned}$$

Therefore, the linearized equation becomes

$$S_{\eta+1} = a_1 S_{\eta-l_1} + b_1 S_{\eta-m_1} + a_2 S_{\eta-l_2} + b_2 S_{\eta-m_2} + ... + a_s S_{\eta-l_s} + b_s S_{\eta-l_s},$$

using Theorem A, we get that the equiliribum point is asympototically stable if

$$|a_1| + |a_1| + ... |a_1| + |b_1| + |b_1| + ... + |b_1| < 1,$$

and hence

$$2\left[|p_1 - q_1| + |p_2 - q_2| + ... + |p_s - q_s|\right] < r + p_1 + q_1 + p_2 + q_2 + ... + p_s + q_s,$$

which means the prove is complete.

2.2. Global Attractor

In this subsection we investigate the global attractivity character of solutions of Eq. (1).

Theorem 2 The equiliribum point of Eq. (1) is global Attractor if $\gamma(1-\alpha) \neq \beta$.

proof: Let a, b are real number and define $f : [a,b]^{2s} \to [a,b]$ a function $f(x_1, y_1, x_2, y_2, ..., x_s, y_s) = r + \frac{p_1 x_1}{y_1} + \frac{q_1 y_1}{x_1} + \frac{p_2 x_2}{y_2} + \frac{q_2 y_2}{x_2} + ... + \frac{p_s x_s}{y_s} + \frac{q_s y_s}{x_s}$. Since $p_1 x_1^2 - q_1 y_1^2 \geq 0$, $p_2 x_2^2 - q_2 y_2^2 \geq 0, ..., p_s x_s^2 - q_s y_s^2 \geq 0$, for $x_1, y_1, x_2, y_2, ...x_s, y_s \geq 0$, the function f is increasing in $x_1, x_2, ..., x_s$ and decreasing in $y_1, y_2, ..., y_s$, hence

$N = f(N, n, N, n, ..., N, n)$ and $n = f(n, N, n, N, ..., n, N)$.

Hence we get

$$N = r + \frac{p_1 N}{n} + \frac{q_1 n}{N} + \frac{p_2 N}{n} + \frac{q_2 n}{N} + ... + \frac{p_s N}{n} + \frac{q_s n}{N},$$
$$n = r + \frac{p_1 n}{N} + \frac{q_1 N}{n} + \frac{p_2 n}{N} + \frac{q_2 N}{n} + ... + \frac{p_s n}{N} + \frac{q_s N}{n},$$

or

$$N^2 n = rNn + p_1 N^2 + q_1 n^2 + p_2 N^2 + q_2 n^2 + ... + p_s N^2 + q_s n^2,$$
$$N n^2 = rNn + p_1 n^2 + q_1 N^2 + p_2 n^2 + q_2 N^2 + ... + p_s n^2 + q_s N^2,$$

subtracting these two equations, we get

$$\begin{aligned}(N-n)Nn &= p_1\left(N^2-n^2\right) + q_1\left(n^2-N^2\right) \\ &+ p_2\left(N^2-n^2\right) + q_2\left(n^2-N^2\right) \\ &+ ... + p_s\left(N^2-n^2\right) + q_s\left(n^2-N^2\right) \\ 0 &= (N-n)\left[Nn + (N+n)\right. \\ &\left.(q_1+q_2+...+q_s-p_1-p_2-...-p_s)\right]\end{aligned}$$

Under the conditions $q_1+q_2+...+q_s \geq p_1+p_2+...+p_s$, we obtain

$$N = n,$$

we obtain by therom (B) that he equiliribum point $\bar{T}$ of Eq.(1) is global Attractor.

2.3. Boundness of solutions

In this subsection we study the boundedness of solutions of Eq. (1).

