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AN OVERVIEW TO ANALYTICITY OF DUAL FUNCTIONS

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Abstract. In this paper, the analyticity conditions of dual functions are clearly examined and the properties of the concept derivative are given in detail. Then, using the dual order relation, the dual analytic regions of dual analytic functions are constructed such that a collection of these regions forms a basis on D^n. Finally, the equivalent of the inverse function theorem in dual space is given by a theorem and proved.

1. Introduction

In 1873, W. K. Clifford originally introduced the theory of algebra of dual numbers as a tool for his geometrical researches. Clifford showed that they constitute an algebra but not a field because only dual numbers with real part not zero have an inverse element [1]. An ordered pair of real numbers $\overline{x} = (x, x^*)$ is called a dual number, where x and x^* are termed by real part and dual part of the dual number, respectively. Dual numbers may be formally stated by $\overline{x} = x + \varepsilon x^*$, where $\varepsilon = (0, 1)$ is entitled by dual unit satisfying the condition that $\varepsilon^2 = 0$. The algebra of dual numbers is derived from this description. If $x = y$, $x^* = y^*$ for $\overline{x} = x + \varepsilon x^*$ and $\overline{y} = y + \varepsilon y^*$, $\overline{x}$ and $\overline{y}$ are equal, and it is indicated as $\overline{x} = \overline{y}$. As for complex numbers, addition and product of two dual numbers are defined as follows, respectively:

$$(x + \varepsilon x^*) + (y + \varepsilon y^*) = x + y + \varepsilon (x^* + y^*),$$

$$(x + \varepsilon x^*) \cdot (y + \varepsilon y^*) = xy + \varepsilon (xy^* + x^* y).$$

The set of all dual numbers which is symbolized as D, i.e.,

$$\mathrm{D} = \left\{ \overline{x} = x + \varepsilon x^* \mid x, x^* \in \mathbb{R},\ \varepsilon^2 = 0 \right\}$$

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is a commutative ring over the real numbers field according to the operators $+$ and $\cdot$. The unit element of multiplication operation $\cdot$ in D is the dual number $\overline{1}$ $= (1,0) = 1+\varepsilon 0$. The dual number $\overline{x} = x+\varepsilon x^*$ that is divided by the dual number $\overline{y} = y+\varepsilon y^*$ providing $y \neq 0$ can be described as

$$\frac{\overline{x}}{\overline{y}} = \frac{x+\varepsilon x^*}{y+\varepsilon y^*} = \frac{x}{y} + \varepsilon\left(\frac{x^* y - x y^*}{y^2}\right)$$

(see [1] and [2]). The dual number has a geometrical meaning which is discussed in detail in Yaglom [3]. It has contemporary applications within the curve design methods in computer aided geometric design and computer modeling of rigid bodies, linkages, robots, modelling human body dynamics, mechanism design, etc. [4]. The dual vectors were improved by A. P. Kotelnikov in the early part of the twentieth century [5]. After W. K. Clifford, E. Study applied dual numbers and dual vectors to his study on kinematics and line geometry [6]. There exist several articles with regard to algebraic study of dual numbers (see [1] and [2]). This nice notion was first performed by Kotelnikov to mechanics. Besides, the notion is often used in several fields of fundamental sciences such as astronomy, algebraic geometry, quantum mechanics and Riemannian geometry. For more details, we refer the reader to [3]-[12].

The set $\mathrm{D}^n = \left\{ \overrightarrow{\overline{x}} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n) \mid \overline{x}_i \in \mathrm{D}, 1 \leq i \leq n \right\}$ is a module over the ring D according to the operators

$$\overrightarrow{\overline{x}} + \overrightarrow{\overline{y}} = (\overline{x}_1 + \overline{y}_1, \overline{x}_2 + \overline{y}_2, ..., \overline{x}_n + \overline{y}_n)$$

and

$$\overline{\lambda}\overrightarrow{\overline{x}} = \left(\overline{\lambda}\overline{x}_1, \overline{\lambda}\overline{x}_2, ..., \overline{\lambda}\overline{x}_n\right).$$

This module is called D$-$module or dual space. The elements of D^n are called dual vectors and a dual vector $\overrightarrow{\overline{x}}$ can be expressed as

$$\overrightarrow{\overline{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x}^*,$$

where $\overrightarrow{x}$ and $\overrightarrow{x}^*$ are real vectors in $\mathbb{R}^n$ [1].

The dual function $\langle , \rangle_{\mathrm{D}} : \mathrm{D}^n \times \mathrm{D}^n \to \mathrm{D}$,

$$\left\langle \overrightarrow{\overline{x}}, \overrightarrow{\overline{y}} \right\rangle_{\mathrm{D}} = \langle \overrightarrow{x}, \overrightarrow{y} \rangle + \varepsilon \left(\langle \overrightarrow{x}, \overrightarrow{y}^* \rangle + \langle \overrightarrow{x}^*, \overrightarrow{y} \rangle \right)$$

is called dual inner product function on D^n, where the notation $\langle , \rangle$ is Euclidean inner product on $\mathbb{R}^n$.

Similar to dual inner product function, dual norm function $\|.\|_{\mathrm{D}} : \mathrm{D}^n \to \mathrm{D}$ is defined as follows:

$$\left\| \overrightarrow{\overline{x}} \right\|_{\mathrm{D}} = \begin{cases} 0 & , \overrightarrow{x} = \overrightarrow{0} \\ \| \overrightarrow{x} \| + \varepsilon \dfrac{\langle \overrightarrow{x}, \overrightarrow{x}^* \rangle}{\| \overrightarrow{x} \|} & , \overrightarrow{x} \neq \overrightarrow{0}, \end{cases}$$

where the notation $\|.\|$ is Euclidean norm on $\mathbb{R}^n$.

Given the vectors $\overrightarrow{\overline{e}}_i = \left(\overline{\delta}_{i1}, \overline{\delta}_{i2}, ..., \overline{\delta}_{in}\right)$, where

$$\overline{\delta}_{ij} = \begin{cases} 1+\varepsilon 0 & , i=j \\ 0+\varepsilon 0 & , i\neq j \end{cases} , \ 1 \leq i,j \leq n,$$

the set $\left\{\overrightarrow{\overline{e}}_1, \overrightarrow{\overline{e}}_2, ..., \overrightarrow{\overline{e}}_n\right\}$ is standard basis of D^n. It turns out that every dual vector $\overrightarrow{\overline{x}} \in \mathrm{D}^n$ can be written in the form

$$\overrightarrow{\overline{x}} = \overline{x}_1 \overrightarrow{\overline{e}}_1 + \overline{x}_2 \overrightarrow{\overline{e}}_2 + ... + \overline{x}_n \overrightarrow{\overline{e}}_n,$$

where $\overrightarrow{\overline{e}}_i = \overrightarrow{e}_i + \varepsilon \overrightarrow{0}$ for $1 \leq i \leq n$.

Consider that $\overline{x} = x + \varepsilon x^*$ and $\overline{y} = y + \varepsilon y^*$ are dual numbers. The relation $\overline{x} <_{\mathrm{D}} \overline{y}$ (resp. $\overline{x} \leq_{\mathrm{D}} \overline{y}$) between these dual numbers is as follows (see [13], [14]):

1) Firstly, one compares the real parts of these dual numbers and must be $x < y$ (resp. $x < y$).

2) If the real parts of these dual numbers are the same, one compares their dual parts and must be $x^* < y^*$ (resp. $x^* \leq y^*$).

We can infer that there exist the following relations:

$$\overline{x} <_{\mathrm{D}} \overline{y} \Leftrightarrow x < y \text{ or } \left(x = y \text{ and } x^* < y^*\right)$$

and

$$\overline{x} \leq_{\mathrm{D}} \overline{y} \Leftrightarrow x < y \text{ or } \left(x = y \text{ and } x^* \leq y^*\right).$$

For the historical development of the term derivative, the expression "The derivative was first used, then discovered, and then studied and developed and finally defined." was used. The reason for using this expression is development process of the derivative starting with P. de Fermat in 1630s, continuing with I. Newton, J. L. Lagrange, G.W. Leibniz, A. L. Cauchy and reaching maturity in the 1870s with K. Weierstrass. The approaches to the derivative put forward by Leibniz and Newton were sufficient to find answers to the questions about the tangent of the curve and the velocity of the bodies. In fact, in the 19th century, this concept reached a consistent and solid foundation with the definition of derivative created by Cauchy using the term limit. It is well known that Cauchy put forward the first popularly acceptable account of the fundamental notions of the calculus. In order to prove the theorems related to the derivative, he used his own definitions. He described the derivative $\xi'(x)$ of a continuous function ξ as the limit when it exists, of the ratio $\dfrac{\xi(x+h) - \xi(x)}{h}$ as h went to zero. The instantaneous rate of change is entitled by the derivative. A comparison of the change in one quantity to the simultaneous change in a second quantity is expressed as a rate of change. Many of today's important problems in several fields such as engineering, biology, chemistry, physics, economics, involve finding the rate at which one quantity changes with respect to another, that is, they involve finding the derivative [15].

Topology is a mathematical discipline which originated at the turn of the 20th century. On the other hand, some isolated results about topology can be traced back several centuries. In mathematics, topology is interested in the properties of a geometric object which is preserved under continuous deformations including twisting, crumpling, stretching and bending. For many years, topology has been one of the most influential and exciting fields of research in modern mathematics. Topology is used for application fields such as physics, computer science, biology, robotics, fiber art, puzzles and games. Besides, topology has lots of applications in several branches of mathematics including differential equations, knot theory, dynamical systems, and Riemann surfaces in complex analysis. It also has some applications for describing the space-time structure of universe and analyzing many biological systems such as nanostructure and molecules, and in string theory in physics (see [16]- [29]).

In this paper, using the order relation on dual numbers, we obtain the topology on D^n denoted by $\overline{\tau}_{\overline{d}}$. Then, how the analyticity conditions of a dual function which is often expressed in other studies are obtained is given clearly. Making use of this topology, dual analytic areas of dual analytic functions are determined. Besides, inner and external operations on the set constituted by dual analytic functions are given. With the help of these operations, some properties regarding dual analytic functions are expressed and proved. The relations between the elements of dual space and real space which will be used to define the basic concepts of differential geometry are examined. The terms dual tangent space, dual directional derivative, dual vector field and dual tangent map which are the basic tools of differential geometry are given in detail. The concepts of injective function, surjective function, inverse function and diffeomorphism in dual space is firstly expressed in this study. Some theorems related to these terms are obtained and proved. The foundation of term surface in dual space is constituted via these terms [13].

2. On Dual Analytic Functions

Firstly, we shall study the concept of topology generating the basic structure of theory of curves and surfaces given by means of the expression of distance function in dual space. Previously, we talked about this basis [13]. After constructing a topology structure in dual space, we will determine the dual analytic regions of dual analytic functions by means of this topology.

Theorem 1. *Given the sets*

$$\begin{aligned}
\overline{B}\left(\overline{a},\overline{r}\right) &= \left\{\overline{x}=x+\varepsilon x^{*}\in \mathrm{D}^{n} \mid \|x-a\|<r,\ x^{*}\in\mathbb{R}\right\} \\
&\quad \cup\left\{\overline{x}=x+\varepsilon x^{*}\in \mathrm{D}^{n} \mid \|x-a\|=r \ \mathit{and}\ \frac{\langle x-a,x^{*}-a^{*}\rangle}{\|x-a\|}<r^{*}\right\} \\
&= U_1\cup U_2 \\
&= U_1\cup C_1\cup \ldots\cup C_k,\ \ (k\in I=\{1,2,\ldots\})
\end{aligned}$$

and

$$U_3=\left\{\overline{x}=x+\varepsilon x^{*}\in \mathrm{D}^{n} \mid x=a',\ m<x_1^{*}<n,\ x_{j+1}^{*}=c_j\in\mathbb{R},\ m,n\in[-\infty,\infty]\right\},$$

then a collection of all the sets $U_1, U_3, C_1, ..., C_k$ $(k \in I)$ *forms a basis* $\overline{\beta}$ *on* D^n, *where* $\overline{a} = a + \varepsilon a^* \in \mathrm{D}^n$, $r \in \mathbb{R}^+$, $r^* \in \mathbb{R}$ *and* $1 \leq j \leq n-1$.

