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SOME COINCIDENCE BEST PROXIMITY POINT RESULTS IN S-METRIC SPACES

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Abstract. In this paper, we introduce the notions of S-proximal Berinde g-cyclic contraction of two nonself mappings and S-proximal Berinde g-contractions of the first kind and second kind in an S-metric space and prove some coincidence best proximity point theorems for these types of nonself mappings in this space. Also, we give two examples to analyze and support our main results. The results presented here generalize some results in the existing literature.

1. Introduction

Let (X, d) be a metric space and $J : X \to X$ be a self mapping. A fixed point problem is to find a point x in X such that $Jx = x$ or $d(x, Jx) = 0$. In this direction, Banach [1] proved his famous result "Banach contraction principle", which states that "let (X, d) be a complete metric space and $J : X \to X$ be a contraction mapping, then J has a unique fixed point". Later, many authors studied the results dealing with "fixed point" in different spaces (see, e.g., [2]-[7]).

Let (X, d) be a metric space, Y and Z be two nonempty subsets of X and $J : Y \to Z$ be a nonself mapping. A point $x \in Y$ is called a best proximity point of J if $d(x, Jx) = \triangle_{YZ}$ where $\triangle_{YZ} = d(Y, Z) = \inf\{d(x, y) : x \in Y,\ y \in Z\}$. Clearly, if J is a self mapping, then the best proximity point problem reduces to a fixed point problem. In this way, the best proximity point problem can be viewed as a natural generalization of a fixed point problem.

A coincidence best proximity point problem is to find a point x in Y such that $d(gx, Jx) = \triangle_{YZ}$, where g is a self mapping on Y. If g is an identity mapping on Y, then it can be observed that a coincidence best proximity point is essentially a best proximity point. Hence, the coincidence best proximity point problem is an extension of the best proximity point problem. There are several results dealing with proximity point problem in different spaces (see [8]-[12]).

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In 2011, Basha [13] studied and established best proximity point theorems for the proximal contractions of the first kind and second kind, and proximal cyclic contractions in a metric space. More recently, Klanarong and Chaiya [14] presented coincidence best proximity point theorems for the proximal Berinde g-contractions of the first kind and second kind, and proximal Berinde g-cyclic contractions which are more general than the nonself mappings considered in [13].

In 2012, Sedghi et al. [15] introduced the notion of S-metric space and investigated the topology of this space. They also characterized some well-known fixed point results in the context of S-metric space. Later, some authors have published the best proximity point and coincidence best proximity point results on the setting of S-metric space (for details, see [16]-[18]).

Inspired and motivated by the above results, in this paper, we introduce the notions of S-proximal Berinde g-cyclic contractions of two nonself mappings and S-proximal Berinde g-contractions of the first kind and second kind in an S-metric space and establish some coincidence best proximity point theorems for these kinds of nonself mappings in this space. We also give two examples to support our results. The results presented in this paper can be regarded as an extension of corresponding results from a metric space to an S-metric space.

2. Preliminaries and lemmas

In this section, we recall some definitions and lemmas which are needed in the sequel.

The notion of an S-metric space is introduced as a generalization of a metric space as follows.

Definition 2.1. *(see* [15, Definition 2.1]*) Let X be a nonempty set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:*

(S1) $S(x, y, z) \geq 0$;

(S2) $S(x, y, z) = 0$ *if and only if* $x = y = z$;

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

for all $x, y, z, a \in X$. Then the function S is called an S-metric on X, and the pair (X, S) is called an S-metric space.

Some geometric examples for S-metric spaces can be seen in [15].

The following lemma can be considered as the symmetry condition and it will be used in the proofs of some theorems.

Lemma 2.1. *(see* [15, Lemma 2.5]*) Let (X, S) be an S-metric space. Then*

$$S(x, x, y) = S(y, y, x) \quad \text{for all } x, y \in X.$$

We need the following result which can easily be derived from Definition 2.1 and Lemma 2.1.

Lemma 2.2. *(see* [18, Remark 2.6]*) Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \leq S(x, x, y) + 2S(y, y, z) \quad \text{for all } x, y, z \in X.$$

Sedghi et al. [15, 19] defined some basic topological concepts in an S-metric space as follow.

Definition 2.2. *(see* [15, Definition 2.6]*) Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, the open ball $B_S(x, r)$ is defined as follows:*

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\}.$$

Definition 2.3. *(see* [15, Definition 2.8 (3)-(5)]*) Let (X, S) be an S-metric space.*

(i) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$,

(ii) A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n, m \to \infty$. That is, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x) < \varepsilon$ for all $n, m \geq n_0$. We write $x_n \to x$ for brevity.