Theorem 3 Every solution of Eq. (1) is boumded and prsists if $r > p_1 + q_1 + p_2 + q_2 + ... + p_s + q_s$.

proof: Sppose $\{T_\eta\}_{-L}^{\infty}$ be solution of Eq. (1). It follows from Eq. (1) that

$$\begin{aligned}T_{\eta+1} &= r + \frac{p_1 T_{\eta-l_1}}{T_{\eta-m_1}} + \frac{q_1 T_{\eta-m_1}}{T_{\eta-l_1}} + \frac{p_2 T_{\eta-l_2}}{T_{\eta-m2}} + \frac{q_2 T_{\eta-m_2}}{T_{\eta-l_2}} \\ &+ ... + \frac{p_s T_{\eta-l_s}}{T_{\eta-m_s}} + \frac{q_s T_{\eta-m_s}}{T_{\eta-l_s}} > r,\end{aligned}$$

thus

$$T_{\eta+1} > r, \qquad \text{for} \quad \eta \geq 0.$$

Also, it follows from Eq. (1) that

$$\begin{aligned}T_{\eta+1} &\leq r + \frac{p_1 T_{\eta-l_1}}{r} + \frac{q_1 T_{\eta-m_1}}{r} + \frac{p_2 T_{\eta-l_2}}{r} + \frac{q_2 T_{\eta-m_2}}{r} \\ &+ ... + \frac{p_s T_{\eta-l_s}}{r} + \frac{q_s T_{\eta-m_s}}{r},\end{aligned}$$

using Comparisons Theroms, we get

$$\lim_{\eta\to\infty} subT_\eta \leq \frac{r^2}{(r - p_1 - p_2 ... - p_s - q_1 - q_2 ... - q_s)}.$$

Therefore $\{T_\eta\}_{-L}^{\infty}$ is bounded and persists.

3. Periodic two solution of Eq. (1):

In this section, we investigate the periodic two solutions of special cases of Eq. (1). We states theorem that gives us necessary and sufficient conditions of the following equation

$$\begin{aligned}T_{\eta+1} &= r + \frac{p_1 T_\eta}{T_{\eta-1}} + \frac{q_1 T_{\eta-1}}{T_\eta} + \frac{p_2 T_{\eta-2}}{T_{\eta-3}} + \frac{q_2 T_{\eta-3}}{T_{\eta-2}} \\ &+ ... + \frac{p_s T_{\eta-2l}}{T_{\eta-(2l+1)}} + \frac{q_s T_{\eta-(2l+1)}}{T_{\eta-2l}}, \qquad \eta = 0, 1, ...,\end{aligned} \tag{6}$$

where $T_{\eta-2l} = ... = T_{\eta-2} = T_\eta = u$, and $T_{\eta-(2l+1)} = ... = T_{\eta-3} = T_{\eta-1} = v$, has a prime period solution of periodic two.

Theorem 4

Assume that $p_1 + p_2 + ... + p_s \neq q_1 + q_2 + ... + q_s$ and $c \in R/\{0, \pm 1\}$, then Eq. (6) has a periodic solution of prime periodic two if and only if $r = q_1 + q_2 + ... + q_s - (p_1 + p_2 + ... + p_s)\left(\frac{c^2+c+1}{c}\right)$, where $c = \frac{u}{v}$ such that $u, v, u, v, ...$ is a periodic solution of Eq. (6).

Proof: From Eq. (6), we obtain

$$u = r + \frac{p_1 v}{u} + \frac{q_1 u}{v} + \frac{p_2 v}{u} + \frac{q_2 u}{v} + ... + \frac{p_s v}{u} + \frac{q_s u}{v}, \text{ and}$$
$$v = r + \frac{p_1 u}{v} + \frac{q_1 v}{u} + \frac{p_2 u}{v} + \frac{q_2 v}{u} + ... + \frac{p_s u}{v} + \frac{q_s v}{u}.$$

