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Research Article

# Actions and semi-direct products in categories of groups with action

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## Abstract

Derived actions in the category of groups with action on itself $\mathbf{Gr}^\bullet$ are defined and described. This category plays a crucial role in the solution of two problems of Loday stated in the literature. A full subcategory of reduced groups with action $\mathbf{rGr}^\bullet$ of $\mathbf{Gr}^\bullet$ is introduced, which is not a category of interest but has some properties, which can be applied in the investigation of action representability in this category; these properties are similar to those, which were used in the construction of universal strict general actors in the category of interest. Semi-direct product constructions are given in $\mathbf{Gr}^\bullet$ and $\mathbf{rGr}^\bullet$ and it is proved that an action is a derived action in $\mathbf{Gr}^\bullet$ (resp. $\mathbf{rGr}^\bullet$) if and only if the corresponding semi-direct product is an object of $\mathbf{Gr}^\bullet$ (resp. $\mathbf{rGr}^\bullet$). The results obtained in this paper will be applied in the forthcoming paper on the representability of actions in the category $\mathbf{rGr}^\bullet$.

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**Keywords.** Group with action, semi-direct product, action, extension

## 1. Introduction

Action theories were developed in many algebraic categories like the categories of groups, associative algebras, (pre)crossed modules, non-associative algebras, in particular Lie, Leibniz, alternative algebras and others and, in more general settings of semi-abelian category [1] and category of interest [12, 13]. There were two different approaches to the definition of action, which turned out to be equivalent. In this paper we give a new example of a category, where action theory can be developed. It is a category of groups with action on itself introduced in [5–7], where it played a main role in the solution of two problems of Loday stated in [10, 11]. This category is neither a category of interest, nor a modified category of interest [2]. It is a category of groups with operations, but doesn't satisfy all conditions stated in [14]. The category $\mathbf{Gr}^\bullet$ is a category of $\Omega$-groups in the sense of Kurosh [9]. Actions are defined in $\mathbf{Gr}^\bullet$ as derived actions from split extensions in this category as it is in the category of interest or in any semi-abelian category. We describe derived action conditions in this category and construct a semi-direct product

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$B \ltimes A$, where $A, B \in \mathbf{Gr}^{\bullet}$ and $B$ has a derived action on $A$. We prove that an action of $B$ on $A$ is a derived action if and only if $B \ltimes A \in \mathbf{Gr}^{\bullet}$ (Theorem 3.2). Then we define a full subcategory $\mathbf{rGr}^{\bullet}$ in $\mathbf{Gr}^{\bullet}$, give examples of its objects including a construction of free objects and describe derived actions in $\mathbf{rGr}^{\bullet}$. Our interest is to investigate the existence of a universal acting object on an object $A \in \mathbf{Gr}^{\bullet}$ applying the results obtained in [3, 4] for categories of interest. Since the category $\mathbf{Gr}^{\bullet}$ is far from being category of interest we found its subcategory $\mathbf{rGr}^{\bullet}$, which is not a category of interest, but has interesting properties which are close to those ones used in the construction of a universal strict general actor for any object of a category of interest in [3, 4]. We prove necessary and sufficient condition for the action of $B$ on $A$, $A, B \in \mathbf{rGr}^{\bullet}$, to be a derived action in terms of the semi-direct product $B \ltimes A$, like we have in $\mathbf{Gr}^{\bullet}$ (Theorem 4.4). Applying the results of this paper, in [8], we prove that under certain conditions on the object $A \in \mathbf{rGr}^{\bullet}$, it has representable actions in the sense of [1], i.e. a universal acting object, which represents all actions on $A$.

## 2. Preliminary definitions and results

Let $G$ be a group which acts on itself from the right side, i.e. we have a map $\varepsilon\colon G \times G \to G$ with

$$\begin{aligned}\varepsilon(g, g' + g'') &= \varepsilon(\varepsilon(g, g'), g'')\\ \varepsilon(g, 0) &= g\\ \varepsilon(g' + g'', g) &= \varepsilon(g' + g) + \varepsilon(g'' + g)\end{aligned}$$

for $g, g', g'' \in G$. Denote $\varepsilon(g, h) = g^h$, for $g, h \in G$.

