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Investigating (p,q)-hybrid Durrmeyer-type Operators in terms of Their Approximation Properties

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Abstract

This study introduces (p,q)-hybrid Durrmeyer-Stancu type linear positive operators, which are generalized forms of q-hybrid Durrmeyer-Stancu-type linear positive operators and examines their approximation properties. The first modulus of continuity on a finite interval is introduced using Peetre's K-functional. Then, the weighted approximation theorem in a weighted space is provided using Gadzhiev's weighted Korovkin-type theorem. Finally, these operators' rates of convergence are obtained for the continuous functions.

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1. INTRODUCTION

In Dinlemez et al. (2014), they introduced q −hybrid Durrmeyer-Stancu type linear positive operators for $0 < q \leq 1$ as

$$H_{m,q}^{\alpha,\beta}(g,x) = \sum_{k=1}^{\infty} s_{m,k,q}(x) \int_0^{\infty/A} b_{m,k-1,q}(t) g\left(\frac{[m]_q t + \alpha}{[m]_q + \beta}\right) d_q t + e^{-[m]_q x} g\left(\frac{\alpha}{[m]_q + \beta}\right), \quad (1)$$

where

$$s_{m,k,q}(x) = \frac{e^{-[m]_q x}[m-1]_q}{[k]_q!}\left([m]_q x\right)^k,$$

and

$$b_{m,k,q}(x) = \begin{bmatrix} m+k-1 \\ k \end{bmatrix}_q q^{k(k-1)} \frac{x^k}{(1+x)_q^{m+k}}.$$

are q-Szász and q-Baskakov basis functions, respectively. A q −analogue of the Bernstein operators was introduced by Lupaş (1987). These operators were based on q-integer and q-binomial coefficients for the first time. Then, a number of interesting generalizations about q-calculus were studied by Jackson (1910), Koelink & Koornwinder (1990), Phillips (1997), Kac & Cheung (2002), De Sole & Kac (2005), Doğru & Gupta (2005, 2006), Gupta & Heping (2008), Gupta & Aral (2010), Gupta & Karsli (2012), Aral et al. (2013), Yüksel (2013).

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Sahai & Yadav (2007), Kanat & Sofyalıoğlu (2018), Sofyalıoğlu et al. (2021) introduced the generalization of (p,q) −calculus. Recently, the series of studies on (p,q)-generalizations with a sequence of linear positive operators have been made by Mursaleen et al. (2015 a,b,c), Acar et al. (2016, 2018), Gupta (2018), Cai et al. (2021), Kanat & Sofyalıoğlu (2021). Our objective is going to obtain the generalization of (p,q) −calculus of hybrid Durrmeyer-Stancu type operators in Dinlemez et al. (2014).

2. PRELIMINARIES AND NOTATIONS

Some basic formulas in (p,q) −calculus in the literature can be obtained using basic q −calculus as follows

$$[m]_{p,q}=\frac{p^m-q^m}{p-q}, \qquad [m]_{p,q}!=[1]_{p,q}[2]_{p,q}\dots.[m]_{p,q},$$

$$(a\oplus b)^{\,m}_{\,p,q}=(a+b)(ap+bq)(ap^2-bq^2)\dots(ap^{m-1}-bq^{m-1}),$$

$$d_{p,q}f(x)=f(px)-f(qx), \qquad [m]_{p,q}=p^{m-1}[m]_{q/p},$$

$$[m]_{p,q}!=p^{\frac{m(m-1)}{2}}[m]_{q/p}!, \qquad (a\oplus b)^{\,m}_{\,p,q}=p^{\frac{m(m-1)}{2}}(a+b)^{\,m}_{\,q/p}.$$

We define the (p,q) −beta functions $B_{p,q}(k,m)$ as follows

$$B_{p,q}(k,m)=p^{\binom{m}{2}}q^{\binom{k}{2}}\int_0^{\infty/A}\frac{t^{k-1}}{(1+t)^{m+k}_{p,q}}d_{p,q}t, A>0 \text{ and } m,k\in\mathbb{N}. \tag{2}$$