(iii) The S-metric space (X, S) is called complete if every Cauchy sequence in (X, S) is convergent in (X, S).

Definition 2.4. *(see* [19, Corollary 2.4]*) Let X and $X^{'}$ be two S-metric spaces, and let $f : X \to X'$ be a function. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ for any sequence $\{x_n\}$ in X such that $x_n \to x$. We say that f is continuous on X if f is continuous at every point $x \in X$.*

Özgür and Taş [20, 21] defined the concepts of cluster point and closed set in an S-metric space.

Definition 2.5. *Let (X, S) be an S-metric space and $Y \subseteq X$ be any subset.*

(i) (see [20, Definition 4.2]*) A point $x \in X$ is a cluster point of Y if*

$$(B_S(x, r) - \{x\}) \cap Y \neq \emptyset$$

for every $r > 0$. The set of all cluster points of Y is denoted by $Y_S^{'}$.

(ii) (see [21, Definition 3.3]*) Let (X, S) be an S-metric space and $Y \subseteq X$. The subset Y is called closed if the set of cluster points of Y is contained by Y, that is, $Y_S^{'} \subset Y$.*

Özgür and Taş [21] also defined the concept of sub-S-metric space and gave a property for closed subsets in complete S-metric spaces.

Definition 2.6. *(see* [21, Definition 3.2]*) Let (X, S) be an S-metric space and Y be a nonempty subset of X. Let a function $S_Y : Y \times Y \times Y \to [0, \infty)$ be defined by*

$$S_Y(x, y, z) = S(x, y, z) \quad \text{for all } x, y, z \in Y.$$

Then S_Y is called a reduced S-metric and (Y, S_Y) is called a sub-S-metric space of (X, S).

Proposition 2.3. *(see* [21, Proposition 3.4]*) If (X, S) is a complete S-metric space and Y is a closed set in (X, S), then (Y, S_Y) is complete.*

The relation between a metric and an S-metric was given in [22] as follows.

Lemma 2.4. *(see* [22, Lemma 1.12]*) Let (X, d) be a metric space. Then the following properties are satisfied:*

1) $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.

2) $x_n \to x$ in (X, d) if and only if $x_n \to x$ in (X, S_d).

3) $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}$ is a Cauchy sequence in (X, S_d).

4) (X, d) is complete if and only if (X, S_d) is complete.

In [23], the function S_d was called an S-metric generated by d. We know some examples of an S-metric which are not generated by any metric (see [22, 23] for more details).

On the other hand, Gupta [24] claimed that every S-metric on X defines a metric d_S on X as follows:

$$d_S(x,y) = S(x,x,y) + S(y,y,x), \quad \forall x,y \in X. \tag{2.1}$$

However, the function $d_S(x,y)$ defined in (2.1) does not always a metric because the triangle inequality is not satisfied for all elements of X everywhere (see [23] for more details).

Khanpanuk [18] defined the following concepts in an S-metric space.

Definition 2.7. *(see* [18, Definition 3.4]*) Let (X,S) be an S-metric space. A mapping $g : X \to X$ is called an isometry if*

$$S(gx, gy, gz) = S(x,y,z), \quad \forall x,y,z \in X.$$

Clearly, a self mapping which is an isometry is continuous.

Definition 2.8. *(see* [18, Definition 3.5]*) Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$ be a mapping and $g : Y \to Y$ be an isometry. The mapping J is said to preserve the isometric distance with respect to g if*

$$S(Jgx, Jgy, Jgz) = S(Jx, Jy, Jz) \quad \forall x,y,z \in Y.$$

Klanarong and Chaiya [14] introduced the following new classes of nonself mappings in a metric space.

Definition 2.9. *(see* [14, Definitions 3.2 and 3.4]*) Let (X,d) be a metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$ and $g : Y \to Y$ be mappings. The mapping J is said to be*

(i) a proximal Berinde g-contraction of the first kind if there exist $\alpha \in [0,1)$ and $L_1 \geq 0$ such that

$$\begin{array}{rcl} d(gu_1, Jx_1) & = & d(gu_2, Jx_2) = \triangle_{YZ} \\ & \Longrightarrow & \\ d(gu_1, gu_2) & \leq & \alpha d(gx_1, gx_2) + L_1 \min\{d(gx_1, gu_2), d(gx_2, gu_1)\} \end{array}$$

for all $x_1, x_2, u_1, u_2 \in Y$,

(ii) a proximal Berinde g-contraction of the second kind if there exist $\beta \in [0,1)$ and $L_2 \geq 0$ such that

$$\begin{array}{rcl} d(gu_1, Jx_1) & = & d(gu_2, Jx_2) = \triangle_{YZ} \\ & \Longrightarrow & \\ d(Jgu_1, Jgu_2) & \leq & \beta d(Jgx_1, Jgx_2) + L_2 \min\{d(Jgx_1, Jgu_2), d(Jgx_2, Jgu_1)\} \end{array}$$

for all $x_1, x_2, u_1, u_2 \in Y$.