We denote the group operation additively, nevertheless the group is not commutative in general. From the third condition on $\varepsilon$ it follows that

$$0^h = 0, \text{ for any } h \in G.$$

If $(G', \varepsilon')$ is another group with action then a homomorphism $(G, \varepsilon) \to (G', \varepsilon')$ is a group homomorphism $\varphi\colon G \to G'$, for which the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\varepsilon} & G \\ {\scriptstyle \varphi\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G' \times G' & \xrightarrow[\varepsilon']{} & G' \end{array}$$

commutes. In other words, we have

$$\varphi\left(g^h\right) = \varphi(g)^{\varphi(h)} \tag{2.1}$$

for all $g, h \in G$.

Note that action defined above is a split derived action in the sense of [12, 13].

According to Kurosh [9] an $\Omega$-group is a group with a system of $n$-ary algebraic operations $\Omega_{n \geq 0}$, which satisfy the condition

$$000 \cdots 0\omega = 0, \tag{2.2}$$

where 0 is the identity element of $G$, and 0 on the left side occurs $n$ times if $\omega$ is an $n$-ary operation. In special cases $\Omega$-groups give groups, rings, associative and non-associative algebras like Lie and Leibniz algebras etc. and groups with action on itself as well. In the latter case $\Omega$ consists of one binary operation which is an action or $\Omega$ consists of only unary operations, which are elements of $G$, and this operation is an action again. In both cases condition (2.2) is satisfied. Denote the category of groups with action on itself by $\mathbf{Gr}^{\bullet}$; here the action is considered as a binary operation and morphisms between the objects in $\mathbf{Gr}^{\bullet}$ are group homomorphisms satisfying condition (2.1).

**Example 2.1.** [5] Every group with trivial action on itself or with an action by conjugation is an object of $\mathbf{Gr}^\bullet$. There are two pairs of adjoint functors between the category of groups and the category $\mathbf{Gr}^\bullet$ [5].

**Example 2.2.** [6] For any set $X$ there exists a free group with action $F(X)$ with the basis $X$ in $\mathbf{Gr}^\bullet$; one can see the construction in [6].

**Example 2.3.** Let $\mathbb{Z}^\bullet$ be an abelian group of integers which acts on itself in the following way;

$$x^y = (-1)^y x$$

for any $x, y \in \mathbb{Z}$. It is easy to check that $\mathbb{Z}^\bullet \in \mathbf{Gr}^\bullet$.

Let $G \in \mathbf{Gr}^\bullet$.

**Definition 2.4.** [5] A non-empty subset $A$ of $G$ is called an ideal of $G$ if it satisfies the following conditions

**1)** $A$ is a normal subgroup of $G$ as a group;
**2)** $a^g \in A$, for any $a \in A$ and $g \in G$;
**3)** $-g + g^a \in A$, for any $a \in A$ and $g \in G$.

Note that the condition 3 in this definition is equivalent to the condition, that $g^a - g \in A$, since $(-g)^a = -g^a$, for any $a \in A$ and $g \in G$. This definition is equivalent to the definition of an ideal given in [9] for $\Omega$-groups in the case where $\Omega$ consists of one binary operation of action, one can see the proof in [5].

## 3. Actions and semi-direct products in $\mathbf{Gr}^\bullet$

Let $A, B \in \mathbf{Gr}^\bullet$. An action of $B$ on $A$ by definition is a triple of mappings $\beta = (\beta_+, \beta_*, \beta_{*^\circ})\colon B \times A \to A$, where $*$ is a binary operation of action, $*^\circ$ is its dual operation in $\mathbf{Gr}^\bullet$, i.e. $\beta_+(b,a) = b \cdot a$, $\beta_*(b,a) = a * b = a^b$ and $\beta_{*^\circ}(b,a) = a *^\circ b = b^a$.

In the category of interest or category of groups with operations there is a condition $0 * g = g * 0 = 0$, for any binary operation $* \in \Omega \backslash \{+\}$, any object $G$ in this category and any element $g \in G$. In the category $\mathbf{Gr}^\bullet$ we have $0^g = 0$, for any $G \in \mathbf{Gr}^\bullet$ and any $g \in G$, but $g^0 \neq 0$ in general. Therefore we modify the definition of derived action due to split extensions [12–14], known for the category of groups with operations or category of interest, for the category $\mathbf{Gr}^\bullet$. Note that the definition of derived action from the split extension agrees with the definition of action in a semi-abelian category [1].

Let $A, B \in \mathbf{Gr}^\bullet$. An extension of $B$ by $A$ is a sequence

$$0 \longrightarrow A \xrightarrow{\;i\;} E \xrightarrow{\;p\;} B \longrightarrow 0 \tag{3.1}$$

in which $p$ is surjective and $i$ is the kernel of $p$. We say that an extension is split if there is a morphism $j\colon B \to E$, such that $pj = 1_B$. We will identify $i(a)$ with $a$.