3. (p,q) −HYBRID OPERATORS

Let $A>0$, $k\in\mathbb{N}$, $m\in\mathbb{N}\setminus\{0\}$, and f is a continuous function with real-value in the interval $[0,\infty)$. Then, (p,q) − hybrid Durrmeyer-Stancu type linear positive operators are written for $0<q<p\le 1$ as follows

$$H^{\alpha,\beta}_{m,p,q}(g,x)=\sum_{k=1}^{\infty}s_{m,k,p,q}(x)\gamma_{m,k}(p,q)\int_0^{\infty/A}b_{m,k-1,p,q}(t)g\left(\frac{p^{-m}[m]_{p,q}t+\alpha}{[m]_{p,q}+\beta}\right)d_{p,q}t+e^{-[m]_{p,q}x}g\left(\frac{\alpha}{[m]_{p,q}+\beta}\right), \tag{3}$$

where

$$s_{m,k,p,q}(x)=\frac{e^{-[m]_{p,q}x}[m-1]_{p,q}}{[k]_{p,q}!}\left([m]_{p,q}x\right)^k,$$

$$b_{m,k,p,q}(x)=\begin{bmatrix}m+k-1\\ k\end{bmatrix}_{p,q}\frac{x^k}{(1+x)^{m+k}_{p,q}},$$

and

$$\gamma_{m,k}(p,q)=q^{k(k-1)}p^{\binom{m-1}{2}}.$$

When we set $p=1$ in (3), the operators $H^{\alpha,\beta}_{m,p,q}$ are reduced to q −hybrid Durrmeyer- Stancu type operators given in (1). Along with the manuscripts, the following notations will be used

$$R_{p,q}(m,\beta)=\left([m]_{p,q}+\beta\right), \qquad T_{p,q}(m,s)=\prod_{i=2}^{s}[m-i]_{p,q}.$$

And now the lemma for the Korovkin test functions can be given as follows:

Lemma1 When $e_r(t)=t^r,\ r=0,1,2$, we get

(i) $H_{m,p,q}^{\alpha,\beta}(e_0,x)=1,$

(ii) $H_{m,p,q}^{\alpha,\beta}(e_1,x)=\frac{p^{-2}[m]_{p,q}^2}{qR_{p,q}(m,\beta)T_{p,q}(m,2)}x+\frac{\alpha}{R_{p,q}(m,\beta)},$

(iii) $H_{m,p,q}^{\alpha,\beta}(e_2,x)=\frac{p^{-3}[m]_{p,q}^4}{q^4\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,3)}x^2+\left\{\frac{p^{-5}[2]_{p,q}[m]_{p,q}^3}{q^3\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,3)}+\frac{2\alpha p^{-3}[m]_{p,q}^2}{q\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,2)}\right\}x+\frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}$

Proof After $(p,q)-$ beta functions in (2) are used, it is obtained as follows

$$\int_0^{\infty/A} b_{m,k-1,p,q}(t)t^r\,d_{p,q}t=\begin{bmatrix}m+k-2\\k-1\end{bmatrix}_{p,q}\int_0^{\infty/A}\frac{t^{k+r-1}}{(1+t)_{p,q}^{m+k-1}}d_{p,q}t$$
$$=\frac{[k+r-1]_{p,q}!\,[m-r-2]_{p,q}!\,q^{-\binom{k+r}{2}}}{[m-1]_{p,q}!\,[k-1]_{p,q}!\,p^{\binom{m-r-1}{2}}}. \tag{4}$$

Then, by using (4) for $r=0$, we obtain

$$\begin{aligned}H_{m,p,q}^{\alpha,\beta}(e_0,x)&=\sum_{k=1}^{\infty}s_{m,k,p,q}(x)\gamma_{m,k}(p,q)\int_0^{\infty/A}b_{m,k-1,p,q}(t)d_{p,q}t+e^{-[m]_{p,q}x}\\&=e^{-[m]_{p,q}x}\sum_{k=0}^{\infty}\frac{\left([m]_{p,q}x\right)^k}{[k]_{p,q}!}q^{-k(k-1)/2}\\&=e^{-[m]_{p,q}x}E_{p,q}^{[m]_{p,q}x}=1,\end{aligned}$$

and the proof of (i) is completed. The following (ii) is obtained by a direct computation