In the case $L_1 = 0$ (or $L_2 = 0$) and $gx = x$ for all $x \in Y$, it is easy to see that a proximal Berinde g-contraction of the first kind (or the second kind) reduces to proximal contraction of the first kind (or the second kind) which was introduced in [13]. But the converse is not true (see [14, Example 3.3]).

Definition 2.10. *(see* [14, Definition 3.5]*) Let (X,d) be a metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z, T : Y \to Z$ and $g : Y \cup Z \to Y \cup Z$*

be mappings. The pair (J,T) is said to be a proximal Berinde g-cyclic contraction if there exist $\gamma \in [0,1)$ and $L \geq 0$ such that

$$\begin{aligned} d(gu_1, Jx_1) &= d(gu_2, Tx_2) = \triangle_{YZ} \\ &\Longrightarrow \\ d(gu_1, gu_2) &\leq \gamma d(gx_1, gx_2) + (1-\gamma) d(Y,Z) + L d(gx_1, gu_1) \end{aligned}$$

for all $x_1, gu_1 \in Y$ and $x_2, gu_2 \in Z$.

In the case $L = 0$ and $gx = x$ for all $x \in Y \cup Z$, it is easy to see that a proximal Berinde g-cyclic contraction reduces to a proximal cyclic contraction which was introduced in [13].

3. Main Results

Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. We define the following sets:

$$\begin{aligned} \triangle^S_{YZ} &= S(Y,Y,Z) = \inf\{S(x,x,y) : x \in Y, y \in Z\}, \\ Y_0 &= \left\{x \in Y : \text{there exists some } y \in Z \text{ such that } S(x,x,y) = \triangle^S_{YZ}\right\}, \\ Z_0 &= \left\{y \in Z : \text{there exists some } x \in Y \text{ such that } S(x,x,y) = \triangle^S_{YZ}\right\}. \end{aligned}$$

Definition 3.1. *Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$ and $g : Y \to Y$ be mappings. A point $x \in Y$ is said to be a coincidence best proximity point of the pair (g,J) if $S(gx,gx,Jx) = \triangle^S_{YZ}$.*

Note that if g is the identity mapping on Y in Definition 3.1, then the point x is the best proximity point of J.

Definition 3.2. *Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$, $T : Z \to Y$ and $g : Y \cup Z \to Y \cup Z$ be mappings. An element $(x,y) \in Y \times Z$ is called a coincidence best proximity point of the triple (g,J,T) if $(gx,gy) \in Y \times Z$ and $S(gx,gx,Jx) = S(gy,gy,Ty) = S(x,x,y) = \triangle^S_{YZ}$.*

Note that if g is the identity mapping on $Y \cup Z$ in Definition 3.2, then the point x and y is the best proximity point of J and T, respectively.

Now, we introduce the S-proximal Berinde g-contractions of the first kind and second kind in an S-metric space.

Definition 3.3. *Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$ and $g : Y \to Y$ be mappings. The mapping J is said to be*

(i) an S-proximal Berinde g-contraction of the first kind if there exist $\alpha \in [0,1)$ and $L_1 \geq 0$ such that

$$\begin{aligned} S(gu_1, gu_1, Jx_1) &= S(gu_2, gu_2, Jx_2) = \triangle^S_{YZ} \\ &\Longrightarrow \\ S(gu_1, gu_1, gu_2) &\leq \alpha S(gx_1, gx_1, gx_2) \\ &\quad + L_1 \min\{S(gx_1, gx_1, gu_2), S(gx_2, gx_2, gu_1)\} \end{aligned} \tag{3.1}$$

for all $x_1, x_2, u_1, u_2 \in Y$,

(ii) an S-proximal Berinde g-contraction of the second kind if there exist $\beta \in [0,1)$ and $L_2 \geq 0$ such that

$$\begin{aligned} S(gu_1, gu_1, Jx_1) &= S(gu_2, gu_2, Jx_2) = \triangle^S_{YZ} \\ &\Longrightarrow \\ S(Jgu_1, Jgu_1, Jgu_2) &\leq \beta S(Jgx_1, Jgx_1, Jgx_2) + L_2 \min\{S(Jgx_1, Jgx_1, Jgu_2), \\ & \quad S(Jgx_2, Jgx_2, Jgu_1)\} \end{aligned} \tag{3.2}$$

for all $x_1, x_2, u_1, u_2 \in Y$.

