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On the dynamical behaviors and periodicity of difference equation of order three

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Abstract — The major target of our research paper is to demonstrate the boundedness, stability and periodicity of the solutions of the following third- order difference equation

$$w_{n+1} = \alpha w_n + \frac{\beta + \gamma w_{n-2}}{\delta + \zeta w_{n-2}}, \quad n = 0, 1, 2, \ldots$$

where w_{-2}, w_{-1}, and w_0 are arbitrary real numbers and the values α, β, γ, δ, and ζ are defined as positive constants.

Subject Classification (2020): 39A10, 39A11.

1. Introduction

Difference equations are basis in many fields of life which model several varied phenomena in ecology, biology, physics, engineering, etc. since have an important status in applied sciences. Recently, researchers have concentrated on studying the behaviors of rational difference equations of order greater than one which deserves further consideration. The study of the behaviors of difference equations of a higher order is quite challenging and valuable due to the importance of rational difference equations and its applications. Moreover, there are many recent published research paper in this area. For examples, Alayachi et al. [1] explored the qualitative behavior of the solutions of the following recursive equation

$$Y_{n+1} = AY_{n-1} + \frac{BY_{n-1}Y_{n-3}}{CY_{n-3} + DY_{n-5}}$$

In [2], Elabbasy et al. studied the periodicity of solutions, boundedness and stability of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}$$

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In [3], they analyzed some behaviors of the solution of the following equation

$$x_{n+1} = \frac{ax_{n-1} + bx_{n-2}}{c + dx_{n-1}x_{n-2}}$$

Alayachi et al.[4], gave a description about the stability and periodicity of the following difference equation:

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}$$

Almatrafi et al. [5], investigated the following difference equation:

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}$$

Elsayed et al. [6] researched the global stability, periodicity character of the following second order difference equation

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}}$$

The dynamical behaviors of the following difference equation

$$U_{n+1} = \zeta U_{n-8} + \frac{\epsilon U_{n-8}^2}{\mu U_{n-8} + kU_{n-17}}$$

was investigated by Alshareef et al. in [7].

Also, the authors in [8] considered the stability, periodicity character of the following third order rational difference equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e + fx_{n-2}}$$

Aloqeili [9] has described the behaviors of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}$$

For more linked results on this side can be found in [10–27].

The purpose of this research paper is to investigate the following new rational difference equation

$$w_{n+1} = \alpha w_n + \frac{\beta + \gamma w_{n-2}}{\delta + \zeta w_{n-2}}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

where w_{-2}, w_{-1}, and w_0 are arbitrary real numbers and the values α, β, γ, δ, and ζ are defined as positive constants.

2. Some Basic Theorems

In this part, we recall some basic theorems that we use in this paper.

Theorem 2.1. [5] Suppose that $p_i \in R$, $i = 1, 2, \ldots$ and $k \in \{0, 1, 2, \ldots\}$. Then,

$$\sum_{i=1}^{k} |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$w_{n+k} + p_1 w_{n+k-1} + \cdots + p_k w_n = 0, \quad n = 0, 1, 2, \ldots$$

The next theorem will be useful to prove the global attractor of the fixed point.

Theorem 2.2. [5] Suppose that $g : [a,b]^{l+1} \to [a,b]$ be a continuous function, where l is a positive integer and $[a,b]$ be an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-l}), \;\; n = 0, 1, \ldots$$

Assume that g satisfying the following :

(1) For each integer i with $1 \le i \le l+1$, the function $g(z_1, z_2, \ldots, z_{l+1})$ is weakly monotonic in z_i for each $z_1, z_2, \ldots, z_{l+1}$.

(2) If (m, M) be a solution of the system

$$m = g(m_1, m_2, \ldots, m_{l+1}) \quad \text{and} \quad M = g(M_1, M_2, \ldots, M_{l+1})$$

then, for each $i = 1, 2, \ldots, l+1$

$$m = M$$

Then, there exists a unique equilibrium point $\overline{x}$ and every solution converges to $\overline{x}$.

3. Local Stability of Equation (1.1)

This section is devoted to discuss the local stability of the solution of Equation (1.1).