A split extension induces a triple of actions of $B$ on $A$ corresponding to the operation of addition, action and its dual operation in $\mathbf{Gr}^\bullet$. From the split extension (3.1) for any $b \in B$ and $a \in A$ we define

$$b \cdot a = j(b) + a - j(b) \tag{3.2}$$
$$b^a = j(b)^a - j(b) \tag{3.3}$$
$$a^b = a^{j(b)} \tag{3.4}$$

Actions defined by (3.2)-(3.4) will be called derived actions of $B$ on $A$ as it is in the case of groups with operations or category of interest. Note that (3.3) differs from what we have in the noted known cases, since as we have mentioned above $b^0 \neq 0$ in $B$.

**Proposition 3.1.** *Let $A, B, \in \mathbf{Gr}^\bullet$. Derived actions of $B$ on $A$ satisfy the following conditions:*

**(a)** *well-known group action conditions for the dot left action:*

$$
\begin{aligned}
b\cdot(a_1+a_2) &= b\cdot a_1 + b\cdot a_2\\
(b_1+b_2)\cdot a &= b_1\cdot(b_2\cdot a)\\
0\cdot a &= a
\end{aligned}
$$

*where* $a, a_1, a_2 \in A$ *and* $b, b_1, b_2 \in B$,

**(b)** ${0_A}^b = 0_A$, ${0_B}^a = 0_A$, $b^{0_A} = 0_A$, $a^{0_B} = a$ *where* $0_A$ *and* $0_B$ *denote the zero elements of* $A$ *and* $B$ *respectively. For any* $a, a' \in A$ *and* $b, b' \in B$,

- **($1_A$)** $(a+a')^b = a^b + (a')^b$,
- **($2_A$)** $(b+b')^a = b^a + b\cdot((b')^a)$,
- **($3_A$)** $(b\cdot a)^{a'} + b^{a'} = b^{a'} + b\cdot\left(a^{a'}\right)$,
- **($4_A$)** $(b\cdot a)^{b'} = b^{b'}\cdot a^{b'}$,
- **($1_B$)** $b^{(a+a')} = (b^a)^{a'} + b^{a'}$,
- **($2_B$)** $a^{b+b'} = \left(a^b\right)^{b'}$
- **($3_B$)** $\left(a^{(b\cdot a')}\right)^b = \left(a^b\right)^{a'}$,
- **($4_B$)** $\left(b^{(b'\cdot a)}\right)^{b'} = \left(b^{b'}\right)^a$.

Note that as it will be shown in the proof of Theorem 3.2, all properties noted in **(b)** except $a^{0_B} = a$ follow from **($1_A$)**, **($2_A$)** and **($1_B$)**. Nevertheless we preferred for explicitness to state in the theorem these properties separately.

***Proof.*** **(a)** This is obvious.

**(b)** Follows from the action properties (${0_E}^a = 0_E, a^{0_E} = a$, for all $a \in A$) and the definition of the derived action corresponding to the action operation and its dual (3.3) and (3.4).

**($1_A$)** Let $a, a' \in A$ and $b \in B$; then

$$
\begin{aligned}
(a+a')^b &= (a+a')^{j(b)}\\
&= a^{j(b)} + (a')^{j(b)}\\
&= a^b + (a')^b
\end{aligned}
$$

**($2_A$)** Let $a \in A$ and $b, b' \in B$; then

$$
\begin{aligned}
(b+b')^a &= \left(j(b+b')\right)^a - j\left(b+b'\right)\\
&= j\left(b\right)^a + j\left(b'\right)^a - j\left(b'\right) - j\left(b\right)\\
&= j\left(b\right)^a - j\left(b\right) + j\left(b\right) + j\left(b'\right)^a - j\left(b'\right) - j\left(b\right)\\
&= b^a + b\cdot\left((b')^a\right)
\end{aligned}
$$

**($3_A$)** Let $a, a' \in A$ and $b \in B$; then

$$
\begin{aligned}
(b\cdot a)^{a'} + b^{a'} &= (j(b) + a - j(b))^{a'} + j(b)^{a'} - j(b)\\
&= j(b)^{a'} + a^{a'} - j(b)^{a'} + j(b)^{a'} - j(b)\\
&= j(b)^{a'} - j(b) + j(b) + a^{a'} - j(b)\\
&= b^{a'} + b\cdot\left(a^{a'}\right)
\end{aligned}
$$