Now, we define the S-proximal Berinde g-cyclic contraction in an S-metric space.

Definition 3.4. *Let (X, S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z, T : Z \to Y$ and $g : Y \cup Z \to Y \cup Z$ be mappings. The pair (J, T) is said to be an S-proximal Berinde g-cyclic contraction if there exist $\gamma \in [0, 1)$ and $L \geq 0$ such that*

$$\begin{aligned} S(gu_1, gu_1, Jx_1) &= S(gu_2, gu_2, Tx_2) = \triangle^S_{YZ} \\ &\Longrightarrow \\ S(gu_1, gu_1, gu_2) &\leq \gamma S(gx_1, gx_1, gx_2) + (1-\gamma)\triangle^S_{YZ} + LS(gx_1, gx_1, gu_1) \end{aligned}$$

for all $x_1, gu_1 \in Y$ and $x_2, gu_2 \in Z$.

Next, we give the following coincidence best proximity point result in an S-metric space.

Theorem 3.1. *Let (X, S) be a complete S-metric space and Y, Z be two nonempty closed subsets of X. Let $J : Y \to Z$, $T : Z \to Y$ and $g : Y \cup Z \to Y \cup Z$ satisfy the following conditions:*

(i) J and T are S-proximal Berinde g-contractions of the first kind, i.e., there exist $\alpha, \beta \in [0, 1)$ and $L_1, L_2 \geq 0$ such that J and T satisfy the condition (3.1), respectively;

(ii) $J(Y_0) \subseteq Z_0$ and $T(Z_0) \subseteq Y_0$;

(iii) g is an isometry with $\emptyset \neq Y_0 \subseteq g(Y_0)$ and $Z_0 \subseteq g(Z_0)$.

(iv) The pair (J, T) is an S-proximal Berinde g-cyclic contraction.

Then, there exists a point $x \in Y$ and there exists a point $y \in Z$ such that

$$S(gx, gx, Jx) = S(gy, gy, Ty) = S(x, x, y) = \triangle^S_{YZ}. \tag{3.3}$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \text{for all } n \in \mathbb{N}$$

converges to the element x, and for any fixed $y_0 \in Z_0$, the sequence $\{y_n\}$ defined by

$$S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \text{for all } n \in \mathbb{N}$$

converges to the element y. In addition, if $\alpha + L_1 < 1$ and $\beta + L_2 < 1$, then the there exists unique element x and there exists unique element y which satisfy the equation (3.3).

Proof. Let $x_0 \in Y_0$ be given. Since $J(Y_0) \subseteq Z_0$, $Jx_0 \in Z_0$. Hence there is $z_1 \in Y$ such that $S(z_1, z_1, Jx_0) = \triangle^S_{YZ}$ which implies that $z_1 \in Y_0$. As $Y_0 \subseteq g(Y_0)$, there exists $x_1 \in Y_0$ such that $gx_1 = z_1$, so $S(gx_1, gx_1, Jx_0) = S(z_1, z_1, Jx_0) = \triangle^S_{YZ}$. In a similar way, there is $x_2 \in Y_0$ such that $S(gx_2, gx_2, Jx_1) = \triangle_{YZ}^S$. Inductively, we can construct a sequence $\{x_n\}$ in Y_0 such that

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle_{YZ}^S, \quad \forall n \in \mathbb{N}.$$

Since J is an S-proximal Berinde g-contraction of the first kind, for $x_{n-1}, x_n, x_{n+1} \in Y_0$, $S(gx_n, gx_n, Jx_{n-1}) = S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle_{YZ}^S$ implies that