Since $c = \frac{u}{v} \neq 0, \pm 1$. Then, it follows

$$u = r + \frac{p_1}{c} + q_1 c + \frac{p_2}{c} + q_2 + ... + \frac{p_s}{c} + q_s c, \tag{7}$$

and

$$v = r + p_1 c + \frac{q_1}{c} + p_2 c + \frac{q_2}{c} + ... + p_s c + \frac{q_s}{c},$$

or

$$vc = rc + p_1 c^2 + q_1 + p_2 c^2 + q_2 + ... + p_s c^2 + q_s, \quad (8)$$

subtracting Eq. (8) from Eq. (7) gives the following equation

$$u - vc = r(c-1) + (p_1 + p_2 + ... + p_s)\left(c^2 - \frac{1}{c}\right) + (q_1 + q_2 + ... + q_s)(1-c),$$

hence

$$r(c-1) + (p_1 + p_2 + ... + p_s)\left(\frac{c^3 - 1}{c}\right) + (q_1 + q_2 + ... + q_s)(1-c) = 0.$$

Since $c \neq 0$, we conclude

$$r = q_1 + q_2 + ... + q_s - (p_1 + p_2 + ... + p_s)\left(\frac{c^2 + c + 1}{c}\right), \quad (9)$$

which is the condition of this theorem holds.
Furthermore, we rewrite Eqs. (8) and Eq. (7) as follows

$$\begin{aligned} u &= q_1 + q_2 + ... + q_s - (p_1 + p_2 + ... + p_s)\left(\frac{c^2 + c + 1}{c}\right) \\ &+ \frac{p_1}{c} + q_1 c + \frac{p_2}{c} + q_2 + ... + \frac{p_s}{c} + q_s c, \\ &= (q_1 + q_2 + ... + q_s)(c+1) \\ &- (p_1 + p_2 + ... + p_s)(c+1) \\ &= [(q_1 + q_2 + ... + q_s) - (p_1 + p_2 + ... + p_s)](c+1), \end{aligned} \quad (10)$$

and

$$\begin{aligned} v &= \left(q_1 + q_2 + ... + q_s - (p_1 + p_2 + ... + p_s)\left(\frac{c^2 + c + 1}{c}\right)\right) \\ &+ p_1 c + \frac{q_1}{c} + p_2 c + \frac{q_2}{c} + ... + p_s c + \frac{q_s}{c} \\ &= \left((q_1 + q_2 + ... + q_s)(1 + \frac{1}{c}) \right. \quad (11) \\ &- (p_1 + p_2 + ... + p_s)\left(\frac{c+1}{c}\right) \\ &= [(q_1 + q_2 + ... + q_s) - (p_1 + p_2 + ... + p_s)]\left(\frac{c+1}{c}\right), \end{aligned} \quad (12)$$

therefore, u, v distinct real numbers. Let $T_{\eta-2l} = ... = T_{\eta-2} = T_\eta = u$, and $T_{\eta-(2l+1)} = ... = T_{\eta-3} = T_{\eta-1} = v$. Acooording Eq. (6), we staste

$$T_1 = u, T_2 = v.$$

$$\begin{aligned} T_1 &= r + \frac{p_1 u}{v} + \frac{q_1 v}{u} + \frac{p_2 u}{v} + \frac{q_2 v}{u} + ... + \frac{p_s u}{v} + \frac{q_s v}{u} \\ &= q_1 + q_2 + ... + q_s - (p_1 + p_2 + ... + p_s)\left(\frac{c^2 + c + 1}{c}\right) \\ &+ \frac{(p_1 + p_2 + ... + p_s)}{c} + c(q_1 + q_2 + ... + q_s) \\ &= (c+1)[(q_1 + q_2 + ... + q_s) - (p_1 + p_2 + ... + p_s)] \\ &= u, \end{aligned}$$

$$\begin{aligned} T_2 &= r + \frac{p_1 v}{u} + \frac{q_1 u}{v} + \frac{p_2 v}{u} + \frac{q_2 u}{v} + ... + \frac{p_s v}{u} + \frac{q_s u}{v} \\ &= q_1 + q_2 + ... + q_s - (p_1 + p_2 + ... + p_s)\left(\frac{c^2 + c + 1}{c}\right) \\ &+ c(p_1 + p_2 + ... + p_s) + \frac{(q_1 + q_2 + ... + q_s)}{c} \\ &= \frac{(c+1)}{c}[(q_1 + q_2 + ... + q_s) - (p_1 + p_2 + ... + p_s)] \\ &= v. \end{aligned}$$

Hence simmlar T_1, T_2, we get $T_{2\eta+1} = u, T_{2\eta} = v$, for $\eta \geq 0$, therefore the proof is completed.