Proof. It is enough to remark that two conditions given in definition of the term basis are satisfied.

i) It is easily seen that

$$\bigcup_{\overline{A} \in \overline{\beta}} \overline{A} = \mathrm{D}^n.$$

ii) The set $\overline{A}_1 \cap \overline{A}_2$ is an arbitrary union of some sets belonging to class $\overline{\beta}$ for all $\overline{A}_1, \overline{A}_2 \in \overline{\beta}$ expect for $\overline{A}_1 \cap \overline{A}_2 = \varnothing$. Now, let us show that this expression is correct. Suppose that $\overline{y}$ belongs to $\overline{A}_1 \cap \overline{A}_2$. Taking into account the sets $\overline{B}_1, \overline{B}_2, U_3'$ and U_3'', the following situations hold, where

$$\begin{aligned}
\overline{B}_1\left(\overline{a}_1, \overline{r}_1\right) &= \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a_1\| < r_1,\ x^* \in \mathbb{R}^n\right\} \\
&\quad \cup \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a_1\| = r_1 \text{ and } \frac{\langle x - a_1, x^* - a_1^* \rangle}{\|x - a_1\|} < r_1^*\right\} \\
&= U_1 \cup C_1' \cup ... \cup C_l',
\end{aligned}$$

$$\begin{aligned}
\overline{B}_2\left(\overline{a}_2, \overline{r}_2\right) &= \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a_2\| < r_2,\ x^* \in \mathbb{R}^n\right\} \\
&\quad \cup \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a_2\| = r_2 \text{ and } \frac{\langle x - a_2, x^* - a_2^* \rangle}{\|x - a_2\|} < r_2^*\right\} \\
&= U_1' \cup C_1'' \cup ... \cup C_{l'}'',
\end{aligned}$$

$$U_3' = \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid x = b',\ m_1 < x_1^* < n_1,\ x_{j+1}^* = c_j' \in \mathbb{R},\ m_1, n_1 \in [-\infty, \infty]\right\}$$

and

$$U_3'' = \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid x = b'',\ m_2 < x_1^* < n_2,\ x_{j+1}^* = c_j'' \in \mathbb{R},\ m_2, n_2 \in [-\infty, \infty]\right\}.$$

1) Suppose that $\overline{y} \in U_3' \cap U_3''$. The following set can be written:

$$U_3' \cap U_3'' = \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid x = a,\ m < x_1^* < n,\ x_{j+1}^* = c_j \in \mathbb{R},\ m, n \in [-\infty, \infty]\right\} \in \overline{\beta},$$

where $y = b' = b'' = a$, $m < y_1^* < n$, $y_{j+1}^* = c_j' = c_j'' = c_j \in \mathbb{R}$, $m = \max\{m_1, m_2\}$ and $n = \min\{n_1, n_2\}$.

2) Assume that $\overline{y} \in U_1 \cap U_3''$. Hence, it is clear that $U_1 \cap U_3'' = U_3'' \in \overline{\beta}$.

3) Suppose that $\overline{y} \in C_l' \cap U_3''$ for any $l \in I$. In this case, the set $C_l' \cap U_3''$ can be written as

$$U_3^j = \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid x = a_j,\ m_j < x_1^* < n_j,\ x_{j+1}^* = c_j' \in \mathbb{R},\ m_j, n_j \in [-\infty, \infty]\right\}.$$

Therefore, $C_l' \cap U_3'' \in \overline{\beta}$.

4) Assume that $\overline{y} \in U_1 \cap U_1'$. The set $U_1 \cap U_1'$ can be written as an arbitrary union of the sets

$$U = \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a\| < r,\ x^* \in \mathbb{R}^n\right\}.$$

5) Suppose that $\overline{y} \in U_1 \cap C''_{l'}$ for any $l' \in I$. It is easy to check that $U_1 \cap C''_{l'} = C''_{l'} \in \overline{\beta}$.

6) Assume that $\overline{y} \in C'_l \cap C''_{l'}$ for any $l, l' \in I$. The set $C'_l \cap C''_{l'}$ is expressed as $C_l \in \overline{\beta}$, for $l \in I$ or an arbitrary union of the sets U_3 belonging to class $\overline{\beta}$.

With these conventions, we have

$$\overline{A}_1 \cap \overline{A}_2 = \bigcup_{\overline{A} \in \mathring{A} \subseteq \overline{\beta}} \overline{A}$$

for all $\overline{A}_1, \overline{A}_2 \in \overline{\beta}$ expect for $\overline{A}_1 \cap \overline{A}_2 = \varnothing$, where the class $\mathring{A}$ is a class of some sets belonging to the class $\overline{\beta}$. $\square$

Definition 1. *The class $\overline{\beta}$ given in the above mentioned theorem is called dual basis on D^n. The topology obtained from this basis is symbolized as $\overline{\tau}_{\overline{d}}$. Each element of this topology is termed by dual open set.*

Theorem 2. *Suppose that the class of the sets*

$$\begin{aligned} \overline{U} &= \{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a\| < r, x^* \in \mathbb{R}^n\} \\ &= U \times \mathbb{R}^n \end{aligned}$$

belonging to the topology $\overline{\tau}_{\overline{d}}$ is symbolized as $\overline{\beta}_1$, where U is open set with respect to the standard topology of $\mathbb{R}^n$. Then the class $\overline{\beta}_1$ also constitutes a basis on D^n and the relationship between the topology $\overline{\tau}$ obtained from this basis and the topology $\overline{\tau}_{\overline{d}}$ is $\overline{\tau} \subseteq \overline{\tau}_{\overline{d}}$.

For example; let us study the topology $\overline{\tau}_{\overline{d}}$ on D. Assume that

$$\begin{aligned} \overline{B}\left(\overline{a}, \overline{r}\right) &= \{\overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid |x - a| < r, x^* \in \mathbb{R}\} \\ &\quad \cup \left\{ \overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid |x - a| = r \text{ and } \frac{(x - a)\left(x^* - a^*\right)}{|x - a|} < r^* \right\} \\ &= U_1 \cup U_2 \\ &= U_1 \cup C_1 \cup C_2, \end{aligned}$$

where $\overline{a} = a + \varepsilon a^* \in \mathrm{D}$, $r \in \mathbb{R}^+$, $r^* \in \mathbb{R}$ and

$$\begin{aligned} U_2 &= \{\overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid x = a + r,\ x^* < a^* + r^*\} \\ &\quad \cup \{\overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid x = a - r,\ x^* > a^* - r^*\} \\ &= C_1 \cup C_2. \end{aligned}$$

Taking into consideration the set

$$U_3 = \{\overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid x = a',\ m < x^* < n,\ m, n \in [-\infty, \infty]\},$$

the collection of the sets U_1, C_1, C_2 and U_3 forms a basis on D. The topology obtained from this basis is symbolized as $\overline{\tau}_{\overline{d}}$. Besides, the collection of the sets $U_1 \cup U_2$ and U_3 is also a basis on D and the topology generated by this basis is also $\overline{\tau}_{\overline{d}}$.

Observe that

$$B=\left\{\widetilde{x}=(x,x^{*})\in\mathbb{R}^{2}\mid a<x<b,\ c<x^{*}<d,\ a,b,c,d\in\mathbb{R}\right\}.$$

The collection of all the sets B forms a basis on $\mathbb{R}^2$. If the topology generated by this basis is symbolized as τ_1, the relationship between $\overline{\tau}_{\overline{d}}$ and τ_1 is $\tau_1\subseteq\overline{\tau}_{\overline{d}}$. On the other hand, if the topology derived from the collection of only the sets U_1 is symbolized as $\overline{\tau}$, then there exists the following relationship:

$$\overline{\tau}\subseteq\tau_1\subseteq\overline{\tau}_{\overline{d}}.$$

Definition 2. *Let $\overline{x}=x+\varepsilon x^{*}$ be a dual variable. The function $\overline{\xi}:\mathrm{D}\to\mathrm{D}$ of the dual variable $\overline{x}=x+\varepsilon x^{*}$ is defined as follows:*

$$\overline{\xi}\left(\overline{x}\right)=\xi\left(x,x^{*}\right)+\varepsilon\xi^{0}\left(x,x^{*}\right),$$

where ξ and ξ^0 are real functions of the two real variables x and x^*.

In the following theorem, by eliminating the deficiencies in other studies, we shall discuss analyticity conditions of dual functions.

Theorem 3. *The dual function $\overline{\xi}:\overline{U}\subseteq\mathrm{D}\to\mathrm{D}$, $\overline{\xi}\left(\overline{x}=x+\varepsilon x^{*}\right)=\xi\left(x,x^{*}\right)+\varepsilon\xi^{0}\left(x,x^{*}\right)$ is said to be analytic at the point $\overline{x}\in\overline{U}$ if and only if the functions ξ and ξ^0 have continuous partial derivatives ξ_x and ξ^0_x and there exist the equalities $\xi_{x^*}=0$ and $\xi^0_{x^*}=\xi_x$, where $\xi_x=\dfrac{\partial\xi}{\partial x}$.*

Proof. Firstly, let the dual function $\overline{\xi}$ be analytic at the point $\overline{x}\in\overline{U}$. Thus, this assumption permits us to write the following relation:

$$\frac{d\overline{\xi}}{d\overline{x}}=\lim_{\overline{h}\to\overline{0}}\frac{\overline{\xi}\left(\overline{x}+\overline{h}\right)-\overline{\xi}\left(\overline{x}\right)}{\overline{h}}. \tag{1}$$

Observe that $\overline{x}=x+\varepsilon x^{*}$ and $\overline{h}=h+\varepsilon h^{*}$. By definition of dual variable functions and $\varepsilon^2=0$, the following equality holds:

$$\begin{aligned}
\frac{d\overline{\xi}}{d\overline{x}} &= \lim_{\overline{h}\to\overline{0}}\frac{\overline{\xi}\left(\overline{x}+\overline{h}\right)-\overline{\xi}\left(\overline{x}\right)}{\overline{h}}\\
&= \lim_{(h,h^*)\to(0,0)}\frac{\xi\left(x+h,x^*+h^*\right)+\varepsilon\xi^0\left(x+h,x^*+h^*\right)-\xi\left(x,x^*\right)-\varepsilon\xi^0\left(x,x^*\right)}{h+\varepsilon h^*}\\
&= \lim_{(h,h^*)\to(0,0)}\frac{\xi\left(x+h,x^*+h^*\right)-\xi\left(x,x^*\right)}{h}\\
&\quad+\lim_{(h,h^*)\to(0,0)}\varepsilon\begin{pmatrix}\frac{\xi^0(x+h,x^*+h^*)-\xi^0(x,x^*)}{h}\\ -\frac{h^*}{h^2}\left(\xi\left(x+h,x^*+h^*\right)-\xi\left(x,x^*\right)\right)\end{pmatrix}\\
&= \frac{\partial\xi}{\partial x}+\varepsilon\frac{\partial\xi^0}{\partial x}.
\end{aligned}$$

In view of equation (1), it is seen that the limit for $(h,h^*)\to(0,0)$ of real part of the expression $\dfrac{\overline{\xi}\left(\overline{x}+\overline{h}\right)-\overline{\xi}\left(\overline{x}\right)}{\overline{h}}$ is $\dfrac{\partial\xi}{\partial x}$. Then, it is easy to check that

$$\begin{aligned}&\frac{\xi\left(x+h,x^*+h^*\right)-\xi\left(x,x^*\right)}{h}\\ =\;&\frac{\xi\left(x+h,x^*+h^*\right)-\xi\left(x,x^*+h^*\right)}{h}+\frac{\xi\left(x,x^*+h^*\right)-\xi\left(x,x^*\right)}{h}.\end{aligned}\tag{2}$$

From the hypothesis and the equality (2), we have

$$\lim_{(h,h^*)\to(0,0)}\frac{\xi\left(x,x^*+h^*\right)-\xi\left(x,x^*\right)}{h}=0.$$

If this limit exists and equals to zero, it is obvious from discussion that

$$\xi\left(x,x^*+h^*\right)-\xi\left(x,x^*\right)=0$$

such that $\xi\left(x,x^*\right)=\xi\left(x\right)$. Thus, the function ξ depends only on the variable x, i.e., $\dfrac{\partial\xi}{\partial x^*}=0$. It is well known from equation (1) that the limit for $(h,h^*)\to(0,0)$ of dual part of the expression $\dfrac{\overline{\xi}\left(\overline{x}+\overline{h}\right)-\overline{\xi}\left(\overline{x}\right)}{\overline{h}}$ is $\dfrac{\partial\xi^0}{\partial x}$. By some calculations, the following equality holds:

$$\begin{aligned}&\frac{\xi^0\left(x+h,x^*+h^*\right)-\xi^0\left(x,x^*\right)}{h}-\frac{\xi\left(x+h,x^*+h^*\right)-\xi\left(x,x^*\right)}{h}\frac{h^*}{h}\\ =\;&\frac{\xi^0\left(x+h,x^*+h^*\right)-\xi^0\left(x,x^*+h^*\right)}{h}\\ &+\frac{\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)}{h}-\frac{\xi\left(x+h\right)-\xi\left(x\right)}{h}\frac{h^*}{h}.\end{aligned}\tag{3}$$

From the hypothesis and the equality (3), we get

$$\lim_{h^*\to0}\left(\lim_{h\to0}\frac{h\left(\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)\right)-h^*\left(\xi\left(x+h\right)-\xi\left(x\right)\right)}{h^2}\right)=0.\tag{4}$$

Since the statement

$$\lim_{h\to0}\frac{h\left(\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)\right)-h^*\left(\xi\left(x+h\right)-\xi\left(x\right)\right)}{h^2}$$

has the indefiniteness $\left(\frac{0}{0}\right)$, we write the following equality:

$$\lim_{h^*\to0}\left(\lim_{h\to0}\frac{\left(\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)\right)-h^*\xi_x\left(x+h\right)}{2h}\right)=0.\tag{5}$$

From (5), we obtain

$$\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)=h^*\xi_x\left(x\right).$$

Therefore, it is possible to express that

$$\frac{\xi^0(x, x^* + h^*) - \xi^0(x, x^*)}{h^*} = \xi_x(x),$$

where $h^* \neq 0$. The limit of both sides of this identity for $h^* \to 0$ is $\xi^0_{x^*} = \xi_x$.