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &\leq \alpha S(gx_{n-1}, gx_{n-1}, gx_n) \\ &\quad + L_1 \min\{S(gx_{n-1}, gx_{n-1}, gx_{n+1}), S(gx_n, gx_n, gx_n)\} \\ &= \alpha S(gx_{n-1}, gx_{n-1}, gx_n) \end{aligned}$$

for all $n \in \mathbb{N}$. It follows from g being an isometry that

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(gx_n, gx_n, gx_{n+1}) \\ &\leq \alpha S(gx_{n-1}, gx_{n-1}, gx_n) \\ &\leq \alpha^2 S(gx_{n-2}, gx_{n-2}, gx_{n-1}) \\ &\quad \vdots \\ &\leq \alpha^n S(gx_0, gx_0, gx_1) \\ &= \alpha^n S(x_0, x_0, x_1) \end{aligned} \tag{3.4}$$

for all $n \in \mathbb{N}$. Since $\alpha \in [0, 1)$, then we have

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0.$$

For positive integers m and n with $m > n$, it follows that

$$\begin{aligned} & S(x_n, x_n, x_m) \\ \leq\ & 2S(x_{m-1}, x_{m-1}, x_m) + S(x_n, x_n, x_{m-1}) \\ \leq\ & 2S(x_{m-1}, x_{m-1}, x_m) + 2S(x_{m-2}, x_{m-2}, x_{m-1}) + S(x_n, x_n, x_{m-2}) \\ \leq\ & 2S(x_{m-1}, x_{m-1}, x_m) + 2S(x_{m-2}, x_{m-2}, x_{m-1}) + ... + S(x_n, x_n, x_{n+1}). \end{aligned}$$

Now, for $m = n + r; r \geq 1$ and (3.4), we obtain

$$S(x_n, x_n, x_{n+r}) \leq 2\alpha^{n+r-1} S(x_0, x_0, x_1) + 2\alpha^{n+r-2} S(x_0, x_0, x_1) + ... + \alpha^n S(x_0, x_0, x_1).$$

By taking limit as $n \to \infty$, we deduce

$$\lim_{n\to\infty} S(x_n, x_n, x_m) = 0.$$

That is, $\{x_n\}$ is a Cauchy sequence in Y. Since (Y, S_Y) is a complete S-metric space, so there exists $x \in Y$ such that $x_n \to x$ as $n \to \infty$. Similarly, since $T(Z_0) \subseteq Y_0$ and $Z_0 \subseteq g(Z_0)$, there exists a sequence $\{y_n\}$ in Z_0 such that

$$S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle_{YZ}^S, \quad \forall n \in \mathbb{N},$$

and which converges to some element $y \in Z$. Since the pair (J, T) is an S-proximal Berinde g-cyclic contraction and

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle_{YZ}^S = S(gy_{n+1}, gy_{n+1}, Ty_n), \quad \forall n \in \mathbb{N},$$

there exist $\gamma \in [0, 1)$ and $L \geq 0$ such that

$$S(gx_{n+1}, gx_{n+1}, gy_{n+1}) \leq \gamma S(gx_n, gx_n, gy_n) + (1 - \gamma)\triangle_{YZ}^S + LS(gx_n, gx_n, gx_{n+1}).$$

It implies that

$$S(x_{n+1}, x_{n+1}, y_{n+1}) \leq \gamma S(x_n, x_n, y_n) + (1 - \gamma)\triangle_{YZ}^S + LS(x_n, x_n, x_{n+1}).$$

Taking limit as $n \to \infty$, we have

$$S(x,x,y) \leq \gamma S(x,x,y) + (1-\gamma)\triangle^S_{YZ} + LS(x,x,x)$$

yields that

$$S(x,x,y) \leq \triangle^S_{YZ}.$$

Then $S(x,x,y) = \triangle^S_{YZ}$, that is, $x \in Y_0$ and $y \in Z_0$. Since $J(Y_0) \subseteq Z_0$ and $T(Z_0) \subseteq Y_0$, then $Jx \in Z_0$ and $Ty \in Y_0$. Hence there exists $w \in Y_0$ and $z \in Z_0$ such that

$$S(gw,gw,Jx) = \triangle^S_{YZ} = S(gv,gv,Ty)$$

since $Y_0 \subseteq g(Y_0)$ and $Z_0 \subseteq g(Z_0)$. Since J is an S-proximal Berinde g-contraction of the first kind and

$$S(gw,gw,Jx) = S(gx_{n+1},gx_{n+1},Jx_n) = \triangle^S_{YZ},$$

we obtain that

$$\begin{aligned} S(gw,gw,gx_{n+1}) &\leq \alpha S(gx,gx,gx_n) + L_1 \min\{S(gx,gx,gx_{n+1}), S(gx_n,gx_n,gw)\} \\ &= \alpha S(gx,gx,gx_n) + L_1 S(gx,gx,gx_{n+1}) \end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$, by the continuity of g, we get $S(gw,gw,gx) = 0$, and so, $gx = gw$. It is implies that