$$\begin{aligned}H_{m,p,q}^{\alpha,\beta}(e_1,x)&=\sum_{k=1}^{\infty}s_{m,k,p,q}(x)\gamma_{m,k}(p,q)\int_0^{\infty/A}b_{m,k-1,p,q}(t)\frac{p^{-m}[m]_{p,q}t+\alpha}{R_{p,q}(m,\beta)}d_{p,q}t+\frac{\alpha e^{-[m]_{p,q}x}}{R_{p,q}(m,\beta)}\\&=\frac{p^{-m}[m]_{p,q}}{R_{p,q}(m,\beta)T_{p,q}(m,2)}\sum_{k=1}^{\infty}\frac{\left([m]_{p,q}x\right)^k}{[k-1]_{p,q}!}q^{k(k-3)/2}p^{m-3}e^{-[m]_{p,q}x}\\&+\frac{\alpha}{R_{p,q}(m,\beta)}\sum_{k=1}^{\infty}\frac{\left([m]_{p,q}x\right)^k}{[k]_{p,q}!}q^{k(k-1)/2}e^{-[m]_{p,q}x}+\frac{\alpha e^{-[m]_{p,q}x}}{R_{p,q}(m,\beta)}\\&=\frac{p^{-2}[m]_{p,q}^2}{qR_{p,q}(m,\beta)T_{p,q}(m,2)}x+\frac{\alpha}{R_{p,q}(m,\beta)}\end{aligned}$$

Using the following equality

$$[s]_{p,q}=q^{s-r}[r]_{p,q}+p^r[s-r]_{p,q},\qquad 0\le r\le s, \tag{5}$$

we get

$$
\begin{aligned}
H_{m,p,q}^{\alpha,\beta}(e_2,x) &= \sum_{k=1}^{\infty} s_{m,k,p,q}(x)\gamma_{m,k}(p,q)\int_0^{\infty/A} b_{m,k-1,p,q}(t)\left(\frac{p^{-m}[m]_{p,q}t+\alpha}{R_{p,q}(m,\beta)}\right)^2 d_{p,q}t + \frac{\alpha^2 e^{-[m]_{p,q}x}}{\left(R_{p,q}(m,\beta)\right)^2} \\
&= \frac{p^{-2m}\left([m]_{p,q}\right)^2}{\left(R_{p,q}(m,\beta)\right)^2}\sum_{k=1}^{\infty} s_{m,k,p,q}(x)\gamma_{m,k}(p,q)\int_0^{\infty/A} b_{m,k-1,p,q}(t)t^2 d_{p,q}t \\
&+ \frac{2\alpha p^{-m}[m]_{p,q}}{\left(R_{p,q}(m,\beta)\right)^2}\sum_{k=1}^{\infty} s_{m,k,p,q}(x)\gamma_{m,k}(p,q)\int_0^{\infty/A} b_{m,k-1,p,q}(t)t d_{p,q}t \\
&+ \frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}\sum_{k=1}^{\infty} s_{m,k,p,q}(x)\gamma_{m,k}(p,q)\int_0^{\infty/A} b_{m,k-1,p,q}(t) d_{p,q}t \\
&+ \frac{\alpha^2 e^{-[m]_{p,q}x}}{\left(R_{p,q}(m,\beta)\right)^2} \\
&= \frac{p^{-3}[m]_{p,q}^4}{q^4\left(R_{p,q}(m,\beta)\right)^2 T_{p,q}(m,3)}x^2 \\
&+ \left\{\frac{p^{-5}[2]_{p,q}[m]_{p,q}^3}{q^3\left(R_{p,q}(m,\beta)\right)^2 T_{p,q}(m,3)} + \frac{2\alpha p^{-3}[m]_{p,q}^2}{q\left(R_{p,q}(m,\beta)\right)^2 T_{p,q}(m,2)}\right\}x + \frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}.
\end{aligned}
$$

Thus the proof of (iii) is completed.

For the main results of the study, we need to compute the second moment.

Lemma 2 Assuming that $0 < q < p \leq 1$ and $m > 3$, we obtain the following inequality

$$
H_{m,p,q}^{\alpha,\beta}((t-x)^2,x) \leq \left(\frac{2(1-p^{-2}q^3)}{q^4} + \frac{288(\alpha+\beta+1)^2[m]_{p,q}}{q^4 T_{p,q}(m,3)}\right)(x^2+x) + \frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}.
$$