Conversely, suppose that the functions ξ and ξ^0 have continuous partial derivatives ξ_x and ξ^0_x and there are the equalities $\xi_{x^*} = 0$ and $\xi^0_{x^*} = \xi_x$. The expression of dual function $\overline{\xi}$ is simplified to the following form

$$\overline{\xi}(\overline{x}) = \xi(x) + \varepsilon\left(x^*\xi'(x) + \widetilde{\xi}(x)\right), \tag{6}$$

where $\xi \in C^2$, $\widetilde{\xi} \in C^1$. Given a point $\overline{x} \in \overline{U}$, we must show that the expression $\lim\limits_{\overline{h}\to\overline{0}} \frac{\overline{\xi}(\overline{x}+\overline{h}) - \overline{\xi}(\overline{x})}{\overline{h}}$ exists. From the equality (6), the derivative of the dual function $\overline{\xi}$ with respect to dual variable $\overline{x}$ can be expressed as follows:

$$I = \lim_{\overline{h}\to\overline{0}} \frac{\overline{\xi}(\overline{x}+\overline{h}) - \overline{\xi}(\overline{x})}{\overline{h}} = \lim_{(h,h^*)\to(0,0)} \left[\begin{array}{c} \frac{\xi(x+h)-\xi(x)}{h} \\ +\varepsilon\left(\begin{array}{c} x^*\left(\frac{\xi'(x+h)-\xi'(x)}{h}\right) + \frac{\widetilde{\xi}(x+h)-\widetilde{\xi}(x)}{h} \\ +\frac{h^*}{h}\xi'(x+h) - \frac{\xi(x+h)-\xi(x)}{h}\frac{h^*}{h} \end{array}\right) \end{array}\right].$$

From the hypothesis, we have

$$\begin{aligned}
I_1 &= \lim_{(h,h^*)\to(0,0)} \frac{\xi(x+h)-\xi(x)}{h} = \xi'(x), \\
I_2 &= \lim_{(h,h^*)\to(0,0)} x^*\left(\frac{\xi'(x+h)-\xi'(x)}{h}\right) = x^*\xi''(x), \\
I_3 &= \lim_{(h,h^*)\to(0,0)} \frac{\widetilde{\xi}(x+h)-\widetilde{\xi}(x)}{h} = \widetilde{\xi}'(x), \\
I_4 &= \lim_{(h,h^*)\to(0,0)} \frac{h^*}{h}\xi'(x+h) - \frac{\xi(x+h)-\xi(x)}{h}\frac{h^*}{h} = 0
\end{aligned}$$

such that

$$I = I_1 + \varepsilon(I_2 + I_3 + I_4) = \xi'(x) + \varepsilon\left(x^*\xi''(x) + \widetilde{\xi}'(x)\right).$$

Thus, this obviously completes the proof of the theorem. □

We are now ready to state the following corollaries.

Corollary 1. *Theorem 3 implies that the derivative of dual function $\overline{\xi} : \overline{U} \subseteq \mathrm{D} \to \mathrm{D}$ with respect to dual variable $\overline{x}$ is*

$$\frac{d\overline{\xi}}{d\overline{x}} = \lim_{\Delta\overline{x}\to\overline{0}} \frac{\overline{\xi}(\overline{x}+\Delta\overline{x}) - \overline{\xi}(\overline{x})}{\Delta\overline{x}}.$$

This limit is independent of the ratio $\frac{\Delta x^*}{\Delta x}$ [30].

Corollary 2. *Taking into account Theorem 3, the analyticity conditions of dual function* $\overline{\xi}:\overline{U}\subseteq \mathrm{D}\to\mathrm{D}$, $\overline{\xi}\left(\overline{x}\right)=\xi\left(x,x^{*}\right)+\varepsilon\xi^{0}\left(x,x^{*}\right)$ *are* $\dfrac{\partial\xi}{\partial x^{*}}=0$ *and* $\dfrac{\partial\xi^{0}}{\partial x^{*}}=\dfrac{\partial\xi}{\partial x}$. *Thus, the general representation of dual analytic functions is*

$$\overline{\xi}\left(\overline{x}\right)=\xi\left(x,x^{*}\right)+\varepsilon\xi^{0}\left(x,x^{*}\right)=\xi\left(x\right)+\varepsilon\left(x^{*}\xi'\left(x\right)+\widetilde{\xi}\left(x\right)\right),$$

where $\xi,\widetilde{\xi}:U\subseteq\mathbb{R}\to\mathbb{R}$ *and* $\xi\in C^{2}$, $\widetilde{\xi}\in C^{1}$. *In the proof of Theorem 3, it is clearly seen that the derivative of this analytic function* $\overline{\xi}$ *with respect to dual variable* $\overline{x}$ *is*

$$\frac{d\overline{\xi}}{d\overline{x}}=\frac{\partial\xi}{\partial x}+\varepsilon\frac{\partial\xi^{0}}{\partial x}=\xi'\left(x\right)+\varepsilon\left(x^{*}\xi''\left(x\right)+\widetilde{\xi}'\left(x\right)\right)$$

[30].

Now, based on Theorem 3, let us determine the analyticity conditions of dual function $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^{n}\to\mathrm{D}$,

$$\overline{\xi}\left(\overline{x}\right)=\xi\left(x_{1},...,x_{n},x_{1}^{*},...,x_{n}^{*}\right)+\varepsilon\xi^{0}\left(x_{1},...,x_{n},x_{1}^{*},...,x_{n}^{*}\right)=\xi+\varepsilon\xi^{0}.$$

The partial derivatives of dual function $\overline{\xi}$ at any dual point $\overline{a}\in\overline{U}\subseteq\mathrm{D}^{n}$ (if there exists) are

$$\frac{\partial\overline{\xi}}{\partial\overline{x}_{i}}\left(\overline{a}\right)=\lim_{\Delta\overline{x}_{i}\to\overline{0}}\frac{\overline{\xi}\left(\overline{a}_{1},...,\overline{a}_{i}+\Delta\overline{x}_{i},...,\overline{a}_{n}\right)-\overline{\xi}\left(\overline{a}_{1},...,\overline{a}_{n}\right)}{\Delta\overline{x}_{i}},\ 1\leq i\leq n.$$

The above formula is simplified to the following form:

$$\frac{\partial\overline{\xi}}{\partial\overline{x}_{i}}\left(\overline{a}\right)=\frac{d}{d\overline{x}_{i}}\overline{\xi}\left(\overline{a}_{1},...,\overline{x}_{i},...,\overline{a}_{n}\right)|_{\overline{x}_{i}=\overline{a}_{i}}=\lim_{\overline{x}_{i}\to\overline{a}_{i}}\frac{\overline{\mu}\left(\overline{x}_{i}\right)-\overline{\mu}\left(\overline{a}_{i}\right)}{\overline{x}_{i}-\overline{a}_{i}},$$

where $\overline{\mu}\left(\overline{x}_{i}\right)=\overline{\xi}\left(\overline{a}_{1},...,\overline{x}_{i},...,\overline{a}_{n}\right)$. When Theorem 3 is taken into consideration, one can check that if this limit exists, for $1\leq i\leq n$, then the functions ξ and ξ^{0} have continuous partial derivatives $\xi_{x_{i}}$ and $\xi^{0}_{x_{i}}$ at any dual point $\overline{a}\in\overline{U}$ and these relations $\dfrac{\partial\xi}{\partial x_{i}^{*}}=0$ and $\dfrac{\partial\xi^{0}}{\partial x_{i}^{*}}=\dfrac{\partial\xi}{\partial x_{i}}$ are satisfied. From Theorem 3, it is easy to see that the reverse exists. This result follows by proceeding as in the proof of the first assertion. Thus, these conventions permit us to write the following relation:

$$\frac{\partial\overline{\xi}}{\partial\overline{x}_{i}}\left(\overline{a}\right)=\frac{\partial\xi}{\partial x_{i}}\left(a_{1},...,a_{n},a_{1}^{*},...,a_{n}^{*}\right)+\varepsilon\frac{\partial\xi^{0}}{\partial x_{i}}\left(a_{1},...,a_{n},a_{1}^{*},...,a_{n}^{*}\right).$$

Besides, the expression $\lim\limits_{\Delta\overline{x}_{i}\to\overline{0}}\dfrac{\Delta\overline{\xi}}{\Delta\overline{x}_{i}}$ is independent of the ratio $\dfrac{\Delta x_{i}^{*}}{\Delta x_{i}}$. Note that the analyticity conditions of dual function $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^{n}\to\mathrm{D}$ are $\dfrac{\partial\xi}{\partial x_{i}^{*}}=0$ and $\dfrac{\partial\xi^{0}}{\partial x_{i}^{*}}=\dfrac{\partial\xi}{\partial x_{i}}$ $(1\leq i\leq n)$. In view of these equalities, we can write the following expressions:

$$\xi\left(x_{1},...,x_{n},x_{1}^{*},...,x_{n}^{*}\right)=\xi\left(x_{1},...,x_{n}\right)$$

and

$$\xi^0 (x_1, ..., x_n, x_1^*, ..., x_n^*) = \sum_{i=1}^{n} x_i^* \frac{\partial \xi}{\partial x_i} + \widetilde{\xi} (x_1, ..., x_n),$$

where $\xi \in C^2$, $\widetilde{\xi} \in C^1$. By definition of the analyticity conditions of dual function $\overline{\xi} : \overline{U} \subseteq \mathrm{D}^n \to \mathrm{D}$, the general representation of these dual analytic functions is

$$\overline{\xi} (\overline{x}) = \xi (x_1, ..., x_n) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial \xi}{\partial x_i} + \widetilde{\xi} (x_1, ..., x_n) \right). \tag{7}$$

The partial derivatives of this function with respect to dual variables $\overline{x}_j$ are

$$\frac{\partial \overline{\xi}}{\partial \overline{x}_j} = \frac{\partial \xi}{\partial x_j} + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial^2 \xi}{\partial x_j \partial x_i} + \frac{\partial \widetilde{\xi}}{\partial x_j} \right)$$

$(1 \leq j \leq n)$. Throughout this paper, the functions ξ and $\widetilde{\xi}$ will be considered as belonging to $C^\infty-$class. Note that the sets of the topology $\overline{\tau}$ mentioned in Theorem 2 is dual analytic regions of dual analytic functions. The set of dual analytic functions is symbolized as $C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right)$. Therefore, the following expression holds:

$$C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right) = \left\{ \overline{\xi} \mid \overline{\xi} : \overline{U} \subseteq \mathrm{D}^n \to \mathrm{D},\ \overline{\xi} (\overline{x}) = \xi (x) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial \xi}{\partial x_i} + \widetilde{\xi} (x) \right) \right\}.$$

Given the dual functions $\overline{\xi} : \overline{U} \subseteq \mathrm{D}^n \to \mathrm{D}^m$, $\overline{\xi} = \left(\overline{\xi}_1, ..., \overline{\xi}_m \right)$, we conclude that if the dual functions $\overline{\xi}_j : \overline{U} \subseteq \mathrm{D}^n \to \mathrm{D}, (1 \leq j \leq m)$ are dual analytic, then the dual function $\overline{\xi}$ is dual analytic. When the above information is taken into consideration, the following functions can be defined:

i) $+_C : C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right) \times C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right) \to C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right)$, for $\overline{\xi}, \overline{\mu} \in C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right)$ and $\overline{x} \in \overline{U} \subseteq \mathrm{D}^n$, we have

$$\left(\overline{\xi} +_C \overline{\mu} \right) (\overline{x}) = \overline{\xi} (\overline{x}) + \overline{\mu} (\overline{x}) = \xi (x) + \mu (x) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial (\xi + \mu)}{\partial x_i} + \widetilde{\xi} (x) + \widetilde{\mu} (x) \right).$$

ii) $\cdot_C : \mathrm{D} \times C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right) \to C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right)$, for $\overline{\xi} \in C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right)$, $\overline{\lambda} = \lambda + \varepsilon \lambda^* \in \mathrm{D}$ and $\overline{x} \in \overline{U} \subseteq \mathrm{D}^n$, we have

$$\left(\overline{\lambda} \cdot_C \overline{\xi} \right) (\overline{x}) = \overline{\lambda} \cdot \overline{\xi} (\overline{x}) = \lambda \xi (x) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial (\lambda \xi)}{\partial x_i} + \lambda \widetilde{\xi} (x) + \lambda^* \xi (x) \right).$$

iii) $\cdot_{1_C} : C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right) \times C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right) \to C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right)$, for $\overline{\xi}, \overline{\mu} \in C \left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D} \right)$ and $\overline{x} \in \overline{U} \subseteq \mathrm{D}^n$, we have

$$\left(\overline{\xi}\cdot_{1_C}\overline{\mu}\right)(\overline{x})=\overline{\xi}(\overline{x})\cdot\overline{\mu}(\overline{x})=\xi(x)\mu(x)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial(\xi\mu)}{\partial x_i}+\xi(x)\widetilde{\mu}(x)+\widetilde{\xi}(x)\mu(x)\right)$$

[31].