$$S(gx,gx,Jx) = \triangle^S_{YZ}.$$

Similarly, it is easy to verify that $S(gy,gy,Ty) = \triangle^S_{YZ}$. Thus, we can conclude that

$$S(gx,gx,Jx) = S(gy,gy,Ty) = S(x,x,y) = \triangle^S_{YZ}$$

Therefore, the pair (x,y) is a coincidence best proximity point of the triple (g,J,T). Next, we will show that the pair (x,y) is unique. Suppose that $\alpha + L_1 < 1, \beta + L_2 < 1$ and there exists $x \neq x^* \in Y$ such that

$$S(gx^*,gx^*,Jx^*) = \triangle^S_{YZ}.$$

Since J is an S-proximal Berinde g-contraction of the first kind, it follows that

$$\begin{aligned} S(gx,gx,gx^*) &\leq \alpha S(gx,gx,gx^*) + L_1 \min\{S(gx,gx,gx^*), S(gx^*,gx^*,gx)\} \\ &= (\alpha + L_1)\, S(gx,gx,gx^*). \end{aligned}$$

Since $\alpha + L_1 < 1$, then we have $S(gx,gx,gx^*) = 0$. It follows that $x = x^*$, which implies that there exists a unique $x \in Y$ such that $S(gx,gx,Jx) = \triangle^S_{YZ}$. Similarly, we can show that there exists a unique $y \in Z$ such that $S(gy,gy,Ty) = \triangle^S_{YZ}$. Therefore, the pair (x,y) is the unique coincidence best proximity point of the triple (g,J,T).

Now, we give an example to illustrate Theorem 3.1.

Example 3.1. *Let $(\mathbb{R}^2, d)$ be the Euclidean metric space. Define*

$$S(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}.$$

Then $(\mathbb{R}^2, S)$ is an S-metric space. Let $Y = \{(0,y); -1 \leq y \leq 1\}$ and $Z = \{(1,y); -1 \leq y \leq 1\}$. Then $\triangle^S_{YZ} = 1$, $Y_0 = Y$ and $Z_0 = Z$. Define the mappings $J : Y \to Z, T : Z \to Y$ and $g : Y \cup Z \to Y \cup Z$ by

$$J(0,y) = \left(1, \frac{y}{2}\right), T(1,y) = \left(0, \frac{y}{2}\right) \text{ and } g(x,y) = (x,-y).$$

Clearly, $Y_0 = g(Y_0)$, $Z_0 = g(Z_0), J(Y_0) = \left\{\left(1, \frac{y}{2}\right); -1 \leq y \leq 1\right\} \subset Z_0$, $T(Z_0) = \left\{\left(0, \frac{y}{2}\right); -1 \leq y \leq 1\right\} \subset Y_0$ *and the mapping* g *is an isometry. Obviously, the mappings* J *and* T *are* S*-proximal Berinde* g*-contractions of the first kind and the pair* (J, T) *is an* S*-proximal Berinde* g*-cyclic contraction. Hence, the all conditions of Theorem 3.1 are satisfied and the element* $\{(0,0),(1,0)\}$ *in* $Y \times Z$ *is the unique coincidence best proximity point of the triple* (g, J, T)*.*

If we take $L_1 = 0, L_2 = 0$ and $L = 0$ in Theorem 3.1, then we obtain the following coincidence best proximity theorem.

Theorem 3.2. *Let* X, Y, Z, Y_0, Z_0, J, T *and* g *satisfy the hypotheses of Theorem 3.1. Then, there exists a unique point* $x \in Y$ *and there exists a uniqe point* $y \in Z$ *such that*

$$S(gx, gx, Jx) = S(gy, gy, Ty) = S(x, x, y) = \triangle^S_{YZ}.$$

Moreover, for any fixed $x_0 \in Y_0$*, the sequence* $\{x_n\}$ *defined by*

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element x*, and for any fixed* $y_0 \in Z_0$*, the sequence* $\{y_n\}$ *defined by*

$$S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element y*.*

If we take $gx = x$ for all $x \in Y \cup Z$ in Theorem 3.1, then we immediately obtain the following theorem.