4. Numerical results:

Example 1 For confirming the results of subsection (2.1), we consider difference equation

$$T_{\eta+1} = 2 + \frac{T_{\eta-2}}{T_{\eta-3}} + \frac{4T_{\eta-3}}{T_{\eta-2}} + \frac{2T_{\eta-1}}{T_\eta} + \frac{5T_\eta}{T_\eta}, \quad (13)$$

with the initial conditions $T_{-3} = 14.5$, $T_{-2} = 13.5, T_{-1} = 14.5$ and $T_0 = 13.5$, where the equilibrium point is $\bar{T} = 14$. (See Fig. 1).

Example 2 For confirming the results of subsection (2.1), we consider difference equation

$$T_{\eta+1} = 2 + \frac{T_{\eta-2}}{T_{\eta-3}} + \frac{4T_{\eta-3}}{T_{\eta-2}}, \quad (14)$$

with the initial conditions $T_{-3} = 7.5$, $T_{-2} = 6.5, T_{-1} = 7.4$ and $T_0 = 6.5$, where the equilibrium point is $\bar{T} = 7$. (See Fig. 2).

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Example 3 For confirming the results of subsection (2.2), we consider numerical example for Eq. (13) with the initial conditions
IC1: $T_{-3} = 14, T_{-2} = 13, T_{-1} = 14, T_0 = 13,$
IC2: $T_{-3} = 13, T_{-2} = 12, T_{-1} = 11, T_0 = 10,$
IC3: $T_{-3} = 12, T_{-2} = 11, T_{-1} = 10, T_0 = 9,$
IC4: $T_{-3} = 11, T_{-2} = 10, T_{-1} = 9, T_0 = 8.$
(See Fig. 3).

Example 4 For confirming the results of subsection (2.2), we consider numerical example for Eq. (14) with the initial conditions IC1-IC4. (See Fig. 4).

Example 5 For confirming the results of section (3), we consider difference equation

$$T_{\eta+1} = r + \frac{p_1 T_\eta}{T_{\eta-1}} + \frac{q_1 T_{\eta-1}}{T_\eta} + \frac{p_2 T_{\eta-2}}{T_{\eta-3}} + \frac{q_2 T_{\eta-3}}{T_{\eta-2}} + \frac{p_s T_{\eta-4}}{T_{\eta-5}} + \frac{q_s T_{\eta-5}}{T_{\eta-4}}, \quad (15)$$

where $p_1 = 2, q_1 = 6, p_2 = 3, q_2 = 7, p_3 = 4, q_3 = 8, c = 3$, with the initial condition $T_{-5} = 48,\ T_{-4} = 16, T_{-3} = 48,\ T_{-2} = 16, T_{-1} = 48$ and $T_0 = 16$. (See Fig. 5).

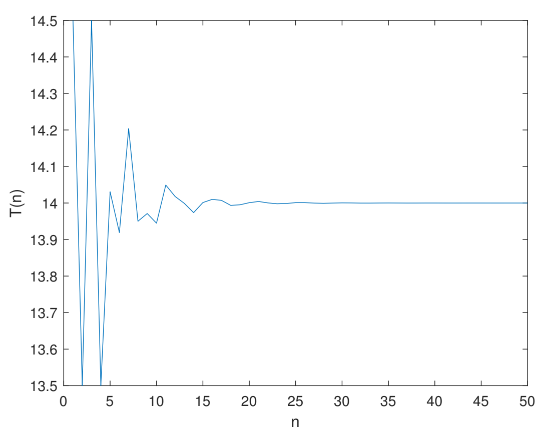

Figure 1: The figure shows the local stability of $\bar{T} = 14$ in Eq. (13).