Proof To write the second moment, we use the result of Lemma 1 and the linearity of $H_{m,p,q}^{\alpha,\beta}$ operators;

$$
\begin{aligned}
H_{m,p,q}^{\alpha,\beta}((t-x)^2,x) &= \left\{\frac{p^{-3}[m]_{p,q}^4}{q^4\left(R_{p,q}(m,\beta)\right)^2 T_{p,q}(m,3)} - \frac{2p^{-2}[m]_{p,q}^2}{qR_{p,q}(m,\beta)T_{p,q}(m,2)} + 1\right\}x^2 \\
&+ \left\{\frac{p^{-5}[2]_{p,q}[m]_{p,q}^3 + 2\alpha q^2 p^{-3}[m-3]_{p,q}[m]_{p,q}^2}{q^3\left(R_{p,q}(m,\beta)\right)^2 T_{p,q}(m,3)} - \frac{2\alpha}{R_{p,q}(m,\beta)}\right\}x \\
&+ \frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2} \\
&\leq \left\{\frac{[m]_{p,q}^4(p^{-3}+q^4) - 2q^3p^{-2}[m-3]_{p,q}^4}{q^4\left(R_{p,q}(m,\beta)\right)^2 T_{p,q}(m,3)}\right. \\
&+ \frac{q^4\left(q^{m-3}[3]_{p,q} + p^3[m-3]_{p,q} + \beta\right)^2\left(q^{m-3} + p[m-3]_{p,q}\right)[m-3]_{p,q}}{q^4\left(R_{p,q}(m,\beta)\right)^2 T_{p,q}(m,3)}
\end{aligned}
$$

$$+\frac{p^{-5}[2]_{p,q}[m]_{p,q}^3+2\alpha q^2p^{-3}[m-3]_{p,q}[m]_{p,q}^2}{q^3\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,3)}\Bigg\}(x^2+x)$$

$$+\frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}.$$

From (4), we have

$$\begin{aligned}H_{m,p,q}^{\alpha,\beta}((t-x)^2,x) \leq & \Bigg\{\frac{2(1+p^{-2}q^3)[m-3]_{p,q}^4-2q^3p^{-2}[m-3]_{p,q}^4}{q^4\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,3)}\\
&+\frac{\left(q^{m+1}p^6+2[3]_{p,q}p^4q^{m+1}+2p^4q^4\beta+4[3]_{p,q}p^6q^{m-3}\right)[m-3]_{p,q}^3}{q^4\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,3)}\\
&+\frac{\left(pq^4\beta^2+2\beta pq^{m+1}\left(p^2+[3]_{p,q}\right)+2[3]_{p,q}p^3q^{2m-2}\right.}{q^4\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,3)}\\
&+\frac{\left.[3]_{p,q}^2pq^{2m-2}(1+6p^3q^{-4})\right)[m-3]_{p,q}^2}{q^4\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,3)}\\
&+\frac{\left(4[3]_{p,q}^3q^{3m-9}+[3]_{p,q}^2q^{3m-5}+\beta^2q^{m+1}+2\beta q^{2m-2}[3]_{p,q}\right)[m-3]_{p,q}}{q^4\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,3)}\\
&+\frac{[3]_{p,q}^4p^{-3}q^{4m-12}+[m]_{p,q}^3[2]_{p,q}qp^{-5}+2\alpha p^{-3}q^3[m]_{p,q}^2[m-3]_{p,q}}{q^4\left(R_{p,q}(m,\beta)\right)^2T_{p,q}(m,3)}\Bigg\}(x^2+x)\\
&+\frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}\\
\leq & \left(\frac{2(1-p^{-2}q^3)}{q^4}+\frac{288(\alpha+\beta+1)^2[m]_{p,q}}{q^4T_{p,q}(m,3)}\right)(x^2+x)+\frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}\end{aligned}$$

And the proof of the Lemma 2 is now completed.

Assume that, $B[0,\infty)$ denotes the set of all bounded functions from $[0,\infty)$ to $\mathbb{R}$. Having the norm $\|g\|_B=\sup\{|g(x)|: x\in[0,\infty)\}, B[0,\infty)$ is a normed space. For all continuous functions in $B[0,\infty)$, the subspace is denoted by $C_B[0,\infty)$. The first modulus of continuity on finite interval $[0,b],\ b>0$ is denoted as follows;

$$w_{[0,b]}(g,\delta)=\sup_{0<h\leq\delta, x\in[0,b]}|g(x+h)-g(x)|. \tag{6}$$

The Peetre's K-functional is defined with the help of the following representation

$$K_2(g,\delta)=\inf\{\|g-f\|_B+\delta\|f''\|_B: f\in W_\infty^2\}, \qquad \delta>0 \tag{7}$$

where $W_\infty^2=\{f\in C_B[0,\infty): f',\ f''\in C_B[0,\infty)\}$. There is a positive constant C at Theorem 2.4 on p.177 in Gadzhiev (1976), such that

$$K_2(g,\delta) \le C w_2(g,\sqrt{\delta}) \tag{8}$$

where

$$w_2(g,\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{x\in[0,b]} |g(x+2h) - g(x+h) - g(x)|. \tag{9}$$

In Gadzhiev (1976), Gadzhiev proved the weighted Korovkin-type theorems. Let $\sigma(x) = 1 + x^2$.