We are interested now to some properties regarding dual analytic functions.

Proposition 1. *Consider $\overline{\mu}:\overline{I}\subseteq \mathrm{D}\to\mathrm{D}^n$ and $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D}$ are dual analytic functions, where the functions $\overline{\mu}$ and $\overline{\xi}$ are as below:*

$$\overline{\mu}\left(\overline{t}\right)=\mu(t)+\varepsilon\left(t^*\mu'(t)+\widetilde{\mu}(t)\right)$$

and

$$\overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\xi}{\partial x_i}+\widetilde{\xi}(x)\right)$$

such that the functions $\xi,\widetilde{\xi},\mu$ and $\widetilde{\mu}$ belong to $C^\infty-$class. If the functions $\overline{\xi}$ and $\overline{\mu}$ are dual analytic at the dual points $\overline{\mu}\left(\overline{t}\right)$ and $\overline{t}$, respectively, then the composition of $\overline{\mu}$ and $\overline{\xi}$,i.e., $\overline{\xi}\circ\overline{\mu}$ is dual analytic function. The derivative of this dual analytic function with respect to dual variable $\overline{t}$ is

$$\frac{d}{d\overline{t}}\left(\overline{\xi}\circ\overline{\mu}\right)\left(\overline{t}\right)=(\xi\circ\mu)'(t)+\varepsilon\left(\begin{array}{c} t^*(\xi\circ\mu)''(t)+\left\langle\widetilde{\mu}(t),\sum_{i=1}^{n}\left(\frac{\partial\xi}{\partial x_i}\circ\mu\right)(t)\overrightarrow{e}_i\right\rangle' \\ +\left(\widetilde{\xi}\circ\mu\right)'(t)\end{array}\right),$$

where $(\xi\circ\mu)'(t)=\dfrac{d}{dt}(\xi\circ\mu)(t)$.

Theorem 4. *Let $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to$ D be dual analytic function. Then the following identity holds*

$$\frac{\partial^2\overline{\xi}}{\partial\overline{x}_k\partial\overline{x}_j}=\frac{\partial^2\overline{\xi}}{\partial\overline{x}_j\partial\overline{x}_k}\quad(1\leq j,k\leq n),$$

for any dual point of $\overline{U}\subseteq\mathrm{D}^n$.

Proof. Let $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D}$ be dual analytic function. From the equality (7), we can write

$$\overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\xi}{\partial x_i}+\widetilde{\xi}(x)\right),$$

where $\xi,\widetilde{\xi}\in C^\infty$. The partial derivatives of dual function $\overline{\xi}$ with respect to dual variable $\overline{x}_j$ are

$$\begin{aligned}\frac{\partial\overline{\xi}}{\partial\overline{x}_j} &= \frac{\partial\xi}{\partial x_j}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2\xi}{\partial x_j\partial x_i}+\frac{\partial\widetilde{\xi}}{\partial x_j}\right)\\ &= \frac{\partial\xi}{\partial x_j}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2\xi}{\partial x_i\partial x_j}+\frac{\partial\widetilde{\xi}}{\partial x_j}\right).\end{aligned}$$

The above formula are simplified to the following form

$$\frac{\partial\overline{\xi}}{\partial\overline{x}_j}=\mu\left(x\right)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\mu}{\partial x_i}+\widetilde{\mu}\left(x\right)\right),$$

where $\dfrac{\partial\xi}{\partial x_j}=\mu\left(x\right)$ and $\dfrac{\partial\widetilde{\xi}}{\partial x_j}=\widetilde{\mu}\left(x\right)$, i.e., $\mu,\widetilde{\mu}\in C^{\infty}$. Thus, we deduce that $\dfrac{\partial\overline{\xi}}{\partial\overline{x}_j}\in C\left(\overline{U}\subseteq \mathrm{D}^n,\mathrm{D}\right)$. In analogous to the derivative $\dfrac{\partial\overline{\xi}}{\partial\overline{x}_j}$, the partial derivatives of dual function $\dfrac{\partial\overline{\xi}}{\partial\overline{x}_j}$ with respect to dual variable $\overline{x}_k$ are

$$\begin{aligned}\frac{\partial^2\overline{\xi}}{\partial\overline{x}_k\partial\overline{x}_j} &= \frac{\partial\mu}{\partial x_k}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2\mu}{\partial x_k\partial x_i}+\frac{\partial\widetilde{\mu}}{\partial x_k}\right)\\ &= \frac{\partial\mu}{\partial x_k}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2\mu}{\partial x_i\partial x_k}+\frac{\partial\widetilde{\mu}}{\partial x_k}\right),\end{aligned}$$

where $\dfrac{\partial\mu}{\partial x_k}=\dfrac{\partial^2\xi}{\partial x_k\partial x_j}=\dfrac{\partial^2\xi}{\partial x_j\partial x_k}$ and $\dfrac{\partial\widetilde{\mu}}{\partial x_k}=\dfrac{\partial^2\widetilde{\xi}}{\partial x_k\partial x_j}=\dfrac{\partial^2\widetilde{\xi}}{\partial x_j\partial x_k}$. Therefore, this yields

$$\frac{\partial^2\overline{\xi}}{\partial\overline{x}_k\partial\overline{x}_j}=\frac{\partial^2\xi}{\partial x_j\partial x_k}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial}{\partial x_i}\left(\frac{\partial^2\xi}{\partial x_j\partial x_k}\right)+\frac{\partial^2\widetilde{\xi}}{\partial x_j\partial x_k}\right). \tag{8}$$

On the other hand, it is easy to compute

$$\frac{\partial^2\overline{\xi}}{\partial\overline{x}_j\partial\overline{x}_k}=\frac{\partial^2\xi}{\partial x_j\partial x_k}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial}{\partial x_i}\left(\frac{\partial^2\xi}{\partial x_j\partial x_k}\right)+\frac{\partial^2\widetilde{\xi}}{\partial x_j\partial x_k}\right). \tag{9}$$

Comparing these two equations (8) and (9), we have $\dfrac{\partial^2\overline{\xi}}{\partial\overline{x}_k\partial\overline{x}_j}=\dfrac{\partial^2\overline{\xi}}{\partial\overline{x}_j\partial\overline{x}_k}$. Thus, this achieves the proof. $\square$

Remark 1. *On the set $\mathbb{R}^n\times\mathbb{R}^n=\{(x,x^*)\mid x,x^*\in\mathbb{R}^n\}$, the equality, inner operation and external operation can be defined as follows:*

(i) For any $(x,x^*),(y,y^*)\in\mathbb{R}^n\times\mathbb{R}^n$, we get

$$(x,x^*)=(y,y^*)\Leftrightarrow x=y \text{ and } x^*=y^*.$$

(ii) $+_1:(\mathbb{R}^n\times\mathbb{R}^n)\times(\mathbb{R}^n\times\mathbb{R}^n)\to\mathbb{R}^n\times\mathbb{R}^n$, for $(x,x^*),(y,y^*)\in\mathbb{R}^n\times\mathbb{R}^n$, we get

$$(x,x^*)+_1(y,y^*)=(x+y,x^*+y^*).$$

(iii) $\cdot_1:\mathrm{D}\times(\mathbb{R}^n\times\mathbb{R}^n)\to\mathbb{R}^n\times\mathbb{R}^n$, for $(x,x^*)\in\mathbb{R}^n\times\mathbb{R}^n$ and $\overline{\lambda}=\lambda+\varepsilon\lambda^*\in\mathrm{D}$, we get

$$\overline{\lambda}\cdot_1(x,x^*)=(\lambda x,\lambda x^*+\lambda^*x).$$

According to the above operations, the set $(\mathbb{R}^n \times \mathbb{R}^n, +_1, \cdot_1)$ *constitutes a module over the set* $(\mathrm{D}, +, \cdot)$.

We are now ready to express the following theorem:

Theorem 5. *Let the sets* $(\mathbb{R}^n \times \mathbb{R}^n, +_1, \cdot_1)$ *and* $(\mathrm{D}^n, +, \cdot)$ *be modules over the set* $(\mathrm{D}, +, \cdot)$. *Then the function* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{D}^n$, $f(x, x^*) = x + \varepsilon x^*$ *is a (module) isomorphism.*

Proof. It is easy to check that f is bijective function. Now, for $(x, x^*), (y, y^*) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\overline{\lambda} = \lambda + \varepsilon\lambda^* \in \mathrm{D}$, the following equality can be written

$$\begin{aligned} f\left(\overline{\lambda} \cdot_1 (x, x^*) +_1 (y, y^*)\right) &= f(\lambda x + y, \lambda x^* + \lambda^* x + y^*) \\ &= \lambda x + y + \varepsilon(\lambda x^* + \lambda^* x + y^*) \\ &= (\lambda + \varepsilon\lambda^*)(x + \varepsilon x^*) + (y + \varepsilon y^*) \\ &= \overline{\lambda} f(x, x^*) + f(y, y^*) \end{aligned}$$

such that f is a (module) linear function. In view of these conventions, we deduce that f is a (module) isomorphism. This permits us to conclude the proof. □

Theorem 6. *The real vector space* $\mathbb{R}^n$ *is isomorphic to a subset of* D^n *defined as* $\overline{A} = \left\{ \overrightarrow{\overline{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{0} \mid \overrightarrow{x} \in \mathbb{R}^n \right\}$ *[32].*

Definition 3. *Let* $\{x_1, ..., x_n, x_1^*, ..., x_n^*\}$ *be coordinate functions of* $\mathbb{R}^{2n}$ *and* $\widetilde{p} = (p_1, ..., p_n, p_1^*, ..., p_n^*) \in \mathbb{R}^{2n}$. *Then we have*

$$\widetilde{x}_i = (x_i, x_i^*) : \mathbb{R}^{2n} \to \mathbb{R} \times \mathbb{R},\ \widetilde{x}_i(\widetilde{p}) = (x_i(\widetilde{p}), x_i^*(\widetilde{p})),$$

where $x_i : \mathbb{R}^{2n} \to \mathbb{R}$, $x_i(\widetilde{p}) = p_i$ *and* $x_i^* : \mathbb{R}^{2n} \to \mathbb{R}$, $x_i^*(\widetilde{p}) = p_i^*$. *Since the function* $h_n : \mathrm{D}^n \to \mathbb{R}^{2n}$, $h_n(\overline{p}) = \widetilde{p}$ *is bijective function, we can write the following diagram:*

$$\begin{array}{ccc} \mathrm{D}^n & \xrightarrow{\overline{x}_i} & \mathrm{D} \\ h_n \downarrow & & \downarrow h_1 \\ \mathbb{R}^{2n} & \xrightarrow[\widetilde{x}_i]{} & \mathbb{R} \times \mathbb{R} \end{array}$$

such that dual coordinate functions $\overline{x}_i$ *can be stated by* $\overline{x}_i = h_1^{-1} \circ \widetilde{x}_i \circ h_n$. *Therefore, for dual coordinate functions* $\overline{x}_i$ $(1 \le i \le n)$, *we obtain*

$$\overline{x}_i(\overline{p}) = x_i(\widetilde{p}) + \varepsilon x_i^*(\widetilde{p}) = p_i + \varepsilon p_i^* = \overline{p}_i,$$

where $\overline{p} = (\overline{p}_1, ..., \overline{p}_n) \in \mathrm{D}^n$ *and* $\overline{p}_i = p_i + \varepsilon p_i^* \in \mathrm{D}$.