Theorem 3.3. *Let* X, Y, Z, Y_0, Z_0, J *and* T *satisfy the hypotheses of Theorem 3.1. Then, there exists a point* $x \in Y$ *and there exists a point* $y \in Z$ *such that*

$$S(x, x, Jx) = S(y, y, Ty) = S(x, x, y) = \triangle^S_{YZ}.$$

Moreover, for any fixed $x_0 \in Y_0$*, the sequence* $\{x_n\}$ *defined by*

$$S(x_{n+1}, x_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element x*, and for any fixed* $y_0 \in Z_0$*, the sequence* $\{y_n\}$ *defined by*

$$S(y_{n+1}, y_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element y*. In addition, if* $\alpha + L_1 < 1$ *and* $\beta + L_2 < 1$*, then the point* x *and* y *is the unique best proximity point of* J *and* T*, respectively.*

If we suppose that J and T are continuous mappings instead of the condition (iv) in Theorem 3.1, then we obtain the following theorem.

Theorem 3.4. *Let* (X, S) *be a complete* S*-metric space and* Y*,* Z *be two nonempty closed subsets of* X*. Let* $J : Y \to Z$*,* $T : Z \to Y$ *and* $g : Y \cup Z \to Y \cup Z$ *satisfy the following conditions:*

(i) J *and* T *are* S*-proximal Berinde* g*-contractions of the first kind, i.e., there exist* $\alpha, \beta \in [0, 1)$ *and* $L_1, L_2 \geq 0$ *such that* J *and* T *satisfy the condition (3.1), respectively;*

(ii) J *and* T *are continuous mappings such that* $J(Y_0) \subseteq Z_0$ *and* $T(Z_0) \subseteq Y_0$*;*

(iii) g *is an isometry with* $\emptyset \neq Y_0 \subseteq g(Y_0)$ *and* $Z_0 \subseteq g(Z_0)$*.*

Then, there exists a point $x \in Y$ *and there exists a point* $y \in Z$ *such that*

$$S(gx, gx, Jx) = S(gy, gy, Ty) = \triangle^S_{YZ}. \tag{3.5}$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element x, and for any fixed $y_0 \in Z_0$, the sequence $\{y_n\}$ defined by

$$S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element y. In addition, if $\alpha + L_1 < 1$ and $\beta + L_2 < 1$, then the there exists unique element x and there exists unique element y which satisfy the equation (3.5).

Proof. By the proof of Theorem 3.1, we get that the sequences $\{x_n\}$ in Y_0 and $\{y_n\}$ in Z_0 such that

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}. \tag{3.6}$$

converge to some elements $x \in Y$ and $y \in Z$, respectively. Since J, T and g are continuous mappings, then we have that $Jx_n \to Jx, Ty_n \to Ty$ and $gx_{n+1} \to gx, gy_{n+1} \to gy$. Taking limit in (3.6) as $n \to \infty$, we conclude that

$$S(gx, gx, Jx) = S(gy, gy, Ty) = \triangle^S_{YZ}.$$

The proof of uniqueness of the elements x and y follows as in Theorem 3.1.

Next, we establish a coincidence best proximity point result for an S-proximal Berinde g-contraction of the first kind and second kind in an S-metric space.

Theorem 3.5. *Let (X, S) be a complete S-metric space and Y, Z be two nonempty closed subsets of X. Let $J : Y \to Z$ and $g : Y \to Y$ satisfy the following conditions:*

(i) J is an S-proximal Berinde g-contraction of the first kind and second kind, i.e., there exist $\alpha, \beta \in [0, 1)$ and $L_1, L_2 \geq 0$ such that J satisfies the conditions (3.1) and (3.2), respectively;

(ii) J preserves the isometric distance with respect to g and $J(Y_0) \subseteq Z_0$;

(iii) g is an isometry with $\emptyset \neq Y_0 \subseteq g(Y_0)$.

Then, there exists a point $x \in Y$ such that

$$S(gx, gx, Jx) = \triangle^S_{YZ}. \tag{3.7}$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}$$

converges to the element x. In addition, if $\alpha + L_1 < 1$ and $\beta + L_2 < 1$, then the there exists unique element x which satisfy the equation (3.7).