$B_\sigma[0,\infty)$ denotes the set of all functions g, from $[0,\infty)$ to $\mathbb{R}$ that meets the growth condition $|g(x)| \le M_g \sigma(x)$.

In this inequality, M_g is a constant depending only on g. $B_\sigma[0,\infty)$ is a normed space with the norm

$\|g\|_\sigma = \sup\left\{\frac{|g(x)|}{\sigma(x)} : x \in [0,\infty)\right\}$. $C_\sigma[0,\infty)$ denotes the subspace of all continuous functions in $B_\sigma[0,\infty)$ and $C_\sigma^*[0,\infty)$ denotes the subspace of all functions $g \in C_\sigma[0,\infty)$ whose following limit exists finitely

$$\lim_{|x|\to\infty} \frac{|g(x)|}{\sigma(x)}.$$

Now, the direct results can be given. Because the following lemma is a routine, its proof is omitted.

Lemma 3 Let

$$\overline{H}_{m,p,q}^{\alpha,\beta}(g,x) = H_{m,p,q}^{\alpha,\beta}(g,x) - g\left(\frac{p^{-2}[m]_{p,q}^2}{qR_{p,q}(m,\beta)T_{p,q}(m,2)}x + \frac{\alpha}{R_{p,q}(m,\beta)}\right) + g(x). \tag{10}$$

For the operators (10), the following equalities are asserted:

(i) $\overline{H}_{m,p,q}^{\alpha,\beta}(1,x) = 1,$

(ii) $\overline{H}_{m,p,q}^{\alpha,\beta}(t,x) = x,$

(iii) $\overline{H}_{m,p,q}^{\alpha,\beta}(t-x,x) = 0.$

Lemma 4 Let $0 < q < p \le 1$ and $m > 3$. Then $g'' \in C_B[0,\infty)$, we have the following inequality

$$\left|\overline{H}_{m,p,q}^{\alpha,\beta}(g,x) - g(x)\right| \le \zeta_{m,p,q}^{\alpha,\beta}(x)\|g''\|_B$$

where $\zeta_{m,p,q}^{\alpha,\beta}(x) = \left(\frac{2(1-p^{-2}q^3)}{q^4} + \frac{332(\alpha+\beta+1)^2}{q^4T_{p,q}(m,2)}\right)(x^2+x) + \frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}$.

Proof Using Taylor's expansion

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du$$

and Lemma 3, we obtain

$$\overline{H}_{m,p,q}^{\alpha,\beta}(g,x) - g(x) = \overline{H}_{m,p,q}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du\,, x\right).$$