Definition 4. *Suppose that* $\overline{p} \in \mathrm{D}^n$ *is a dual point and* $\overrightarrow{\overline{x}} \in \mathrm{D}^n$ *is a dual vector. On the set*

$$T_{\overline{p}}\mathrm{D}^n = \{\overline{p}\} \times \mathrm{D}^n = \left\{ \left(\overline{p}, \overrightarrow{\overline{x}}\right) \mid \overrightarrow{\overline{x}} \in \mathrm{D}^n \right\},$$

equality, inner operation and external operation can be determined as follows:

(i) For any $\left(\overline{p}, \overrightarrow{\overline{x}}\right)$ *and* $\left(\overline{q}, \overrightarrow{\overline{y}}\right)$*, we have*

$$\left(\overline{p}, \overrightarrow{\overline{x}}\right) = \left(\overline{q}, \overrightarrow{\overline{y}}\right) \Leftrightarrow \overline{p} = \overline{q} \text{ and } \overrightarrow{\overline{x}} = \overrightarrow{\overline{y}}.$$

(ii) $\oplus : T_{\overline{p}}\mathrm{D}^n \times T_{\overline{p}}\mathrm{D}^n \to T_{\overline{p}}\mathrm{D}^n$*, for* $\left(\overline{p}, \overrightarrow{\overline{x}}\right), \left(\overline{p}, \overrightarrow{\overline{y}}\right) \in T_{\overline{p}}\mathrm{D}^n$*, we have*

$$\left(\overline{p}, \overrightarrow{\overline{x}}\right) \oplus \left(\overline{p}, \overrightarrow{\overline{y}}\right) = \left(\overline{p}, \overrightarrow{\overline{x}} + \overrightarrow{\overline{y}}\right).$$

(iii) $\odot : \mathrm{D} \times T_{\overline{p}}\mathrm{D}^n \to T_{\overline{p}}\mathrm{D}^n$ *for* $\left(\overline{p}, \overrightarrow{\overline{x}}\right) \in T_{\overline{p}}\mathrm{D}^n$ *and* $\overline{\lambda} \in \mathrm{D}$*, we have*

$$\overline{\lambda} \odot \left(\overline{p}, \overrightarrow{\overline{x}}\right) = \left(\overline{p}, \overline{\lambda}\overrightarrow{\overline{x}}\right).$$

Corollary 3. *Taking into account the operations* $\oplus$ *and* $\odot$ *defined on the set* $T_{\overline{p}}\mathrm{D}^n = \{\overline{p}\} \times \mathrm{D}^n = \left\{\left(\overline{p}, \overrightarrow{\overline{x}}\right) \mid \overrightarrow{\overline{x}} \in \mathrm{D}^n\right\}$*, this set generates a module over the set* $(\mathrm{D}, +, \cdot)$*. This module* $(T_{\overline{p}}\mathrm{D}^n, \oplus, (\mathrm{D}, +, .), \odot)$ *is called dual tangent space and every element of this module is entitled by dual tangent vector.*

Corollary 4. *When above defined operations* $\oplus$ *and* $\odot$ *is taken into consideration, every element* $\overrightarrow{\overline{x}}_{\overline{p}} = \left(\overline{p}, \overrightarrow{\overline{x}}\right)$ *of* $T_{\overline{p}}\mathrm{D}^n$ *can be expressed by*

$$\overrightarrow{\overline{x}}_{\overline{p}} = \left(\overline{p}, \overrightarrow{x} + \varepsilon \overrightarrow{0}\right) \oplus \varepsilon \odot \left(\overline{p}, \overrightarrow{x}^* + \varepsilon \overrightarrow{0}\right),$$

where $\overrightarrow{\overline{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x}^* \in \mathrm{D}^n$.

Corollary 5. *Let us define the sets*

$$\Phi = \left\{\left(\overline{p}, \overrightarrow{x} + \varepsilon \overrightarrow{0}\right) \mid \overline{p} \in \mathrm{D}^n, \overrightarrow{x} \in \mathbb{R}^n\right\}$$

and

$$\Psi = \left\{(\widetilde{p}, (x_1, ..., x_n, 0, ..., 0)) \mid \widetilde{p} \in \mathbb{R}^{2n}, x_i \in \mathbb{R}\right\}.$$

The inner operation on the set Φ *(resp.* Ψ*) is*

$$\begin{aligned}
\left(\overline{p}, \overrightarrow{x} + \varepsilon \overrightarrow{0}\right) +_\Phi \left(\overline{p}, \overrightarrow{y} + \varepsilon \overrightarrow{0}\right) &= \left(\overline{p}, \overrightarrow{x} + \overrightarrow{y} + \varepsilon \overrightarrow{0}\right), \\
(\widetilde{p}, (x_1, ..., x_n, 0, ..., 0)) +_\Psi (\widetilde{p}, (y_1, ..., y_n, 0, ..., 0)) &= (\widetilde{p}, (x_1 + y_1, ..., x_n + y_n, 0, ..., 0))
\end{aligned}$$

and for $\lambda \in \mathbb{R}$*, the external operation on the set* Φ *(resp.* Ψ*) is*

$$\begin{aligned}
\lambda \cdot_\Phi \left(\overline{p}, \overrightarrow{x} + \varepsilon \overrightarrow{0}\right) &= \left(\overline{p}, \lambda \overrightarrow{x} + \varepsilon \overrightarrow{0}\right), \\
\lambda \cdot_\Psi (\widetilde{p}, (x_1, ..., x_n, 0, ..., 0)) &= (\widetilde{p}, (\lambda x_1, ..., \lambda x_n, 0, ..., 0))
\end{aligned}$$

such that the sets $(\Phi, +_\Phi, \cdot_\Phi)$ *and* $(\Psi, +_\Psi, \cdot_\Psi)$ *are* n*-dimensional vector spaces over the field* $(\mathbb{R}, +, \cdot)$.

With these conventions, the following theorem can be given:

Theorem 7. *Consider that*

$$\Phi=\left\{\left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)\mid\overline{p}\in\mathrm{D}^{n},\overrightarrow{x}\in\mathbb{R}^{n}\right\}$$

and

$$\Psi=\left\{\left(\widetilde{p},\left(x_{1},...,x_{n},0,...,0\right)\right)\mid\widetilde{p}\in\mathbb{R}^{2n},x_{i}\in\mathbb{R}\right\}.$$

Then the function $g:\left(\Phi,+_{\Phi},\cdot_{\Phi}\right)\rightarrow\left(\Psi,+_{\Psi},\cdot_{\Psi}\right),\ g\left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)=\left(\widetilde{p},\left(x_{1},...,x_{n},0,...,0\right)\right)$ *is a isomorphism.*

Corollary 6. *From Theorem 7 and* $\overrightarrow{x}=\left(x_{1},...,x_{n}\right)\widetilde{=}\left(x_{1},...,x_{n},0,...,0\right)$, *every dual vector* $\overrightarrow{\overline{x}}_{\overline{p}}=\left(\overline{p},\overrightarrow{\overline{x}}\right)\in T_{\overline{p}}\mathrm{D}^{n}$ *can be written as*

$$\begin{aligned}\overrightarrow{\overline{x}}_{\overline{p}}&=\left(\overline{p},\overrightarrow{\overline{x}}\right)\\&=\left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)\oplus\varepsilon\odot\left(\overline{p},\overrightarrow{x}^{*}+\varepsilon\overrightarrow{0}\right)\\&=\left(\widetilde{p},\overrightarrow{x}\right)\oplus\varepsilon\odot\left(\widetilde{p},\overrightarrow{x}^{*}\right)\\&=\overrightarrow{x}_{\widetilde{p}}\oplus\varepsilon\odot\overrightarrow{x}^{*}_{\widetilde{p}}.\end{aligned}$$

For simplicity, throughout this paper, the operations + and · is used instead of the operations ⊕ and ⊙, respectively. Thus, this means that

$$\overrightarrow{\overline{x}}_{\overline{p}}=\overrightarrow{x}_{\widetilde{p}}+\varepsilon\overrightarrow{x}^{*}_{\widetilde{p}}.$$

Also, it is possible to write the following equality:

$$\overrightarrow{\overline{x}}_{\overline{p}}=\left(\overline{p},\overrightarrow{\overline{x}}\right)=\left(\overline{p},\overline{x}_{1}\overrightarrow{e}_{1}+...+\overline{x}_{n}\overrightarrow{e}_{n}\right)=\overline{x}_{1}\overrightarrow{e}_{1\overline{p}}+...+\overline{x}_{n}\overrightarrow{e}_{n\overline{p}},$$

where $\overrightarrow{e}_{i\overline{p}}=\left(\overline{p},\overrightarrow{e}_{i}+\varepsilon\overrightarrow{0}\right)$. *Moreover, the equation* $\overline{\lambda}_{1}\overrightarrow{e}_{1\overline{p}}+...+\overline{\lambda}_{n}\overrightarrow{e}_{n\overline{p}}=\overrightarrow{\overline{0}}_{\overline{p}}$ *can only be satisfied by* $\overline{\lambda}_{i}=\overline{0}$ *for* $1\leq i\leq n$. *Thus, the set* $\left\{\overrightarrow{e}_{1\overline{p}},...,\overrightarrow{e}_{n\overline{p}}\right\}$ *forms a basis of dual tangent space* $T_{\overline{p}}\mathrm{D}^{n}$.

Theorem 8. *Assume that* $\overline{\xi}\in C\left(\overline{U}\subseteq\mathrm{D}^{n},\mathrm{D}\right)$ *and* $\overrightarrow{\overline{x}}_{\overline{p}}\in T_{\overline{p}}\mathrm{D}^{n}$. *The derivative of dual analytic function* $\overline{\xi}$ *in the direction of dual tangent vector* $\overrightarrow{\overline{x}}_{\overline{p}}$ *is*

$$\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\ \right]=\frac{d}{d\overline{t}}\overline{\xi}\left(\overline{p}+\overline{t}\,\overrightarrow{\overline{x}}\right)_{|\overline{t}=\overline{0}}=\overrightarrow{x}_{\widetilde{p}}\left[\xi\right]+\varepsilon\left(\sum_{i=1}^{n}p_{i}^{*}\overrightarrow{x}_{\widetilde{p}}\left[\xi_{x_{i}}\right]+\overrightarrow{x}_{\widetilde{p}}\left[\widetilde{\xi}\right]+\overrightarrow{x}^{*}_{\widetilde{p}}\left[\xi\right]\right),$$

where $\overrightarrow{x}_{\widetilde{p}}\left[\xi\right]=\sum\limits_{i=1}^{n}\dfrac{\partial\xi}{\partial x_{i}}\left(x\left(\widetilde{p}\right)\right)x_{i}$.

Proof. The proof can be easily made using definitions of dual tangent vector and composition of dual analytic functions. □

Theorem 9. *For* $\overline{\xi},\overline{\mu}\in C\left(\overline{U}\subseteq \mathrm{D}^n,\mathrm{D}\right)$, $\overrightarrow{\overline{x}}_{\overline{p}}$, $\overrightarrow{\overline{y}}_{\overline{p}}\in T_{\overline{p}}\mathrm{D}^n$ *and* $\overline{\lambda}\in\mathrm{D}$, *the following equalities exist:*

(i) $\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}+\overline{\mu}\ \right]=\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\ \right]+\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\mu}\ \right]$

(ii) $\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\lambda}\cdot\overline{\xi}\ \right]=\overline{\lambda}\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\ \right]$

(iii) $\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\cdot\overline{\mu}\ \right]=\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\ \right]\overline{\mu}\left(\overline{p}\right)+\overline{\xi}\left(\overline{p}\right)\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\mu}\ \right]$

(iv) $\left(\overrightarrow{\overline{x}}_{\overline{p}}+\overrightarrow{\overline{y}}_{\overline{p}}\right)\left[\ \overline{\xi}\ \right]=\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\ \right]+\overrightarrow{\overline{y}}_{\overline{p}}\left[\ \overline{\xi}\ \right]$.

Definition 5. *A dual vector field* $\overline{X}$ *on* D^n *is a function that assigns to each dual point* $\overline{p}$ *of* D^n *a dual tangent vector* $\overrightarrow{\overline{X}}_{\overline{p}}$ *to* D^n *at* $\overline{p}$, *i.e.,* $\overline{X}:\mathrm{D}^n\to T\mathrm{D}^n$,

$$\overline{X}\left(\overline{p}\right)=\overrightarrow{\overline{X}}_{\overline{p}}=\overrightarrow{X}_{\widetilde{p}}+\varepsilon\overrightarrow{X}^{*}_{\widetilde{p}},$$

where $\overrightarrow{\overline{X}}=\overrightarrow{X}+\varepsilon\overrightarrow{X}^{*}$. *Suppose that* $\overline{a}_i:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D}$, $\overline{a}_i=a_i+\varepsilon a_i^0$ $(1\leq i\leq n)$ *are dual analytic functions. When the dual vector field can be written in the form* $\overline{X}\left(\overline{x}\right)=\left(\overline{a}_1\left(\overline{x}\right),...,\overline{a}_n\left(\overline{x}\right)\right)$, *the equality can be rearranged as follows:*

$$\overline{X}\left(\overline{x}\right)=X\left(x\right)+\varepsilon\left(\sum_{j=1}^{n}x_j^{*}X_{x_j}+\widetilde{X}\left(x\right)\right),$$

where $X\left(x\right)=\left(a_1\left(x\right),...,a_n\left(x\right)\right)$, $\widetilde{X}\left(x\right)=\left(\widetilde{a}_1\left(x\right),...,\widetilde{a}_n\left(x\right)\right)$ *and the functions* a_i *and* $\widetilde{a}_i$ *belong to* $C^{\infty}-$*class for* $1\leq i\leq n$. *The set of dual analytic vector fields is symbolized as* $\chi\left(\mathrm{D}^n\right)$. *Hence, it is possible to write below expression:*

$$\chi\left(\mathrm{D}^n\right)=\left\{\overline{X}\mid\overline{X}:\mathrm{D}^n\to T\mathrm{D}^n,\ \overrightarrow{\overline{X}}_{\overline{p}}=\overrightarrow{X}_{\widetilde{p}}+\varepsilon\overrightarrow{X}^{*}_{\widetilde{p}}\right\}.$$

We are now ready to introduce that the inner and external operations on $\chi\left(\mathrm{D}^n\right)$ *is described as below:*

(i) $+:\chi\left(\mathrm{D}^n\right)\times\chi\left(\mathrm{D}^n\right)\to\chi\left(\mathrm{D}^n\right)$, *for* $\overrightarrow{\overline{X}},\overrightarrow{\overline{Y}}\in\chi\left(\mathrm{D}^n\right)$ *and* $\overline{p}\in\mathrm{D}^n$, *we have*

$$\left(\overline{X}+\overline{Y}\right)\left(\overline{p}\right)=\overrightarrow{\overline{X}}_{\overline{p}}+\overrightarrow{\overline{Y}}_{\overline{p}}.$$

(ii) $\cdot:\mathrm{D}\times\chi\left(\mathrm{D}^n\right)\to\chi\left(\mathrm{D}^n\right)$, *for* $\overrightarrow{\overline{X}}\in\chi\left(\mathrm{D}^n\right)$, $\overline{\lambda}\in\mathrm{D}$ *and* $\overline{p}\in\mathrm{D}^n$, *we have*

$$\left(\overline{\lambda}\cdot\overline{X}\right)\left(\overline{p}\right)=\overline{\lambda}\cdot\overline{X}\left(\overline{p}\right)=\overline{\lambda}\cdot\overrightarrow{\overline{X}}_{\overline{p}}.$$

In view of above mentioned operations, the set $\left(\chi\left(\mathrm{D}^n\right),+,\cdot\right)$ *forms a module over the set* $\left(\mathrm{D},+,\cdot\right)$.