**(4$_A$)** Let $a \in A$ and $b, b' \in B$; then

$$\begin{aligned}(b \cdot a)^{b'} &= (j(b) + a - j(b))^{b'}\\ &= j(b)^{j(b')} + a^{j(b')} - j(b)^{j(b')}\\ &= j\left(b^{b'}\right) + a^{j(b')} - j\left(b^{b'}\right)\\ &= b^{b'} \cdot a^{b'}\end{aligned}$$

**(1$_B$)** Let $a, a' \in A$ and $b \in B$; then

$$\begin{aligned}b^{(a+a')} &= j(b)^{(a+a')} - j(b)\\ &= (j(b)^a)^{a'} - j(b)^{a'} + j(b)^{a'} - j(b)\\ &= (j(b)^a - j(b))^{a'} + j(b)^{a'} - j(b)\\ &= (b^a)^{a'} + b^{a'}\end{aligned}$$

**(2$_B$)** Let $a \in A$ and $b, b' \in B$; then

$$\begin{aligned}a^{b+b'} &= a^{j(b)+j(b')}\\ &= \left(a^b\right)^{b'}\end{aligned}$$

**(3$_B$)** Let $a, a' \in A$ and $b \in B$; then

$$\begin{aligned}\left(a^{(b \cdot a')}\right)^b &= \left(a^{(j(b)+a'-j(b))}\right)^{j(b)}\\ &= a^{j(b)+a'}\\ &= \left(a^{j(b)}\right)^{a'}\\ &= \left(a^b\right)^{a'}\end{aligned}$$

**(4$_B$)** Let $a, a' \in A$ and $b, b' \in B$; then

$$\begin{aligned}\left(b^{(b' \cdot a)}\right)^{b'} &= \left(j(b)^{(j(b')+a-j(b'))} - j(b)\right)^{j(b')}\\ &= \left(\left(\left(j(b)^{j(b')}\right)^a\right)^{-j(b')}\right)^{j(b')} - j(b)^{j(b')}\\ &= \left(j(b)^{j(b')}\right)^a - j(b)^{j(b')}\\ &= \left(j\left(b^{b'}\right)\right)^a - j\left(b^{b'}\right)\\ &= \left(b^{b'}\right)^a\end{aligned}$$

□

Given a triple of actions of $B$ on $A$ in $\mathbf{Gr}^\bullet$, we can define operations on the product $B \times A$ in the following way:

$$(b, a) + (b', a') = (b + b', a + b \cdot a') \tag{3.5}$$

$$(b, a)^{(b',a')} = (b^{b'}, (a^{a'})^{b'} + (b^{a'})^{b'}) \tag{3.6}$$

for any $(b, a), (b', a') \in B \times A$. This kind of universal algebra will be called semi-direct product and denoted by $B \ltimes A$.

**Theorem 3.2.** *Let $A, B \in \mathbf{Gr}^\bullet$, if $\beta = (\beta_+, \beta_*, \beta_{*\circ})$ is a triple of actions of $B$ on $A$, then the following conditions are equivalent:*

**(1)** *$\beta$ is a triple of derived actions of $B$ on $A$.*

**(2)** *$\beta_+$ satisfies group action conditions, $\beta$ satisfies conditions* $\mathbf{(1_A)}$–$\mathbf{(4_A)}$, $\mathbf{(1_B)}$–$\mathbf{(4_B)}$ *and the condition $a^{0_B} = a$, for any $a \in A$.*

**(3)** *The semi-direct product $B \ltimes A$ is an object in* $\mathbf{Gr}^{\bullet}$.

***Proof.*** **(1)⇒(2):** by Proposition 3.1.

**(2)⇒(3):** First of all we will show that from $\mathbf{(1_A)}$, $\mathbf{(2_A)}$ and $\mathbf{(1_B)}$ follow the conditions of **(b)** except the one $a^{0_B} = a$. From $\mathbf{(1_A)}$ we have

$$0_A{}^b = 0_A + 0_A{}^b = 0_A{}^b + 0_A{}^b$$

and then $0_A{}^b = 0_A$.

From $\mathbf{(2_A)}$ we have

$$0_B{}^a = 0_B + 0_B{}^a = 0_B{}^a + 0_B \cdot \left(0_B{}^b\right) = 0_B{}^a + 0_B{}^a$$

and then $0_B{}^a = 0_A$. Note that we will not use this property in the proof of **(2)⇒(3)**.

