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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 3, Pages 752–758 (2022)
DOI:10.31801/cfsuasmas.1036338
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: December 14, 2021; Accepted: March 16, 2022*

# POWER SERIES METHODS AND STATISTICAL LIMIT SUPERIOR

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Abstract. Given a real bounded sequence $x = (x_j)$ and an infinite matrix $A = (a_{nj})$ the Knopp core theorem is equivalent to study the inequality $\limsup Ax \leq \limsup x$. Recently Fridy and Orhan [6] have considered some variants of this inequality by replacing $\limsup x$ with statistical limit superior $st - \limsup x$. In the present paper we examine similar type of inequalities by employing a power series method $P$, a non-matrix sequence-to-function transformation, in place of $A = (a_{nj})$.

## 1. Introduction

In order to investigate the effect of matrix transformations upon the derived set of a sequence $x = (x_j)$, Knopp [10] introduced the idea of the core of $x$ and proved the well-known Core Theorem. This is equivalent to study the inequality $\limsup Ax \leq \limsup x$ for the finite matrix and bounded sequences $x = (x_j)$ where $Ax := \sum_{j=0}^{\infty} a_{nj}x_j$ ( [12,15]). Based on the recently introduced concept of a statistical cluster point [6], a definition is given for the statistical core by Fridy and Orhan [7]. They have also determined a class of regular matrices for which the inequality $\limsup Ax \leq st - \limsup x$ holds for real bounded sequences.

In the present paper, we consider similar type of inequalities by replacing the sequence to sequence transformation with a power series method which is a sequence to function transformation.

Recall that the core of the sequence $x = (x_j)$ is the closed convex hull of the set of limit points of the sequence $x = (x_j)$.

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2020 *Mathematics Subject Classification.* 40A35, 40G10.
*Keywords.* Natural density, statistical convergence, statistical limit superior, core of a sequence, power series methods.
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Let $(p_j)$ be a non-negative real sequence such that $p_0 > 0$ and the corresponding power series

$$p(t) := \sum_{j=0}^{\infty} p_j t^j$$

has radius of convergence $R$ with $0 < R \leq \infty$.

Let

$$C_P := \left\{ f : (-R, R) \to \mathbb{R} \mid \lim_{0<t\to R^-} \frac{f(t)}{p(t)} \ exists \right\}$$

and

$$C_{P_p} := \left\{ x = (x_j) : p_x(t) := \sum_{j=0}^{\infty} p_j t^j x_j \text{ has radius of convergence } \geq R \text{ and } p_x \in C_p \right\}$$

The functional $P - \lim : C_{P_p} \to \mathbb{R}$ defined by

$$P - \lim x = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j x_j$$

is called a power series method and the sequences $x = (x_j)$ is said to be $P-$ *convergent*. The method $P$ is regular if and only if $\lim\limits_{0<t\to R^-} \frac{p_j t^j}{p(t)} = 0$ for every $j$ (see, e.g. [2]). We note that the Abel method is a particular case of a power series method ( [17]).

From now on we assume that $t \in (0, R)$ and $0 < R \leq \infty$.

In the subsequent sections we give some inequalities by relating $\limsup\limits_{t\to R^-} \frac{p_x(t)}{p(t)}$ to $\limsup x$ and $st$-$\limsup x$. These inequalities are motivated by those of Maddox [2], Orhan [15], and, Fridy and Orhan [7].

## 2. An Inequaility Related to Limit Superior

Let $Q_x(t) := \frac{p_x(t)}{p(t)}$. In this section for real bounded sequences $x = (x_j)$, we consider the inequality

$$\limsup_{t\to R^-} Q_x(t) \leq \limsup_j x_j$$

which may be interpreted as saying that

$$\mathcal{K}{-}core\,\{Q_x(t)\} \subseteq \mathcal{K}{-}core\,\{x\}$$

where $\mathcal{K}{-}core\,\{x\}$ denotes the usual Knopp core of x (see,e.g., [8, p.55]). Let $\ell^\infty$ denote the space of all real bounded sequences and let $L(x) := \limsup\limits_n x_n$ and $l(x) := \liminf\limits_n x_n$. Now we have the following

**Theorem 1.** *For every $x=(x_j)\in\ell^\infty$ we have*

$$\limsup_{t\to R^-} Q_x(t) \le \limsup_j x_j \tag{1}$$

*if and only if $P$ is regular.*

*Proof.* Necessity. Let $x\in c$. Then by $(1)$, we immediately get

$$-\limsup(-x) \le -\limsup_{t\to R^-} Q_{(-x)}(t)$$

Combining this with $(1)$, one can have

$$\liminf x \le \liminf Q_x(t) \le \limsup Q_x(t) \le \limsup x.$$

Since $x\in c$,

$$\lim x = \lim_{t\to R^-} Q_x(t)$$

is obtained, i.e., $P$ is regular.