Now, suppose that $\overline{\xi}\in C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)$. The derivative of dual analytic function $\overline{\xi}$ in the direction of dual analytic vector field $\overline{X}$ is

$$\overline{X}\left[\ \overline{\xi}\ \right]=X\left[\xi\right]+\varepsilon\left(\sum_{j=1}^{n}x_j^{*}\left(X\left[\xi\right]\right)_{x_j}+X\left[\widetilde{\xi}\right]+\widetilde{X}\left[\xi\right]\right),$$

where $X[\xi]=\sum\limits_{i=1}^{n}\dfrac{\partial\xi}{\partial x_i}a_i$, such that $\overline{X}\left[\ \overline{\xi}\ \right]\in C\left(\overline{U}\subseteq \mathrm{D}^n,\mathrm{D}\right)$. We can infer that for $\overline{p}\in\mathrm{D}^n$,

$$\overline{X}_{\overline{p}}\left[\ \overline{\xi}\ \right]=X_{\widetilde{p}}[\xi]+\varepsilon\left(\sum_{j=1}^{n}x_j^*(\widetilde{p})\left(X[\xi]\right)_{x_j}(\widetilde{p})+X_{\widetilde{p}}\left[\widetilde{\xi}\right]+\widetilde{X}_{\widetilde{p}}[\xi]\right).$$

Definition 6. *Suppose that $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D}^m$, $\overline{\xi}=\left(\overline{\xi}_1,...,\overline{\xi}_m\right)$ is a dual analytic function. The function $\overline{\xi}_{*\overline{p}}:T_{\overline{p}}\overline{U}\to T_{\overline{\xi}(\overline{p})}\mathrm{D}^m$ is called a dual tangent map of the function $\overline{\xi}$ at the dual point $\overline{p}$, where*

$$\begin{aligned}\overline{\xi}_{*\overline{p}}\left(\overrightarrow{\overline{x}}_{\overline{p}}\right) &= \left(\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}_1\right],...,\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}_m\right]\right)|_{\overline{q}=\overline{\xi}(\overline{p})}\\ &= \xi_{*\widetilde{p}}\left(\overrightarrow{x}_{\widetilde{p}}\right)+\varepsilon\left(\sum_{j=1}^{n}p_j^*\xi_{x_j*\widetilde{p}}\left(\overrightarrow{x}_{\widetilde{p}}\right)+\widetilde{\xi}_{*\widetilde{p}}\left(\overrightarrow{x}_{\widetilde{p}}\right)+\xi_{*\widetilde{p}}\left(\overrightarrow{x}^*_{\widetilde{p}}\right)\right)\end{aligned}$$

and

$$\xi_{*\widetilde{p}}\left(\overrightarrow{x}_{\widetilde{p}}\right)=\left(\overrightarrow{x}_{\widetilde{p}}[\xi_1],...,\overrightarrow{x}_{\widetilde{p}}[\xi_m]\right).$$

In that case, it turns out that $\overline{\xi}_*:\chi\left(\ \mathrm{D}^n\right)\to\chi\left(\mathrm{D}^m\right)$,

$$\overline{\xi}_*\left(\overline{X}\right)=\xi_*(X)+\varepsilon\left(\sum_{j=1}^{n}x_j^*\left(\xi_*(X)\right)_{x_j}+\widetilde{\xi}_*(X)+\xi_*\left(\widetilde{X}\right)\right),$$

where $\xi_*(X)=\left(X[\xi_1],...,X[\xi_m]\right)$.

Theorem 10. *$\overline{\xi}_{*\overline{p}}:T_{\overline{p}}\mathrm{D}^n\to T_{\overline{\xi}(\overline{p})}\mathrm{D}^m$ is a (module) linear map and the matrix (dual Jacobian matrix) corresponding to this linear map with respect to the bases $\left\{\overrightarrow{\overline{e}}_{1\overline{p}},...,\overrightarrow{\overline{e}}_{n\overline{p}}\right\}$ and $\left\{\overrightarrow{\overline{e}}_{1\overline{q}},...,\overrightarrow{\overline{e}}_{m\overline{q}}\right\}$ is*

$$\begin{aligned}\overline{J}\left(\overline{\xi},\overline{p}\right) &= \begin{bmatrix}\frac{\partial\xi_1}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_1}{\partial x_n}(\widetilde{p})\\ \frac{\partial\xi_2}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_2}{\partial x_n}(\widetilde{p})\\ \vdots & \ddots & \vdots\\ \frac{\partial\xi_m}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_m}{\partial x_n}(\widetilde{p})\end{bmatrix}+\varepsilon\begin{bmatrix}\frac{\partial\xi_1^0}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_1^0}{\partial x_n}(\widetilde{p})\\ \frac{\partial\xi_2^0}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_2^0}{\partial x_n}(\widetilde{p})\\ \vdots & \ddots & \vdots\\ \frac{\partial\xi_m^0}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_m^0}{\partial x_n}(\widetilde{p})\end{bmatrix}\\ &= J\left(\xi,\widetilde{p}\right)+\varepsilon J\left(\xi^0,\widetilde{p}\right)\\ &= J\left(\xi,\widetilde{p}\right)+\varepsilon\left(\sum_{j=1}^{n}p_j^*J\left(\xi_{x_j},\widetilde{p}\right)+J\left(\widetilde{\xi},\widetilde{p}\right)\right),\end{aligned}$$

where $\dfrac{\partial\xi_j^0}{\partial x_i}=\sum\limits_{k=1}^{n}x_k^*\dfrac{\partial^2\xi_j}{\partial x_i\partial x_k}+\dfrac{\partial\widetilde{\xi}_j}{\partial x_i}$.

Remark 2. *Assume that $\overline{\xi} : \overline{U} \subseteq \mathrm{D}^n \to \mathrm{D}$ is dual analytic function, where $\overline{U} = U \times \mathbb{R}^n$. Then we know that*

$$\overline{\xi}\left(\overline{x}\right) = \xi\left(x_1, ..., x_n\right) + \varepsilon\left(\sum_{i=1}^{n} x_i^* \frac{\partial \xi}{\partial x_i}\left(x_1, ..., x_n\right) + \widetilde{\xi}\left(x_1, ..., x_n\right)\right).$$

The value of this function at the dual point $\overline{x} = \overline{p}$ is

$$\begin{aligned} \overline{\xi}\left(\overline{p}\right) &= \xi\left(x_1\left(\widetilde{p}\right), ..., x_n\left(\widetilde{p}\right)\right) + \varepsilon\left(\begin{array}{c} \sum_{i=1}^{n} x_i^*\left(\widetilde{p}\right) \frac{\partial \xi}{\partial x_i}\left(x_1\left(\widetilde{p}\right), ..., x_n\left(\widetilde{p}\right)\right) \\ +\widetilde{\xi}\left(x_1\left(\widetilde{p}\right), ..., x_n\left(\widetilde{p}\right)\right) \end{array}\right) \\ &= \xi\left(p_1, ..., p_n\right) + \varepsilon\left(\sum_{i=1}^{n} p_i^* \frac{\partial \xi}{\partial x_i}\left(p_1, ..., p_n\right) + \widetilde{\xi}\left(p_1, ..., p_n\right)\right). \end{aligned}$$

As a result, the functions ξ and $\widetilde{\xi}$ can be reduced to the functions defined from $U \subseteq \mathbb{R}^n$ to $\mathbb{R}$ such that these functions belong to $C^\infty-$class.

Definition 7. *Assume that $\overline{\xi} : \overline{U} \subseteq \mathrm{D} \to \mathrm{D}$, $\overline{\xi}\left(\overline{x}\right) = \xi\left(x\right) + \varepsilon\left(x^* \xi'\left(x\right) + \widetilde{\xi}\left(x\right)\right)$ is a dual analytic function and $\xi'\left(x\right)$ is not zero for all $x \in U \subseteq \mathbb{R}$. If the equality $\overline{\xi}\left(\overline{x}_1\right) = \overline{\xi}\left(\overline{x}_2\right)$ requires the equality $\overline{x}_1 = \overline{x}_2$ for all $\overline{x}_1, \overline{x}_2 \in \overline{U} \subseteq \mathrm{D}$, then the function $\overline{\xi}$ is called injective function.*

Theorem 11. *Assume that $\overline{\xi} : \overline{U} \subseteq \mathrm{D} \to \mathrm{D}$, $\overline{\xi}\left(\overline{x}\right) = \xi\left(x\right) + \varepsilon\left(x^* \xi'\left(x\right) + \widetilde{\xi}\left(x\right)\right)$ is a dual analytic function and $\xi'\left(x\right)$ is not zero for all $x \in U \subseteq \mathbb{R}$. Then ξ is injective function if and only if the dual analytic function $\overline{\xi}$ is injective function.*

Proof. Suppose that $\overline{\xi} : \overline{U} \subseteq \mathrm{D} \to \mathrm{D}$, $\overline{\xi}\left(\overline{x}\right) = \xi\left(x\right) + \varepsilon\left(x^* \xi'\left(x\right) + \widetilde{\xi}\left(x\right)\right)$ is a dual analytic function, $\xi'\left(x\right)$ is not zero for all $x \in U \subseteq \mathbb{R}$ and ξ is injective function. Assume that there exists the equality $\overline{\xi}\left(\overline{x}_1\right) = \overline{\xi}\left(\overline{x}_2\right)$ for all $\overline{x}_1, \overline{x}_2 \in \overline{U} \subseteq \mathrm{D}$. From the definition of dual analytic functions, the following equality can be written:

$$\overline{\xi}\left(\overline{x}_1\right) = \xi\left(x_1\right) + \varepsilon\left(x_1^* \xi'\left(x_1\right) + \widetilde{\xi}\left(x_1\right)\right) = \xi\left(x_2\right) + \varepsilon\left(x_2^* \xi'\left(x_2\right) + \widetilde{\xi}\left(x_2\right)\right) = \overline{\xi}\left(\overline{x}_2\right)$$

which implies

$$\xi\left(x_1\right) = \xi\left(x_2\right)$$

and

$$x_1^* \xi'\left(x_1\right) + \widetilde{\xi}\left(x_1\right) = x_2^* \xi'\left(x_2\right) + \widetilde{\xi}\left(x_2\right).$$

From the hypothesis, since ξ is an injective function, it is clear that $x_1 = x_2$. On the other hand, we have $\widetilde{\xi}\left(x_1\right) = \widetilde{\xi}\left(x_2\right)$, since $\widetilde{\xi}$ is a well-defined function. Hence, since $\xi'\left(x\right)$ is not zero for all $x \in U \subseteq \mathbb{R}$ and $\widetilde{\xi}\left(x_1\right) = \widetilde{\xi}\left(x_2\right)$, it is easily seen that $x_1^* = x_2^*$. That is to say, $\overline{x}_1 = \overline{x}_2$. Therefore, $\overline{\xi}$ is an injective function.

Conversely, we shall prove this part of theorem by means of contrapositive method. Assume that ξ is not an injective function. That is to say, the equality $\xi\left(x_1\right) = \xi\left(x_2\right)$ requires the inequality $x_1 \neq x_2$ for at least $x_1, x_2 \in U \subseteq \mathbb{R}$.