Then, using Lemma1 and the following inequality

$$\left|\int_x^t (t-u)g''(u)du\right| \le \|g''\|_B \frac{(t-x)^2}{2},$$

we get

$$\begin{aligned}\left|\overline{H}^{\alpha,\beta}_{m,p,q}(g,x)-g(x)\right| &\le \left|H^{\alpha,\beta}_{m,p,q}\left(\int_x^t (t-u)g''(u)du\,,x\right)\right.\\ &\left.-\int_x^{\frac{p^{-2}[m]^2_{p,q}}{qR_{p,q}(m,\beta)T_{p,q}(m,2)}x+\frac{\alpha}{R_{p,q}(m,\beta)}}\left(\frac{p^{-2}[m]^2_{p,q}}{qR_{p,q}(m,\beta)T_{p,q}(m,2)}x+\frac{\alpha}{R_{p,q}(m,\beta)}-u\right)g''(u)du\right|\\ &\le \frac{\|g''\|_B}{2}H^{\alpha,\beta}_{m,p,q}((t-x)^2,x)+\frac{\|g''\|_B}{2}\left(\left(\frac{p^{-2}[m]^2_{p,q}}{qR_{p,q}(m,\beta)T_{p,q}(m,2)}-1\right)+\frac{\alpha}{R_{p,q}(m,\beta)}\right)^2\\ &\le \frac{\|g''\|_B}{2}\left\{\left(\frac{2(1-p^{-2}q^3)}{q^4}+\frac{288(\alpha+\beta+1)^2[m]_{p,q}}{q^4T_{p,q}(m,3)}\right)(x^2+x)+\frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}\right\}\\ &+\frac{\|g''\|_B}{2}\left\{\left(\frac{p^{-4}[m]^4_{p,q}}{\left(qR_{p,q}(m,\beta)T_{p,q}(m,2)\right)^2}-\frac{2p^{-2}q[m]^2_{p,q}\left([m]_{p,q}+\beta\right)[m-2]_{p,q}}{\left(qR_{p,q}(m,\beta)T_{p,q}(m,2)\right)^2}\right.\right.\\ &\left.-\frac{q^2\left([m]_{p,q}+\beta\right)^2[m-2]^2_{p,q}}{\left(qR_{p,q}(m,\beta)T_{p,q}(m,2)\right)^2}\right)x^2+\frac{2\alpha\left(p^{-2}[m]^2_{p,q}-q\left([m]_{p,q}+\beta\right)[m-2]_{p,q}\right)}{qR^2_{p,q}(m,\beta)T_{p,q}(m,2)}x\\ &\left.+\frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}\right.\\ &\le \left\{\left(\frac{2(1-p^{-2}q^3)}{q^4}+\frac{332(\alpha+\beta+1)^2}{q^4T_{p,q}(m,2)}\right)(x^2+x)+\frac{\alpha^2}{\left(R_{p,q}(m,\beta)\right)^2}\right\}\|g''\|_B.\end{aligned}$$

Finally, the proof of Lemma 4 is completed.

Theorem 1 Let $(p_m),(q_m)\subset(0,1)$ be two sequences with $0<q_m<p_m\le 1$ such that $p_m\to 1, q_m\to 1$ as $m\to\infty$. Then for every $m>3$ and $g\in C_B[0,\infty)$, we have the below inequality

$$\left|H^{\alpha,\beta}_{m,p_m,q_m}(g,x)-g(x)\right|\le 2Cw_2\left(g,\sqrt{\zeta^{\alpha,\beta}_{m,p_m,q_m}(x)}\right)+w\left(g,\eta^{\alpha,\beta}_{m,p_m,q_m}(x)\right),$$

Where $\eta^{\alpha,\beta}_{m,p_m,q_m}(x)=\left(\frac{p_m{}^{-2}[m]^2_{p_m,q_m}}{q_mR_{p_m,q_m}(m,\beta)T_{p_m,q_m}(m,2)}-1\right)x+\frac{\alpha}{R_{p_m,q_m}(m,\beta)}$.

Proof. Based on (10), for any $g\in W^2_\infty$, we obtain the inequality

$$\left|H^{\alpha,\beta}_{m,p_m,q_m}(g,x)-g(x)\right| \le \left|\overline{H}^{\alpha,\beta}_{m,p_m,q_m}(g-f,x)-(g-f)(x)+H^{\alpha,\beta}_{m,p_m,q_m}(f,x)-f(x)\right|$$

$$+\left|g\left(\frac{p_m{}^{-2}[m]_{p_m,q_m}^2 x}{q_m R_{p_m,q_m}(m,\beta)T_{p_m,q_m}(m,2)}+\frac{\alpha}{R_{p_m,q_m}(m,\beta)}\right)-g(x)\right|.$$

From Lemma 4, we get

$$\begin{aligned}\left|H_{m,p_m,q_m}^{\alpha,\beta}(g,x)-g(x)\right| &\leq 2\|g-f\|_B+\zeta_{m,p_m,q_m}^{\alpha,\beta}(x)\|f''\|_B \\ &+\left|g\left(\frac{p_m{}^{-2}[m]_{p_m,q_m}^2 x}{q_m R_{p_m,q_m}(m,\beta)T_{p_m,q_m}(m,2)}+\frac{\alpha}{R_{p_m,q_m}(m,\beta)}\right)-g(x)\right|.\end{aligned}$$

As a result of the equality (6), we have the inequality

$$\left|H_{m,p_m,q_m}^{\alpha,\beta}(g,x)-g(x)\right| \leq 2\|g-f\|_B+\zeta_{m,p_m,q_m}^{\alpha,\beta}(x)\|f''\|_B+w\left(g,\eta_{m,p_m,q_m}^{\alpha,\beta}(x)\right).$$

Taking the infimum over $g \in W_\infty^2$ on the right-hand side of the above inequality and then using the inequality (8), we get the desired result.