From $\mathbf{(1_B)}$ we have

$$b^{0_A} = b^{0_A + 0_A} = \left(b^{0_A}\right)^{0_A} + b^{0_A}$$

since $a^{0_A} = a$ for any $a \in A$, we obtain

$$b^{0_A} = b^{0_A} + b^{0_A}$$

and then $b^{0_A} = 0_A$.

Now we shall prove that the semi-direct product $B \ltimes A \in \mathbf{Gr}^{\bullet}$. Obviously, $B \ltimes A$ is a group as it is in the case of groups. We have to show the following equalities for any $(b, a), (b', a'), (b'', a'') \in B \ltimes A$:

**(a)** $(b, a)^{(b',a')+(b'',a'')} = \left((b, a)^{(b',a')}\right)^{(b'',a'')}$

**(b)** $((b, a) + (b', a'))^{(b'',a'')} = (b, a)^{(b'',a'')} + (b', a')^{(b'',a'')}$

**(c)** $(b, a)^{(0_B,0_A)} = (b, a)$.

First we prove the equality in **(a)**.

**(a)** We have

$$\begin{aligned}
(b, a)^{(b',a')+(b'',a'')} &= (b, a)^{(b'+b'',a'+b'\cdot a'')} \\
&= \left(b^{b'+b''}, \left(a^{a'+b'\cdot a''}\right)^{b'+b''} + \left(b^{a'+b'\cdot a''}\right)^{b'+b''}\right) \\
&= \left(\left(b^{b'}\right)^{b''}, \left(\left(\left(a^{a'}\right)^{b'\cdot a''}\right)^{b'}\right)^{b''} + \left(\left(b^{a'}\right)^{b'\cdot a''} + b^{b'\cdot a''}\right)^{b'+b''}\right) \\
&= \left(\left(b^{b'}\right)^{b''}, \left(\left(\left(a^{a'}\right)^{b'}\right)^{a''}\right)^{b''} + \left(\left(\left(b^{a'}\right)^{b'\cdot a''}\right)^{b'}\right)^{b''} + \left(\left(b^{b'\cdot a''}\right)^{b'}\right)^{b''}\right) \\
&= \left(\left(b^{b'}\right)^{b''}, \left(\left(\left(a^{a'}\right)^{b'}\right)^{a''}\right)^{b''} + \left(\left(\left(b^{a'}\right)^{b'}\right)^{a''}\right)^{b''} + \left(\left(b^{b'}\right)^{a''}\right)^{b''}\right).
\end{aligned}$$

On the other hand

$$\begin{aligned}\left((b,a)^{(b',a')}\right)^{(b'',a'')} &= \left(b^{b'}, \left(a^{a'}\right)^{b'} + \left(b^{a'}\right)^{b'}\right)^{(b'',a'')}\\ &= \left(\left(b^{b'}\right)^{b''}, \left(\left(\left(a^{a'}\right)^{b'} + \left(b^{a'}\right)^{b'}\right)^{a''}\right)^{b''} + \left(\left(b^{b'}\right)^{a''}\right)^{b''}\right)\\ &= \left(\left(b^{b'}\right)^{b''}, \left(\left(\left(a^{a'}\right)^{b'}\right)^{a''}\right)^{b''} + \left(\left(\left(b^{a'}\right)^{b'}\right)^{a''}\right)^{b''} + \left(\left(b^{b'}\right)^{a''}\right)^{b''}\right)\end{aligned}$$

From which we conclude that condition **(a)** holds in $B \ltimes A$.
Now we check condition **(b)**.

**(b)** We have

$$\begin{aligned}((b,a)+(b',a'))^{(b'',a'')} &= (b+b', a+b\cdot a')^{(b'',a'')}\\ &= \left((b+b')^{b''}, \left((a+b\cdot a')^{a''}\right)^{b''} + \left((b+b')^{a''}\right)^{b''}\right)\\ &= \left(b^{b''}+b'^{b''}, \left(a^{a''}\right)^{b''} + \left((b\cdot a')^{a''}\right)^{b''} + \left(b^{a''}\right)^{b''} + \left(b\cdot (b')^{a''}\right)^{b''}\right)\\ &= \left(b^{b''}+b'^{b''}, \left(a^{a''}\right)^{b''} + \left(b^{a''}\right)^{b''} + \left(b\cdot \left(a'^{a''}\right)\right)^{b''} + \left(b\cdot (b')^{a''}\right)^{b''}\right)\end{aligned}$$

Here we apply condition $(\mathbf{3_A})$. We have the following equalities.