We must show that dual analytic function $\overline{\xi}$ is not an injective function. It is enough to remark that the equality $\overline{\xi}\left(\overline{x}_1\right)=\overline{\xi}\left(\overline{x}_2\right)$ requires $\overline{x}_1 \neq \overline{x}_2$ for at least $\overline{x}_1, \overline{x}_2 \in \overline{U} \subseteq \mathrm{D}$. Suppose that there exists the equality $\overline{\xi}\left(\overline{x}_1\right)=\overline{\xi}\left(\overline{x}_2\right)$ for at least $\overline{x}_1, \overline{x}_2 \in \overline{U} \subseteq \mathrm{D}$. Thus, this gives rise to the relation

$$\overline{\xi}\left(\overline{x}_1\right)=\xi\left(x_1\right)+\varepsilon\left(x_1^* \xi'\left(x_1\right)+\widetilde{\xi}\left(x_1\right)\right)=\xi\left(x_2\right)+\varepsilon\left(x_2^* \xi'\left(x_2\right)+\widetilde{\xi}\left(x_2\right)\right)=\overline{\xi}\left(\overline{x}_2\right),$$

i.e.,

$$\xi\left(x_1\right)=\xi\left(x_2\right)$$

and

$$x_1^* \xi'\left(x_1\right)+\widetilde{\xi}\left(x_1\right)=x_2^* \xi'\left(x_2\right)+\widetilde{\xi}\left(x_2\right).$$

As already known, since ξ is not an injective function, the equality $\overline{\xi}\left(\overline{x}_1\right)=\overline{\xi}\left(\overline{x}_2\right)$ requires the expression $\overline{x}_1 \neq \overline{x}_2$ for at least $\overline{x}_1, \overline{x}_2 \in \overline{U} \subseteq \mathrm{D}$, that is, dual analytic function $\overline{\xi}$ is not an injective function. Therefore, the proof is completed. □

Definition 8. *Assume that* $\overline{\xi}: \overline{U} \subseteq \mathrm{D} \rightarrow \overline{V} \subseteq \mathrm{D}, \overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(x^* \xi'(x)+\widetilde{\xi}(x)\right)$ *is a dual analytic function and* $\xi'(x)$ *is not zero for all* $x \in U \subseteq \mathbb{R}$. *If there exists at least one* $\overline{x}=x+\varepsilon x^* \in \overline{U} \subseteq \mathrm{D}$ *satisfying the equality* $\overline{y}=\overline{\xi}(\overline{x})$ *for all* $\overline{y}=y+\varepsilon y^* \in \overline{V} \subseteq \mathrm{D}$, *then the dual analytic function* $\overline{\xi}$ *is called surjective function. That is to say, if* ξ *is surjective function and there exists* $x^* \in \mathbb{R}$ *satisfying the equality* $x^*=\dfrac{y^*-\widetilde{\xi}(x)}{\xi'(x)}$ *for all* $y^* \in \mathbb{R}$, *then dual analytic function* $\overline{\xi}$ *is called surjective function.*

Definition 9. *Assume that* $\overline{\xi}: \overline{U} \subseteq \mathrm{D} \rightarrow \overline{V} \subseteq \mathrm{D}, \overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(x^* \xi'(x)+\widetilde{\xi}(x)\right)$ *is a dual analytic function and* $\xi'(x)$ *is not zero for all* $x \in U \subseteq \mathbb{R}$. *If dual analytic function* $\overline{\xi}$ *is bijective function, there is only one function* $\overline{\mu}: \overline{V} \subseteq \mathrm{D} \rightarrow \overline{U} \subseteq \mathrm{D}$ *satisfying the equalities* $\left(\overline{\mu} \circ \overline{\xi}\right)(\overline{x})=\overline{I}(\overline{x})$ *and* $\left(\overline{\xi} \circ \overline{\mu}\right)(\overline{y})=\overline{I}(\overline{y})$, *where* $\overline{I}$ *is dual unit function. The function* $\overline{\mu}$ *is called inverse function of dual function* $\overline{\xi}$ *and the inverse function is symbolized as* $\overline{\mu}=\overline{\xi}^{-1}$.

Theorem 12. *Assume that* $\overline{\xi}: \overline{U} \subseteq \mathrm{D} \rightarrow \overline{\xi}\left(\overline{U}\right) \subseteq \mathrm{D}, \overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(x^* \xi'(x)+\widetilde{\xi}(x)\right)$ *is a dual analytic function and* $\xi'(x)$ *is not zero for all* $x \in U \subseteq \mathbb{R}$. *If there exists inverse of dual function* $\overline{\xi}$, *it is expressed as*

$$\overline{\xi}^{-1}(\overline{y})=\xi^{-1}(y)+\varepsilon\left(y^*\left(\xi^{-1}\right)'(y)-\left(\widetilde{\xi} \circ \xi^{-1}\right)(y) \cdot\left(\xi^{-1}\right)'(y)\right),$$

where ξ^{-1} *is inverse of real function* ξ.

Proof. Suppose that $\overline{\xi}: \overline{U} \subseteq \mathrm{D} \rightarrow \overline{\xi}\left(\overline{U}\right) \subseteq \mathrm{D}, \overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(x^* \xi'(x)+\widetilde{\xi}(x)\right)$ is a dual analytic function, $\xi'(x)$ is not zero for all $x \in U \subseteq \mathbb{R}$ and there exists inverse of dual analytic function $\overline{\xi}$. Since the function ξ is bijective function, there is inverse of function ξ, i.e., ξ^{-1} such that $\left(\xi^{-1}\right)'(y)=\dfrac{1}{\xi'(x)}$ for all $x\in U\subseteq\mathbb{R}$. Thus, from the hypothesis, we have

$$\begin{aligned}\left(\overline{\xi}^{-1}\circ\overline{\xi}\right)(\overline{x}) &= \left(\xi^{-1}\circ\xi\right)(x)+\varepsilon\left(\begin{array}{c} x^*\left(\left(\xi^{-1}\circ\xi\right)'(x)\right)+\widetilde{\xi}(x)\left(\xi^{-1}\right)'(\xi(x)) \\ -\widetilde{\xi}\left(\xi^{-1}(\xi(x))\right)\left(\xi^{-1}\right)'(\xi(x))\end{array}\right)\\ &= x+\varepsilon x^*\\ &= \overline{I}(\overline{x}),\end{aligned}$$

where $\overline{I}:\mathrm{D}\to\mathrm{D}$, $\overline{I}(\overline{x})=x+\varepsilon x^*$. Similarly, we get $\left(\overline{\xi}\circ\overline{\xi}^{-1}\right)(\overline{y})=\overline{I}(\overline{y})$. From the definition of equality in functions, we can write

$$\overline{\xi}^{-1}\circ\overline{\xi}=\overline{\xi}\circ\overline{\xi}^{-1}=\overline{I}.$$

Hence, this achieves the proof. $\square$

Corollary 7. *If there exists the inverse of dual analytic function* $\overline{\xi}:\overline{U}\subseteq\mathrm{D}\to\overline{\xi}\left(\overline{U}\right)\subseteq\mathrm{D}, \overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(x^*\xi'(x)+\widetilde{\xi}(x)\right)$, *the inverse function is a dual analytic function expressed as follows:*

$$\overline{\xi}^{-1}(\overline{x})=\xi^{-1}(x)+\varepsilon\left(x^*\left(\xi^{-1}\right)'(x)-\left(\widetilde{\xi}\circ\xi^{-1}\right)(x)\left(\xi^{-1}\right)'(x)\right).$$

The derivative of this dual analytic function with respect to dual variable $\overline{x}$ *is*

$$\frac{d\overline{\xi}^{-1}}{d\overline{x}}=\left(\xi^{-1}\right)'(x)+\varepsilon\left(\begin{array}{c} x^*\left(\xi^{-1}\right)''(x)-\left(\widetilde{\xi}\circ\xi^{-1}\right)(x)\left(\xi^{-1}\right)''(x) \\ -\widetilde{\xi}'\left(\xi^{-1}(x)\right)\left(\left(\xi^{-1}\right)'(x)\right)^2\end{array}\right).$$

Definition 10. *Let* $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\overline{V}\subseteq\mathrm{D}^n$ *be a dual analytic function, where*

$$\begin{aligned}\overline{\xi}(\overline{x}) &= \left(\overline{\xi}_1(\overline{x}),...,\overline{\xi}_n(\overline{x})\right)\\ &= \left(\xi_1(x),...,\xi_n(x)\right)+\varepsilon\left(\left(\sum_{i=1}^n x_i^*\frac{\partial\xi_1}{\partial x_i},...,\sum_{i=1}^n x_i^*\frac{\partial\xi_n}{\partial x_i}\right)+\left(\widetilde{\xi}_1(x),...,\widetilde{\xi}_n(x)\right)\right).\end{aligned}$$

If there exists the inverse function $\overline{\xi}^{-1}$ *being dual analytic function, then the dual analytic function* $\overline{\xi}$ *is called dual diffeomorphism.*

Theorem 13. *Assume that* $\overline{\xi}:\mathrm{D}^n\to\mathrm{D}^n$, $\overline{\xi}(\overline{x})=\left(\overline{\xi}_1(\overline{x}),...,\overline{\xi}_n(\overline{x})\right)=\xi+\varepsilon\xi^0$ *is a dual analytic function. If* $rankJ(\xi,q)=n$ *for all* $q\in U$, *where* $\overline{q}=q+\varepsilon q^*\in\mathrm{D}^n$ *and* U *is open set in terms of standard topology of* $\mathbb{R}^n$, *then there is at least one dual open set* $\overline{U}\in\overline{\tau}$ *in* D^n *covering point* $\overline{q}\in\mathrm{D}^n$ *such that* $\overline{\xi}\mid_{\overline{U}}:\overline{U}\to\overline{\xi}\left(\overline{U}\right)$ *is dual diffeomorphism.*

Proof. Assume that $\overline{\xi}:\mathrm{D}^n\to\mathrm{D}^n$, $\overline{\xi}(\overline{x})=\left(\overline{\xi}_1(\overline{x}),...,\overline{\xi}_n(\overline{x})\right)=\xi+\varepsilon\xi^0$ is a dual analytic function and $rankJ(\xi,q)=n$ for all $q\in U$. We know that the functions ξ and $\widetilde{\xi}$ can be reduced to the functions defined from $U\subseteq\mathbb{R}^n$ to $\mathbb{R}$ such that these functions belong to $C^{\infty}-$class. Hence, the function $\overline{\xi}:\mathrm{D}^n\to\mathrm{D}^n$ can be expressed as

$$\overline{\xi}\left(\overline{x}\right)=\xi\left(x\right)+\varepsilon\left(\sum_{j=1}^{n}x_j^*\frac{\partial\xi}{\partial x_j}\left(x\right)+\widetilde{\xi}\left(x\right)\right).$$

Since the function $\xi:\mathbb{R}^n\to\mathbb{R}^n$ belongs to $C^{\infty}-$class and $rankJ\left(\xi,q\right)=n$ for all $q\in U$, $\xi\mid_U:U\to\xi\left(U\right)$ is a diffeomorphism.
Suppose that there is the equality $\overline{\xi}\left(\overline{p}\right)=\overline{\xi}\left(\overline{q}\right)$ for all $\overline{p},\overline{q}\in\overline{U}\subseteq\mathrm{D}^n$ $\left(p,q\in U\subseteq\mathbb{R}^n\right)$. Hence, we can write

$$\xi\left(p\right)+\varepsilon\left(\sum_{j=1}^{n}p_j^*\frac{\partial\xi}{\partial x_j}\left(p\right)+\widetilde{\xi}\left(p\right)\right)=\xi\left(q\right)+\varepsilon\left(\sum_{j=1}^{n}q_j^*\frac{\partial\xi}{\partial x_j}\left(q\right)+\widetilde{\xi}\left(q\right)\right),$$

which implies

$$\xi\left(p\right)=\xi\left(q\right)$$

and

$$\sum_{j=1}^{n}p_j^*\frac{\partial\xi}{\partial x_j}\left(p\right)+\widetilde{\xi}\left(p\right)=\sum_{j=1}^{n}q_j^*\frac{\partial\xi}{\partial x_j}\left(q\right)+\widetilde{\xi}\left(q\right).\tag{10}$$

Since $\xi\mid_U$ is injective function, we have $p=q$. From the equation (10), we get the following equality:

$$\left(p_1^*-q_1^*\right)\frac{\partial\xi}{\partial x_1}+...+\left(p_n^*-q_n^*\right)\frac{\partial\xi}{\partial x_n}=\left(0,...,0\right).$$