Theorem 2 Let $(p_m),(q_m) \subset (0,1)$ be two sequences with $0<q_m<p_m \leq 1$ such that $p_m \to 1,\ q_m \to 1$ as $m \to \infty$. Then $g \in C_\sigma^*[0,\infty)$, we have

$$\lim_{m\to\infty}\left\|H_{m,p_m,q_m}^{\alpha,\beta}(g,x)-g(x)\right\|_\sigma=0.$$

Proof. From Lemma 1, it is obvious that $\left\|H_{m,p_m,q_m}^{\alpha,\beta}(e_0,x)-e_0\right\|_\sigma=0$. Because $\left|\frac{p_m{}^{-2}[m]_{p_m,q_m}^2 x}{q_m R_{p_m,q_m}(m,\beta)T_{p_m,q_m}(m,2)}+\frac{\alpha}{R_{p_m,q_m}(m,\beta)}-x\right| \leq (x+1)o(1)$ and $\frac{1+x}{1+x^2}$ is positive and it is bounded from above for each $x \geq 0$, we get

$$\left\|H_{m,p_m,q_m}^{\alpha,\beta}(e_1,x)-e_1\right\|_\sigma \leq \frac{1+x}{1+x^2}o(1).$$

And then $\lim_{m\to\infty}\left\|H_{m,p_m,q_m}^{\alpha,\beta}(e_1,x)-e_1(x)\right\|_\sigma=0.$

Similarly for every $m>3$, we can write

$$\begin{aligned}\left\|H_{m,p_m,q_m}^{\alpha,\beta}(e_2,x)-e_2(x)\right\|_\sigma &= \sup_{x\in[0,\infty)}\left\{\frac{\left|\frac{p_m{}^{-3}[m]_{p_m,q_m}^4 x^2}{q_m^4\left(R_{p_m,q_m}(m,\beta)\right)^2 T_{p_m,q_m}(m,3)}\right.}{1+x^2}\right. \\ &+\frac{\left\{\frac{p_m{}^{-5}[2]_{p_m,q_m}[m]_{p_m,q_m}^3 x^2}{q_m^4\left(R_{p_m,q_m}(m,\beta)\right)^2 T_{p_m,q_m}(m,3)}+\frac{2\alpha p_m{}^{-3}[m]_{p_m,q_m}^2}{q_m^2\left(R_{p_m,q_m}(m,\beta)\right)^2 T_{p_m,q_m}(m,3)}\right\}x}{1+x^2} \\ &\left.+\frac{\left.\frac{\alpha^2}{\left(R_{p_m,q_m}(m,\beta)\right)^2}-x^2\right|}{1+x^2}\right\}\end{aligned}$$

$$\leq \sup_{x\in[0,\infty)} \frac{1+x+x^2}{1+x^2} o(1),$$

and we get $\lim_{m\to\infty} \left\| H^{\alpha,\beta}_{m,p_m,q_m}(e_2,x) - e_2(x) \right\|_\sigma = 0$. Therefore, by using A. D. Gadzhiev.s Theorem in Gadzhiev (1976), we obtain Theorem 2's result.

Lemma 5 Let $g \in C_\sigma[0,\infty)$, $(p_m),(q_m) \subset (0,1)$ be two sequences with $0 < q_m < p_m \leq 1$ such that $p_m \to 1, q_m \to 1$ as $m \to \infty$ and $w_{[0,d+1]}(g,\delta)$ be its modulus of continuity on the finite interval $[0,d+1]$ $d > 0$. Then for every $m > 3$, there exists a constant $C > 0$ such that the inequality holds

$$\left\| H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x) \right\|_{C[0,d]} \leq C\left\{(d+1)^2 \xi^{\alpha,\beta}_{m,p_m,q_m}(d) + w_{[0,d+1]}\left(g, \sqrt{\xi^{\alpha,\beta}_{m,p_m,q_m}(d)}\right)\right\},$$

where

$$\xi^{\alpha,\beta}_{m,p_m,q_m}(d) = \left(\frac{2(1-p_m^{-2}q_m^3)}{q_m^4} + \frac{288(\alpha+\beta+1)^2[m]_{p_m,q_m}}{q_m^4 T_{p_m,q_m}(m,3)}\right)(d^2+d) + \frac{\alpha^2}{\left(R_{p_m,q_m}(m,\beta)\right)^2}.$$