$$\begin{aligned}(b,a)^{(b'',a'')} + (b',a')^{(b'',a'')} &= \left(b^{b''}, \left(a^{a''}\right)^{b''} + \left(b^{a''}\right)^{b''}\right) + \left(b'^{b''}, \left(a'^{a''}\right)^{b''} + \left(b'^{a''}\right)^{b''}\right)\\ &= \left(b^{b''}+b'^{b''}, \left(a^{a''}\right)^{b''} + \left(b^{a''}\right)^{b''} + b^{b''}\cdot\left(a'^{a''}\right)^{b''} + b^{b''}\cdot\left(b'^{a''}\right)^{b''}\right)\end{aligned}$$

Applying condition $(\mathbf{4_A})$ we obtain

$$b\cdot\left(a'^{a''}\right)^{b''} = b^{b''}\cdot\left(a'^{a''}\right)^{b''}$$

and

$$\left(b\cdot (b')^{a''}\right)^{b''} = b^{b''}\cdot\left(b'^{a''}\right)^{b''}$$

which proves that we have condition **(b)**.
Finally, we check condition **(c)**.

**(c)** Here, if we apply the equalities

$$\left(a^{0_A}\right)^{0_B} = a^{0_B} = a$$

and

$$\left(b^{0_A}\right)^{0_B} = {0_A}^{0_B} = 0_A,$$

then we get

$$(b,a)^{(0_B,0_A)} = \left(b^{0_B}, \left(a^{0_A}\right)^{0_B} + \left(b^{0_A}\right)^{0_B}\right) = (b,a).$$

**(3)⇒(1):** Suppose $B \ltimes A \in \mathbf{Gr}^{\bullet}$, then we have a split extension

$$0 \longrightarrow A \xrightarrow{\ i\ } B \ltimes A \underset{p}{\overset{j}{\rightleftarrows}} B \longrightarrow 0 \tag{3.7}$$

where $p(b,a)=b$, $i(a)=(0,a)$ and $j(b)=(b,0)$. Define derived actions from this extension in a usual way.

$$\begin{aligned} b \mathbin{\bar{\cdot}} a &= j(b)+a-j(b)\\ &=(b,0)+(0,a)-(b,0)\\ &=(b,b\cdot a)+(-b,0)\\ &=(0,b\cdot a), \end{aligned}$$

therefore the derived action corresponding to the addition operation coincides with the given action.

Action corresponding to the action operation, denoted by $*$, is defined by

$$\begin{aligned} a*b &= (0_B,a)^{(b,0_A)}\\ &=\left(0_B{}^b, \left(a^{0_A}\right)^b + \left(0_B{}^{0_A}\right)^b\right)\\ &=\left(0,a^b\right). \end{aligned}$$

As we see this action also coincides with the given action.

For the dual to $*$ operation, i.e. dual action we have

$$\begin{aligned} a*^\circ b &= (b,0_A)^{(0_B,a)}-(b,0_A)\\ &=\left(b^{0_B},(0_A{}^a)^{0_B}+(b^a)^{0_B}\right)-(b,0_A)\\ &=(b,b^a)-(b,0_A)\\ &=(b-b,b^a+b\cdot 0_A)\\ &=(0_B,b^a). \end{aligned}$$

Therefore this action also coincides with the given action of $B$ on $A$, which proves that the given action of $B$ on $A$ is a derived action, which concludes the proof of the theorem.

□

For the examples of derived actions in the category $\mathbf{Gr}^\bullet$ see Section 4, Lemma 4.5 and Corollary 4.6.

## 4. The subcategory $\mathbf{rGr}^\bullet \hookrightarrow \mathbf{Gr}^\bullet$

Consider the objects $A\in\mathbf{Gr}^\bullet$ which satisfy two conditions:

**(1)** $x^y+z=z+x^y$, $y\neq 0$ and
**(2)** $x^{(y^z)}=x^y$,

for any $x,y,z\in A$. This kind of objects will be called *reduced groups with action*, and the corresponding full subcategory of $\mathbf{Gr}^\bullet$ will be denoted by $\mathbf{rGr}^\bullet$.

Derived actions are defined in $\mathbf{rGr}^\bullet$ in analogous way as it is in $\mathbf{Gr}^\bullet$.