Conversely, assume that $P$ is regular. Let $x\in\ell^\infty$ and $\varepsilon>0$. Then choose an index $m$ so that $x_j<L(x)+\varepsilon$ whenever $j\ge m$. Hence we have

$$\begin{aligned}\sum_{j=0}^{\infty} p_j t^j x_j &= \sum_{j<m} p_j t^j x_j + \sum_{j\ge m} p_j t^j x_j \\ &\le \|x\| \sum_{j<m} p_j t^j + (L(x)+\varepsilon)\sum_{j=0}^{\infty} p_j t^j.\end{aligned}$$

Multiplying both sides by $\dfrac{1}{p(t)}$ we get

$$\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j x_j \le \frac{\|x\|}{p(t)}\sum_{j<m} p_j t^j + (L(x)+\varepsilon)$$

Taking limit superior as $t\to R^-$ and using the regularity of $P$ one can observe that

$$\limsup_{t\to R^-} Q_x(t) \le L(x)+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary we conclude that (1) holds, which proves the theorem. □

## 3. An Inequality Concerning Statistical Limit Superior

In this section, replacing limit superior by statistical limit superior of a real bounded sequence we prove an inequality.

Following the concepts of statistical convergence and statistical cluster points of a sequence $x=(x_j)$, Fridy and Orhan [7] have introduced the definition of statistical limit superior and inferior.

We first recall some terminology and notation. If $K \subseteq \mathbb{N}_0$ and $K_n := \{k \leq n : k \in K\}$ then $|K_n|$ denotes the cardinality of $K_n$. If the limit $\delta(K) := \lim_n \frac{1}{n+1}|K_n|$ exists, then we say that $K$ has a natural (asymptotic) density. A sequence $x = (x_j)$ is statistically convergent to $L$, denoted $st - \lim x = L$, if for every $\varepsilon > 0$, $\delta(\{j : |x_j - L| \geq \varepsilon\}) = 0$, (see, e.g., [3,5,14,16]).

The number $\gamma$ is called a statistical cluster point of $x = (x_j)$ if for every $\varepsilon > 0$ the set $\{j : |x_j - \gamma| < \varepsilon\}$ does not have density zero ( [6]).

Note that throughout the paper the statement $\delta(K) \neq 0$ means that either $\delta(K) > 0$ or $K$ does not have natural density.

Following [7] we recall the following definitions and results. For a real number sequence $x = (x_j)$ let $B_x$ denote the set:

$$B_x := \{b \in \mathbb{R} : \delta\{j : x_j > b\} \neq 0\};$$

similarly

$$A_x := \{a \in \mathbb{R} : \delta\{j : x_j < a\} \neq 0\}.$$

Then the statistical limit superior of $x$ is given by

$$st - \limsup x := \begin{cases} \sup B_x & , if \;\; B_x \neq \varnothing \\ -\infty & , if \;\; B_x = \varnothing. \end{cases}$$

Also, the statistical limit inferior of $x$ is given by

$$st - \liminf x := \begin{cases} \inf A_x & , if \;\; A_x \neq \varnothing \\ \infty & , if \;\; A_x = \varnothing. \end{cases}$$

If $\beta := st - \limsup x$ is finite, then for every $\varepsilon > 0$, $\delta\{j : x_j > \beta - \varepsilon\} \neq 0$ and $\delta\{j : x_j > \beta + \varepsilon\} = 0$. We also have that $st - \limsup x \leq \limsup x$.

Recall that, by $W_q\,(q > 0)$, we denote the space of all $x = (x_j)$ such that for some $L$,

$$\frac{1}{n+1}\sum_{j=0}^{n}|x_j - L|^q \to 0 \;\;,\;\; (n \to \infty)$$

If $x \in W_q$ then we say that $x$ is strongly Cesàro convergent with index $q$. When $q = 1$ this space is denoted by $W$ and it is called the space of strong Cesàro convergent sequences ( [13]). It is well-known that strong Cesàro convergence and statistical convergence are equivalent on bounded sequences ( [1,3,9]).

In order to prove an inequality relating $Q_x(t)$ to $st - \limsup x$ we need the following result which is an analog of Theorem 1 of Maddox [13] (see also [4,11]).

Note that $P-$density of $E \subseteq \mathbb{N}$ is defined by

$$\delta_P(E) := \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j \in E} p_j t^j$$

whenever the limit exists (see, [18]).

**Theorem 2.** *The power series method $P$ transforms bounded strongly convergent sequences, leaving the strong limit invariant, into the space of convergent sequences if and only if $P$ is regular and for any subset $E \subseteq \mathbb{N}$ with $\delta(E) = 0$ implies that*

$$\delta_P(E) = 0. \tag{2}$$

*Proof.* Sufficiency. Let $x \in \ell^\infty$ and strongly convergent to $L$. In order to prove the sufficiency it is enough to show that

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j |x_j - L| = 0. \tag{3}$$

Let $\varepsilon > 0$ and let $E_\varepsilon := \{j \in \mathbb{N} : |x_j - L| \geq \varepsilon\}$.