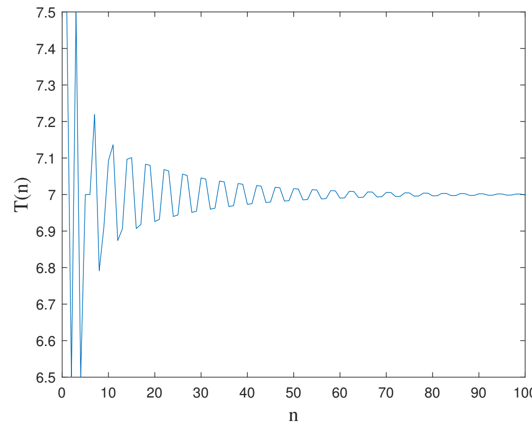

Figure 2: The figure shows the local stability of $\bar{T} = 7$ in Eq. (14).

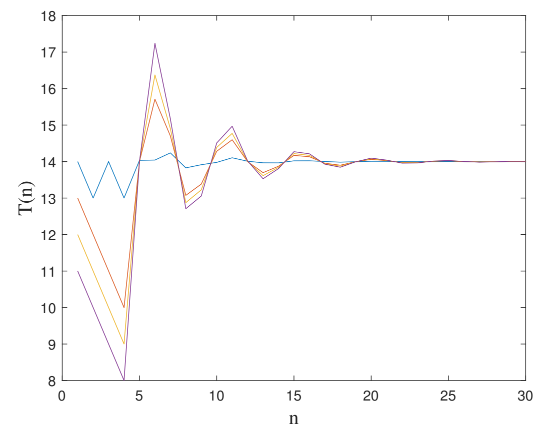

Figure 3: The figure shows the global stability of $\bar{T} = 7$ in Eq. (14).

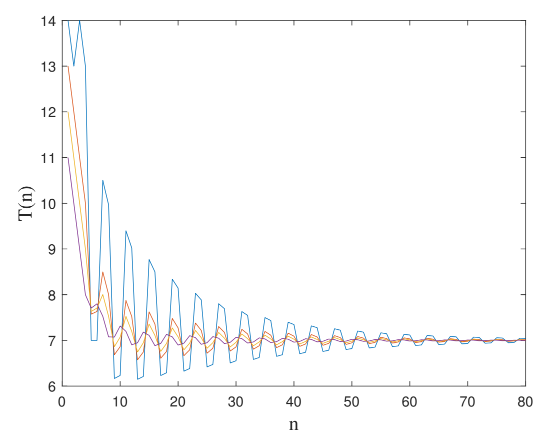

Figure 4: The figure shows the global stability of $\bar{T} = 14$ in Eq. (13).

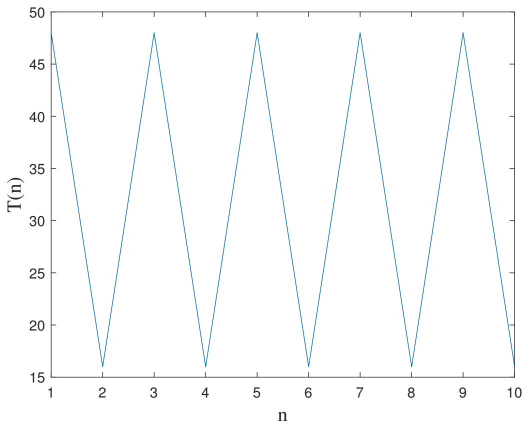

Figure 5: The figure shows Eq. (15) has period two solutions where $p_1, q_1, p_2, q_2, p_3, q_3$ and initial condition satisfies the condition of Theorem 4.

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