**Example 4.1.** For any set $X$ let $F(X)$ be a free group with action with the basis $X$ in $\mathbf{Gr}^\bullet$ (see Example 2.2 in Section 2). Let $R$ be a congruence relation on $F(X)$ generated by the relations

$$x^y+z\sim z+x^y$$

for any $y\neq 0$ and

$$x^{(y^z)}\sim x^y$$

for any $x,y,z\in F(X)$. Then the quotient object ${F(X)}/{R}$ by the $R$ obviously is an object of $\mathbf{rGr}^\bullet$ and it is a free object in $\mathbf{rGr}^\bullet$ with the basis $X$.

**Example 4.2.** An easy checking shows that the object $\mathbb{Z}^\bullet$ in Example 2.3 in Section 2 is an object of $\mathbf{rGr}^\bullet$.

**Example 4.3.** Any abelian group with trivial action on itself is an object of $\mathbf{rGr}^{\bullet}$.

**Theorem 4.4.** *Let $A, B \in \mathbf{rGr}^{\bullet}$ and $\beta = (\beta_{+}, \beta_{*}, \beta_{*\circ})\colon B \times A \to A$ be a triple of actions of $B$ on $A$ in $\mathbf{rGr}^{\bullet}$. Then the following conditions are equivalent:*

**(1)** *$\beta$ is a triple of derived actions in $\mathbf{rGr}^{\bullet}$.*

**(2)** *$\beta$ satisfies condition **(2)** of Theorem 3.2 and the following conditions*

$$\begin{aligned} b\cdot a^{a'} &= a^{a'} \text{ for } a'\neq 0 & a^{b}+a' &= a'+a^{b} \text{ for } b\neq 0 \\ b\cdot a^{b'} &= a^{b'} \text{ for } b'\neq 0 & a^{\left(a'^{b}\right)} &= a^{a'} \\ b^{b'}\cdot a &= a \text{ for } b'\neq 0 & a^{\left(b^{a'}\right)} &= a \text{ for } a'\neq 0 \\ b^{\left(a^{a'}\right)} &= b^{a} & b^{(b'^{a})} &= 0 \\ a^{\left(b^{b'}\right)} &= a^{b} & b^{\left(a^{b'}\right)} &= b^{a} \end{aligned} \tag{4.1}$$

*for any $a, a' \in A$, $b, b' \in B$. Note that under the conditions* (4.1), **($\mathbf{3_A}$)** *and* **($\mathbf{4_A}$)** *have simpler forms.*

**(3)** *The semi-direct product $B \ltimes A$ is an object in $\mathbf{rGr}^{\bullet}$.*

***Proof.*** **(1)⇒(2):** We will check only the conditions $a^{\left(b^{a'}\right)} = a$, $b^{(b'^{a})} = 0$ and $b^{\left(a^{b'}\right)} = b^{a}$. Other conditions are obvious.

**(i)** $a^{\left(b^{a'}\right)} = a^{\left(j(b)^{a'}\right)-j(b)} = a^{j(b)-j(b)} = a^{0} = a;$

**(ii)** $b^{(b'^{a})} = j(b)^{(j(b')^{a}-j(b'))} - j(b) = \left(j(b)^{j(b')}\right)^{-j(b')} - j(b) = j(b)^{0} - j(b) = 0;$

**(iii)** $b^{\left(a^{b'}\right)} = j(b)^{\left(a^{j(b')}\right)} - j(b) = j(b)^{a} - j(b) = b^{a}.$

**(2)⇒(3):** By Theorem 3.2 we need to prove only that

$$(b,a)^{(b',a')} + (b'',a'') = (b'',a'') + (b,a)^{(b',a')}$$

and

$$(b,a)^{\left((b',a')^{(b'',a'')}\right)} = (b,a)^{(b',a')}$$

for any $(b,a), (b',a'), (b'',a'') \in B \ltimes A$. We have

$$\begin{aligned} (b,a)^{(b',a')} + (b'',a'') &= \left(b^{b'}, \left(a^{a'}\right)^{b'} + \left(b^{a'}\right)^{b'}\right) + (b'',a'') \\ &= \left(b^{b'}+b'', \left(a^{a'}\right)^{b'} + \left(b^{a'}\right)^{b'} + b^{b'}\cdot a''\right) \\ &= \left(b^{b'}+b'', \left(a^{a'}\right)^{b'} + \left(b^{a'}\right)^{b'} + a''\right). \end{aligned}$$