Proof. Let $x \in [0,d]$ and $t > d+1$. Since $t - x > 1$, we have

$$\begin{aligned} |g(t)-g(x)| &\leq M_g(2+t^2+x^2) \\ &\leq 3M_g(1+d)^2(t-x)^2. \end{aligned} \tag{11}$$

Let $x \in [0,d]$ and $t < d+1$ and $\delta > 0$. Then we have

$$|g(t)-g(x)| \leq \left(1+\frac{|t-x|}{\delta}\right) w_{[0,d+1]}(g,\delta). \tag{12}$$

With the help of (11) and (3.12), we can write

$$|g(t)-g(x)| \leq 3M_g(1+d)^2(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right) w_{[0,d+1]}(g,\delta).$$

Then, using Lemma 2 and Cauchy-Schwarz's inequality, we get the following inequalities

$$\begin{aligned} \left|H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x)\right| &\leq 3M_g(1+d)^2 H^{\alpha,\beta}_{m,p_m,q_m}((t-x)^2,x) \\ &\quad + w_{[0,d+1]}(g,\delta)\left[1+\frac{1}{\delta}\left(H^{\alpha,\beta}_{m,p_m,q_m}((t-x)^2,x)\right)^{1/2}\right] \\ &\leq 3M_g(1+d)^2 \xi^{\alpha,\beta}_{m,p_m,q_m}(x) + w_{[0,d+1]}(g,\delta)\left[1+\frac{1}{\delta}\left(\xi^{\alpha,\beta}_{m,p_m,q_m}(x)\right)^{1/2}\right], \end{aligned}$$

where

$$\xi^{\alpha,\beta}_{m,p_m,q_m}(x) = \left(\frac{2(1-p_m^{-2}q_m^3)}{q_m^4} + \frac{288(\alpha+\beta+1)^2[m]_{p_m,q_m}}{q_m^4 T_{p_m,q_m}(m,3)}\right)(x^2+x) + \frac{\alpha^2}{\left(R_{p_m,q_m}(m,\beta)\right)^2}$$

Setting

$$\delta^2 := \xi^{\alpha,\beta}_{m,p_m,q_m}(d) = \left(\frac{2(1-p_m^{-2}q_m^3)}{q_m^4} + \frac{288(\alpha+\beta+1)^2[m]_{p_m,q_m}}{q_m^4 T_{p_m,q_m}(m,3)}\right)(d^2+d) + \frac{\alpha^2}{\left(R_{p_m,q_m}(m,\beta)\right)^2}$$

and $C = \min\{3M_g, 2\}$. Therefore, the proof of Lemma 5 is finished.

Theorem 3 Let $\lambda > 0, (p_m), (q_m) \subset (0,1)$ be two sequences with $0 < q_m < p_m \leq 1$ such that $p_m \to 1$, $q_m \to 1$ as $m \to \infty$ and $g \in C_\sigma^*[0,\infty)$. Then we have

$$\lim_{m\to\infty} \sup_{x\geq 0} \frac{\left|H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x)\right|}{1+x^{2+\lambda}} = 0.$$

Proof. For $\lambda > 0, g \in C_\sigma^*[0,\infty)$ and $b > 1$, the following inequality is ensured

$$\begin{aligned}\sup_{x\geq 0} \frac{\left|H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x)\right|}{1+x^{2+\lambda}} &\leq \sup_{0\leq x<d} \frac{\left|H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x)\right|}{1+x^{2+\lambda}} + \sup_{d\leq x} \frac{\left|H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x)\right|}{1+x^{2+\lambda}} \\ &\leq \left\|H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x)\right\|_{C[0,d]} + \sup_{d\leq x} \frac{\left|H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x)\right|}{1+x^2} \\ &\leq \left\|H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x)\right\|_{C[0,d]} + \left\|H^{\alpha,\beta}_{m,p_m,q_m}(g,x) - g(x)\right\|_{\sigma}.\end{aligned}$$

Using Lemma 5 and Theorem 2, the proof of Theorem 3 is provided.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

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