Since the set $\left\{\dfrac{\partial\xi}{\partial x_1},...,\dfrac{\partial\xi}{\partial x_n}\right\}$ is linearly independent, we have $p_i^*=q_i^*$ for $1\leq i\leq n$, i.e., $p^*=q^*$. That is to say, we can write $\overline{p}=p+\varepsilon p^*=q+\varepsilon q^*=\overline{q}$ such that the dual analytic function $\overline{\xi}\mid_{\overline{U}}$ is injective.
Now, let us show that there exists at least one $\overline{q}\in\overline{U}\subseteq\mathrm{D}^n$ satisfying the equality $\overline{p}=\overline{\xi}\left(\overline{q}\right)$ for all $\overline{p}\in\overline{\xi}\left(\overline{U}\right)\subseteq\mathrm{D}^n$. The equality

$$\overline{p}=p+\varepsilon p^*=\xi\left(q\right)+\varepsilon\left(\sum_{j=1}^{n}q_j^*\frac{\partial\xi}{\partial x_j}\left(q\right)+\widetilde{\xi}\left(q\right)\right)=\overline{\xi}\left(\overline{q}\right)$$

allows us to write

$$p=\xi\left(q\right)$$

and

$$p^*=\sum_{j=1}^{n}q_j^*\frac{\partial\xi}{\partial x_j}\left(q\right)+\widetilde{\xi}\left(q\right).\tag{11}$$

Since $\xi\mid_U$ is bijective function, there exists $q=\xi^{-1}\left(p\right)\in U\subseteq\mathbb{R}^n$. Expanding the equation (11), it is seen that the following linear equation system is obtained:

$$\begin{aligned}
q_1^* \frac{\partial \xi_1}{\partial x_1}(q) + \ldots + q_n^* \frac{\partial \xi_1}{\partial x_n}(q) &= p_1^* - \widetilde{\xi}_1(q) \\
q_1^* \frac{\partial \xi_2}{\partial x_1}(q) + \ldots + q_n^* \frac{\partial \xi_2}{\partial x_n}(q) &= p_2^* - \widetilde{\xi}_2(q) \\
&\vdots \\
q_1^* \frac{\partial \xi_n}{\partial x_1}(q) + \ldots + q_n^* \frac{\partial \xi_n}{\partial x_n}(q) &= p_n^* - \widetilde{\xi}_n(q).
\end{aligned}$$

The matrix form of this linear equation system is

$$\begin{bmatrix}
\frac{\partial \xi_1}{\partial x_1}(q) & \frac{\partial \xi_1}{\partial x_2}(q) & \cdots & \frac{\partial \xi_1}{\partial x_n}(q) \\
\frac{\partial \xi_2}{\partial x_1}(q) & \frac{\partial \xi_2}{\partial x_2}(q) & \cdots & \frac{\partial \xi_2}{\partial x_n}(q) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \xi_n}{\partial x_1}(q) & \frac{\partial \xi_n}{\partial x_2}(q) & \cdots & \frac{\partial \xi_n}{\partial x_n}(q)
\end{bmatrix}
\begin{bmatrix} q_1^* \\ q_2^* \\ \vdots \\ q_n^* \end{bmatrix}
=
\begin{bmatrix} p_1^* - \widetilde{\xi}_1(q) \\ p_2^* - \widetilde{\xi}_2(q) \\ \vdots \\ p_n^* - \widetilde{\xi}_n(q) \end{bmatrix}.$$

If we denote $[A]_{n\times n} = \left[\frac{\partial \xi_i}{\partial x_j}(q)\right]_{1\le i,j\le n}$ and $[B]_{n\times 1} = \left[p_i^* - \widetilde{\xi}_i(q)\right]_{1\le i\le n}$, the above matrix form can be rewritten as

$$[A]_{n\times n}[q^*]_{n\times 1} = [B]_{n\times 1}. \tag{12}$$

Since $rankJ(\xi, q) = n$ for all $q \in U \subseteq \mathbb{R}^n$, there exists an inverse of the matrix $[A]_{n\times n}$ such that $[q^*]_{n\times 1} = \left[A^{-1}\right]_{n\times n}[B]_{n\times 1}$. Therefore, there exists dual point $\overline{q} = q + \varepsilon q^* \in \overline{U} \subseteq \mathrm{D}^n$, that is, dual analytic function $\overline{\xi}\mid_{\overline{U}}$ is surjective.
With these conventions, there exists the inverse of dual analytic function $\overline{\xi}\mid_{\overline{U}}$ and this inverse function is $\overline{\mu} : \overline{\xi}\left(\overline{U}\right) \subseteq \mathrm{D}^n \to \overline{U} \subseteq \mathrm{D}^n$,

$$\overline{\mu}(\overline{y}) = \mu(y) + \varepsilon\left(\sum_{j=1}^{n} y_j^* \frac{\partial \mu}{\partial y_j} + \widetilde{\mu}(y)\right) = \mu + \varepsilon\mu^0,$$

where $\mu = (\xi\mid_U)^{-1}$ and $\widetilde{\mu}_i(y) = \left\langle -\widetilde{\xi}(\mu(y)), \nabla\mu_i(y)\right\rangle$ for $1 \le i \le n$. This fact can be verified as follows:

$$\left(\overline{\xi}\mid_{\overline{U}} \circ \overline{\mu}\right)(\overline{y}) = (\xi\mid_U \circ \mu)(y) + \varepsilon
\begin{pmatrix}
y_1^*\left(\frac{\partial \mu_1}{\partial y_1}\frac{\partial \xi}{\partial x_1}(\mu(y)) + \ldots + \frac{\partial \mu_n}{\partial y_1}\frac{\partial \xi}{\partial x_n}(\mu(y))\right) \\
+\ldots+ \\
y_n^*\left(\frac{\partial \mu_1}{\partial y_n}\frac{\partial \xi}{\partial x_1}(\mu(y)) + \ldots + \frac{\partial \mu_n}{\partial y_n}\frac{\partial \xi}{\partial x_n}(\mu(y))\right) \\
+\widetilde{\mu}_1(y)\frac{\partial \xi}{\partial x_1}(\mu(y)) + \ldots + \widetilde{\mu}_n(y)\frac{\partial \xi}{\partial x_n}(\mu(y)) \\
+\widetilde{\xi}\mid_U(\mu(y))
\end{pmatrix}$$

$$\begin{aligned}
&= (\xi\mid_U \circ\mu)(y)+\varepsilon\left(\begin{array}{c} y_1^*\dfrac{\partial(\xi\circ\mu)}{\partial y_1}+...+y_n^*\dfrac{\partial(\xi\circ\mu)}{\partial y_n} \\ -\widetilde{\xi}_1(\mu(y))\dfrac{\partial(\xi\circ\mu)}{\partial y_1}-...-\widetilde{\xi}_n(\mu(y))\dfrac{\partial(\xi\circ\mu)}{\partial y_n} \\ +\widetilde{\xi}\mid_U(\mu(y)) \end{array}\right) \\
&= y+\varepsilon y^* \\
&= \overline{I}(\overline{y}).
\end{aligned}$$

In analogous to $\left(\overline{\xi}\mid_{\overline{U}}\circ\overline{\mu}\right)(\overline{y})=\overline{I}(\overline{y})$, it is easy to check that $\left(\overline{\mu}\circ\overline{\xi}\mid_{\overline{U}}\right)(\overline{x})=\overline{I}(\overline{x})$. On the other hand, the dual function $\overline{\mu}=\left(\overline{\xi}\mid_{\overline{U}}\right)^{-1}$ is a dual analytic function, since $\dfrac{\partial\mu}{\partial y_i^*}=0$ and $\dfrac{\partial\mu^0}{\partial y_i^*}=\dfrac{\partial\mu}{\partial y_i}$ for $1\leq i\leq n$ and the functions μ and $\widetilde{\mu}$ belong to $C^\infty-$class. That is to say, $\overline{\xi}\mid_{\overline{U}}$ is dual diffeomorphism. □

3. Conclusion

The relation between some sets of the topology constituted in dual space and regions where dual analytic functions are analytic is explained in this paper. Besides, we can assert that it is possible to construct the concept of dual surface via the expression of the inverse function theorem in dual space.

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References

[1] Clifford, W. K., Preliminary sketch of biquaternions, *Proc. London Math. Soc.*, 4 (1873), 381–395.
[2] Kandasamy, W. B. V., Smarandache, F., Dual Numbers, ZIP Publishing, Ohio, 2012.
[3] Yaglom, I. M., A Simple Non-Euclidean Geometry and Its Physical Basis, Springer-Verlag, New York, 1979.
[4] Li, S., Ge, Q. J., Rational Beizer line symmetric motions, *Trans. ASME J. Mech. Design*, 127(2) (2005), 222–226. https://doi.org/10.1115/DETC99/DAC-8654
[5] Kotelnikov, A. P., Screw Calculus and Some of Its Applications in Geometry and Mechanics, Annals of the Imperial University, Kazan, 1895.
[6] Study, E., Geometrie der Dynamen, Druck und Verlag von B.G. Teubner, Leipzig, 1903.
[7] Çöken, A. C., Görgülü, A., On the dual darboux rotation axis of the dual space curve, *Demonstratio Math.*, 35(2) (2002), 385–390. https://doi.org/10.1515/dema-2002-0219

[8] Ercan, Z., Yüce, S., On properties of the dual quaternions, *Eur. J. Pure Appl. Math.*, 4(2) (2011), 142–146.
[9] Pennestri, E., Stafenelli, R., Linear algebra and numerical algorithms using dual numbers, *Multibody Syst. Dyn.*, 18(3) (2007), 323–344. https://doi.org/10.1007/s11044-007-9088-9
[10] Soule, C., Rational K-Theory of The Dual Numbers of A ring of Algebraic Integers, Springer Lec. Notes 854, pp. 402-408, 1981.
[11] Veldkamp, G. R., On the use of dual numbers, vectors and matrices in instantaneous spatial kinematics, *Mechanism and Machine Theory 2*, 11 (1976), 141–156. https://doi.org/10.1016/0094-114X(76)90006-9
[12] Yang, A. T., Freudenstein, F., Application of dual number quaternion algebra to the analysis of spatial mechanisms, *Trans. ASME J. Appl. Mech.*, 31(2) (1964), 300–308. https://doi.org/10.1115/1.3629601
[13] Aktaş, B., Surfaces and Some Special Curves on These Surfaces in Dual Space (Turkish), Doctorial Dissertation, Kırıkkale University, Kırıkkale, 2020.
[14] Aktaş, B., Durmaz, O., Gündoğan, H., On the basic structures of dual space, *Facta Universitatis, Series: Mathematics and Informatics*, 35(1) (2020), 253–272. https://doi.org/10.22190/FUMI2001253A
[15] Zembat, İ. Ö., Özmantar, M. F., Bingölbali, E., Şandır, H., Delice, A., Tanımları ve Tarihsel Gelişimleriyle Matematiksel Kavramlar (1.baskı), Ankara:Pegem-Akademi, 2013.
[16] Croom, F. H., Principles of Topology, Saunders College Publishing, 1989.
[17] Richeson, D., Euler's Gem: The Polyhedron Formula and The Birth of Topology, Princeton University Press, 2008.
[18] Mashaghi, S., Jadidi, T., Koenderink, G., Mashaghi, A., Lipid nanotechnology, *Int. J. Mol. Sci.*, 14(2) (2013), 4242–4282. https://doi.org/10.3390/ijms14024242
[19] Adams, C., The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots, American Mathematical Society, 2004.
[20] Stadler, B. M. R., Stadler, P. F., Wagner, G. P., Fontana, W., The topology of the possible: formal spaces underlying patterns of evolutionary change, *Journal of Theoretical Biology*, 213(2) (2001), 241–274. https://doi.org/10.1006/jtbi.2001.2423
[21] Carlsson, G., Topology and data, *Bulletin (New Series) of the American Mathematical Society*, 46(2) (2009), 255–308. https://doi.org/10.1090/S0273-0979-09-01249-X
[22] Vickers, S., Topology via Logic, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 1996.
[23] Stephenson, C., Lyon, D., Hübler, A., Topological properties of a self assembled electrical network via ab initio calculation, *Scientific Reports,* 7 (2017).
[24] Cambou, A. D., Menon, N., Three dimensional structure of a sheet crumpled into a ball,*Proc. Natl. Acad. Sci. U.S.A.*, 108(36) (2011), 14741–14745. https://doi.org/10.1073/pnas.1019192108
[25] Yau, S., Nadis, S., The Shape of Inner Space, Basic Books, 2010.
[26] Craig, J. J., Introduction to Robotics: Mechanics and Control, 3rd Ed. Prentice-Hall, 2004.
[27] Farber, M., Invitation to Topological Robotics, European Mathematical Society, 2008.
[28] Horak, M., Disentangling topological puzzles by using knot theory, *Mathematics Magazine*, 79(5) (2006), 368–375. https://doi.org/10.1080/0025570X.2006.11953435
[29] Eckman, E., Connect the Shapes Crochet Motifs: Creative Techniques for Joining Motfis of all Shapes, Storey Publishing, 2012.
[30] Dimentberg, F. M., The Screw Calculus and its Applications to Mechanics, Foreing Technology Division, Wright-Patterson Air Force Base, Ohio, 1965.
[31] Durmaz, O., Aktaş, B., Gündoğan, H., New approaches on dual space, *Facta Universitatis, Series: Mathematics and Informatics*, 35(2) (2020), 437–458. https://doi.org/10.22190/FUMI2002437D

[32] Hacısalihoğlu, H. H., Hareket Geometrisi ve Kuaterniyonlar Teorisi, Gazi Üniversitesi Fen Fakültesi Yayınları, Ankara, 1983.