Proof. Following similar arguments to those given in proof of Theorem 3.1, we deduce that the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}$$

is convergent to some $x \in Y$. Since J is an S-proximal Berinde g-contraction of the second kind and preserves the isometric distance with respect to g, then we have

$$\begin{aligned}
& S(Jx_n, Jx_n, Jx_{n+1}) \\
= \;& S(Jgx_n, Jgx_n, Jgx_{n+1}) \\
\leq \;& \beta S(Jgx_{n-1}, Jgx_{n-1}, Jgx_n) \\
& + L_2 \min\{S(Jgx_{n-1}, Jgx_{n-1}, Jgx_{n+1}), S(Jgx_n, Jgx_n, Jgx_n)\} \\
= \;& \beta S(Jgx_{n-1}, Jgx_{n-1}, Jgx_n) \\
= \;& \beta S(Jx_{n-1}, Jx_{n-1}, Jx_n).
\end{aligned}$$

Similarly, in the proof of Theorem 3.1, we can show that $\{Jx_n\}$ is a Cauchy sequence and converges to some element $y \in Z$. Therefore we can conclude that

$$S(gx, gx, y) = \lim_{n\to\infty} S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ},$$

that is, $gx \in Y_0$. Since $Y_0 \subseteq g(Y_0)$, there exists $z \in Y_0$ such that $gx = gz$ and so $S(gx, gx, gz) = 0$. By the fact that g is an isometry, we get $S(x, x, z) = S(gx, gx, gz) = 0$. Hence $x = z \in Y_0$ and so $Jx \in J(Y_0) \subseteq Z_0$. Then there exists $u \in Y_0$ such that

$$S(gu, gu, Jx) = \triangle^S_{YZ}. \tag{3.8}$$

It follows from J being an S-proximal Berinde g-contraction of the first kind that

$$\begin{aligned} S(gu, gu, gx_{n+1}) &\leq \alpha S(gx, gx, gx_n) + L_1 \min\{S(gx, gx, gx_{n+1}), S(gx_n, gx_n, gu)\} \\ &\leq \alpha S(gx, gx, gx_n) + L_1 S(gx, gx, gx_{n+1}) \end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in (3.9), we conclude that $gu = gx$. Therefore, from (3.8), we have

$$S(gx, gx, Jx) = \triangle^S_{YZ},$$

that is, x is a coincidence best proximity point of the pair (g, J). The proof of uniqueness of the element x follows as in Theorem 3.1.

The following example illustrates the preceding coincidence best proximity point theorem.

Example 3.2. *Let $X = \mathbb{R}$ and $S(x, y, z) = \max\{|x-y|, |y-z|, |z-x|\}$. Then $(\mathbb{R}, S)$ is an S-metric space. Let $Y = [-2, 2]$ and $Z = \{-3\} \cup [3, 4]$. Then $\triangle^S_{YZ} = 1$, $Y_0 = \{-2, 2\}$ and $Z_0 = \{-3, 3\}$. Define the mappings $J : Y \to Z$ and $g : Y \to Y$ by*

$$Jx = \begin{cases} 3, & \text{if } x \text{ is rational} \\ 4, & \text{otherwise} \end{cases} \quad \text{and} \quad gx = -x.$$

Clearly, $Y_0 = g(Y_0), J(Y_0) = \{3\} \subset Z_0$ and the mapping g is an isometry. Obviously, the mapping J preserves the isometric distance with respect to g and it is an S-proximal Berinde g-contraction of the first kind and second kind. Thus, the all conditions of Theorem 3.5 are fulfilled and the element -2 in Y is the unique coincidence best proximity point of the pair (g, J).

If we take $L_1 = 0$ and $L_2 = 0$ in Theorem 3.5, then we obtain the following coincidence best proximity theorem.

Theorem 3.6. *Let X, Y, Z, Y_0, Z_0, J and g satisfy the hypotheses of Theorem 3.5. Then, there exists a unique point $x \in Y$ such that*

$$S(gx, gx, Jx) = \triangle^S_{YZ}.$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}$$

converges to the element x.

If we take $gx = x$ for all $x \in Y$ in Theorem 3.5, then we get the following theorem.

Theorem 3.7. *Let X, Y, Z, Y_0, Z_0 and J satisfy the hypotheses of Theorem 3.5. Then, there exists a point $x \in Y$ such that*

$$S(x, x, Jx) = \triangle^S_{YZ}.$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(x_{n+1}, x_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}$$

converges to the element x. In addition, if $\alpha + L_1 < 1$ and $\beta + L_2 < 1$, then the element x is the unique best proximity point of J.

Remark. *Since both the proximal contraction of the first kind and the proximal Berinde g-contraction of the first kind are special cases of the S-proximal Berinde g-contraction of the first kind, Theorems 3.1-3.3 generalize the corresponding results for both the proximal contraction and the proximal Berinde g-contraction of the first kind. Same is the case for Theorems 3.5-3.7 dealing with the S-proximal Berinde g-contraction of the first kind and second kind.*

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