On the other hand

$$\begin{aligned} (b'',a'') + (b,a)^{(b',a')} &= (b'',a'') + \left(b^{b'}, \left(a^{a'}\right)^{b'} + \left(b^{a'}\right)^{b'}\right) \\ &= \left(b''+b^{b'}, a'' + b''\cdot\left(a^{a'}\right)^{b'} + b''\cdot\left(b^{a'}\right)^{b'}\right) \\ &= \left(b^{b'}+b'', a'' + \left(a^{a'}\right)^{b'} + \left(b^{a'}\right)^{b'}\right) \\ &= \left(b^{b'}+b'', \left(a^{a'}\right)^{b'} + \left(b^{a'}\right)^{b'} + a''\right). \end{aligned}$$

which proves the first identity. For the second identity we have

$$\begin{aligned}
(b,a)^{\left((b',a')^{(b'',a'')}\right)} &= (b,a)^{\left(b'^{b''},\left(a'^{a''}\right)^{b''}+\left(b'^{a''}\right)^{b''}\right)}\\
&= \left(b^{\left(b'^{b''}\right)}, \left(a^{\left(\left(a'^{a''}\right)^{b''}+\left(b'^{a''}\right)^{b''}\right)}\right)^{\left(b'^{b''}\right)} + \left(b^{\left(\left(a'^{a''}\right)^{b''}+\left(b'^{a''}\right)^{b''}\right)}\right)^{\left(b'^{b''}\right)}\right)\\
&= \left(b^{b'}, \left(\left(a^{\left(\left(a'^{a''}\right)^{b''}\right)}\right)^{\left(\left(b'^{a''}\right)^{b''}\right)}\right)^{b'} + \left(b^{\left(\left(a'^{a''}\right)^{b''}+\left(b'^{a''}\right)^{b''}\right)}\right)^{b'}\right)\\
&= \left(b^{b'}, \left(a^{a'}\right)^{b'} + \left(\left(b^{\left(a'^{a''}\right)}\right)^{\left(\left(b'^{a''}\right)^{b''}\right)} + b^{\left(\left(b'^{a''}\right)^{b''}\right)}\right)^{b'}\right)\\
&= \left(b^{b'}, \left(a^{a'}\right)^{b'} + \left(\left(b^{a'}\right)^{\left(b'^{a''}\right)}\right)^{b'} + \left(b^{\left(b'^{a''}\right)}\right)^{b'}\right)\\
&= \left(b^{b'}, \left(a^{a'}\right)^{b'} + \left(b^{a'}\right)^{b'}\right)\\
&= (b,a)^{(b',a')}
\end{aligned}$$

which proves the second identity. Here we applied that $\left(b^{\left(b'^{a''}\right)}\right)^{b'} = 0$, which follows from (4.1), where we have $b^{(b'^{a})} = 0$, for any $a \in A$, in particular for $a = a''$ in our case, and the fact that $0^{b'} = 0$ (3.1 (b)).

**(3)⇒(1):** The proof is the same as of the one in Theorem 3.2 and therefore we omit. □

**Lemma 4.5.** *Let $A \in$ **Gr**$^\bullet$ (resp. $A \in$ **rGr**$^\bullet$). An action of $A$ on itself defined by $a \cdot a' = a + a' - a, a' * a = a'^{\triangleright a} = a'^{a}$ and $a' *^{\circ} a = a^{\circ a'} = a^{a'} - a$, for $a, a' \in A$, is a derived action in **Gr**$^\bullet$ (resp. **rGr**$^\bullet$).*

***Proof.*** Easy but careful checking of the conditions given in Theorem 3.2 (resp. Theorem 4.4). □

Note, that an action of $A$ on itself defined by $a \cdot a' = a + a' - a, a'^{\triangleright a} = a'^{a}$ and $a^{\circ a'} = a^{a'}$, for $a, a' \in A$, is not a derived action in **Gr**$^\bullet$ and therefore in **rGr**$^\bullet$. It is obvious that conditions $(\mathbf{2_A})$ and $(\mathbf{1_B})$ are not satisfied.

**Corollary 4.6.** *Let $A \in$ **Gr**$^\bullet$ (resp. $A \in$ **rGr**$^\bullet$) and let $I \subset A$ be an ideal of $A$. Then the action of $A$ on $I$ defined by $a \cdot i = a + i - a, i^{\triangleright a} = i^{a}$ and $a^{\circ i} = a^{i} - a$, $i \in I$, $a \in A$ is a derived action in $A \in$ **Gr**$^\bullet$ (resp. in $A \in$ **rGr**$^\bullet$).*

Lemma 4.5 and Corollary 4.6 give examples of derived actions in the categories **Gr**$^\bullet$ and **rGr**$^\bullet$.

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