Since $x = (x_j)$ bounded and strongly convergent to $L$, it is statistically convergent to $L$ (see [3, 9]). Hence $\delta(E_\varepsilon) = 0$. This implies, by the hypothesis that, $\delta_P(E_\varepsilon) = 0$. From

$$\begin{aligned} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j |x_j - L| &= \frac{1}{p(t)} \sum_{j \in E_\varepsilon} p_j t^j |x_j - L| + \frac{1}{p(t)} \sum_{j \in E_\varepsilon^c} p_j t^j |x_j - L| \\ &\leq \sup_j |x_j - L| \frac{1}{p(t)} \sum_{j \in E_\varepsilon} p_j t^j + \varepsilon, \end{aligned}$$

we have

$$\begin{aligned} \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j |x_j - L| &\leq \|x - Le\|_\infty \frac{1}{p(t)} \sum_{j \in E_\varepsilon} p_j t^j + \varepsilon \\ &\leq \varepsilon \end{aligned}$$

because

$$\delta_P(E_\varepsilon)) := \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j \in E_\varepsilon} p_j t^j = 0.$$

We obtain that (3) is true.

Necessity. Note that any convergent sequence is statistically convergent to the same value. Since statistical convergence and strong Cesàro convergence are equivalent on the space of bounded sequences, we observe that P is regular. Assume now that there is a subset $E \subseteq \mathbb{N}$ with $\delta(E) = 0$ such that (2) fails. This implies that $E$ is an infinite set.

So we may write $E = \{k_j : j \in \mathbb{N}\} = \{k_1, k_2, ...\}$. Since the continuous method is regular the corresponding matrix method is also regular. Hence by the Schur theorem there exists a bounded sequences $z = \left(z_{k_1}, z_{k_2}, ...z_{k_j}, ...\right)$ which is not summable by the regular matrix method. Now define a bounded sequence, $x = (x_k)$ as follows: $x_k = z_k$ if $k = k_j$ $(j = 0, 1, 2, ...)$ and $x_k = 0$ otherwise. Since $\delta(E) = 0$, it follows from the fact that

$$\frac{1}{n+1}\sum_{k=0}^{n}|x_k-0| = \frac{1}{n+1}\sum_{k=0}^{n}|x_k| \leq \sup_k |x_k| \frac{1}{n+1}\sum_{k=0}^{n}\chi_E(k) \to 0,\ (n\to\infty)$$

i.e., the sequence $x=(x_k)$ is a bounded statistically convergent sequence which is not summable by the regular discrete method. So it is not summable by the continuous method either. This contradicts the hypothesis. □

In the rest of the paper we use the following notation:

$$\alpha(x) := st-\liminf x \text{ and } \beta(x) := st-\limsup x$$

**Theorem 3.** *For every $x=(x_k)\in\ell^\infty$ we have*

$$\limsup_{t\to R^-} Q_x(t) \leq st-\limsup x \tag{4}$$

*if and only if $P$ is regular and that* (2) *holds.*

*Proof.* Let $x\in\ell^\infty$. Suppose that (4) holds. Since $\beta(x)\leq\limsup x$ it follows from (4) and Theorem 1 that $P$ is regular. On the other hand (4) implies that

$$-\beta(-x) \leq \liminf_{t\to R^-} Q_x(t) \leq \limsup_{t\to R^-} Q_x(t) \leq \beta(x). \tag{5}$$

If $x=(x_k)$ is a bounded statistically convergent sequence, (5) implies that

$$P-\lim x = st-\lim x.$$

Hence by Theorem 2, we observe that (2) holds.

Conversely, assume $P$ is regular and (2) holds. Let $x$ be bounded. Then $\beta(x)$ is finite. Given $\varepsilon>0$ let $E:=\{k\in\mathbb{N}: x_j>\beta(x)+\varepsilon\}$. Hence $\delta(E)=0$ and if $k\notin E$ then $x_j\leq\beta(x)+\varepsilon$.

For a fixed positive integer $m$ we write

$$\begin{aligned} Q_x(t) &= \frac{1}{p(t)}\sum_{j<m}p_jt^jx_j + \frac{1}{p(t)}\sum_{j\geq m}p_jt^jx_j \\ &\leq \|x\|\frac{1}{p(t)}\sum_{j<m}p_jt^j + \frac{1}{p(t)}\sum_{\substack{j\geq m\\ j\notin E}}p_jt^jx_j + \frac{1}{p(t)}\sum_{\substack{j\geq m\\ j\in E}}p_jt^jx_j \\ &\leq \|x\|\frac{1}{p(t)}\sum_{j<m}p_jt^j + (\beta(x)+\varepsilon)\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j + \|x\|\frac{1}{p(t)}\sum_{j\in E}p_jt^j \end{aligned}$$

Taking the limit superior as $t\to R^-$ and using the regularity of $P$ we get that

$$\limsup_{t\to R^-} Q_x(t) \leq (\beta(x)+\varepsilon) + \|x\|\,\delta_P(E).$$

Recall that $\delta_P(E) = 0$ by (2). Since $\varepsilon$ is arbitrary we conclude that (4) holds. This proves the theorem. □

**Declaration of Competing Interests** The author declares that there are no conflicts of interest about the publication of this paper.

**Acknowledgements** The author thanks the referees for their valuable suggestions which made this paper much improved.

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