Equation (1.1) has the unique equilibrium point which given by

$$\overline{w} = \alpha \overline{w} + \frac{\beta + \gamma \overline{w}}{\delta + \zeta \overline{w}}$$

or

$$\zeta(1-\alpha)\overline{w}^2 + (\delta - \delta\alpha - \gamma)\overline{w} - \beta = 0$$

Then, the unique equilibrium point is

$$\overline{w} = \frac{(\gamma - \delta + \delta\alpha) + \sqrt{(\gamma - \delta + \delta\alpha)^2 + 4\beta\zeta(1-\alpha)}}{2\zeta(1-\alpha)}$$

Theorem 3.1. The equilibrium point $\overline{w}$ is locally asymptotically stable if and only if

$$(\delta - \zeta\overline{w})^2 > \frac{|\gamma\delta - \beta\zeta|}{(1-\alpha)}, \quad \alpha < 1$$

Proof.

Let us define the function $f : (0,\infty)^2 \longrightarrow (0,\infty)$ as

$$f(u,v) = \alpha u + \frac{\beta + \gamma v}{\delta + \zeta v}$$

Thus,

$$\frac{\partial f(u,v)}{\partial u} = \alpha \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = \frac{\gamma\delta - \beta\zeta}{(\delta + \zeta v)^2}$$

So, we see that at $\overline{w}$,

$$\begin{aligned} \frac{\partial f(\overline{w},\overline{w})}{\partial u} &= \alpha = p_0 \\ \frac{\partial f(\overline{w},\overline{w})}{\partial u} &= \frac{\gamma\delta - \beta\zeta}{(\delta + \zeta\overline{w})^2} = p_1 \end{aligned}$$

Then, by linearization of Equation (1.1) around $\overline{w}$ we have

$$y_{n+1} - p_0 y_n - p_1 y_{n-2} = 0$$

By Theorem 2.1, Equation (1.1) is asymptotically stable if and only if

$$|p_0| + |p_1| < 1$$

Thus,

$$|\alpha| + \left|\frac{\gamma\delta - \beta\zeta}{(\delta + \zeta\overline{w})^2}\right| < 1$$

and so,

$$\left|\frac{\gamma\delta - \beta\zeta}{(\delta + \zeta\overline{w})^2}\right| < 1 - \alpha, \qquad \alpha < 1$$

or

$$\frac{|\gamma\delta - \beta\zeta|}{|1 - \alpha|} < (\delta + \zeta\overline{w})^2, \qquad \alpha < 1$$

which proved the required.

4. Boundedness of Solutions

In this part, the discussion centres on the boundedness of solutions of Equation (1.1). The following theorems provide the bounded and unbounded solution under a specific condition.

Theorem 4.1. Every solution of Equation (1.1) is bounded if $\alpha < 1$.

Proof.
Let $\{w_n\}_{n=-2}^{\infty}$ be a solution of Equation (1.1). From Equation (1.1) we note that

$$\begin{aligned} w_{n+1} &= \alpha w_n + \frac{\beta + \gamma w_{n-2}}{\delta + \zeta w_{n-2}} \\ &= \alpha w_n + \frac{\beta}{\delta + \zeta w_{n-2}} + \frac{\gamma w_{n-2}}{\delta + \zeta w_{n-2}} \end{aligned}$$

Then,

$$\begin{aligned} w_{n+1} &\leq \alpha w_n + \frac{\beta}{\delta} + \frac{\gamma w_{n-2}}{\zeta w_{n-2}} \\ &= \alpha w_n + \frac{\beta}{\delta} + \frac{\gamma}{\zeta}, \qquad \text{for all} \quad n \geq 2 \end{aligned}$$

In order to handle the right hand side by using a comparison we have that

$$z_{n+1} = \alpha z_n + \frac{\beta}{\delta} + \frac{\gamma}{\zeta}$$

so ,we can write

$$z_n = \alpha^n z_0 + \text{constant}$$

and this equation is locally asymptotically stable since $\alpha < 1$ and converges to the equilibrium point $\overline{z} = \frac{\beta\zeta+\gamma\delta}{\delta\zeta(1-\alpha)}$.

Therefore,

$$\lim_{n\to\infty} \sup\ w_n \leq \frac{\beta\zeta+\gamma\delta}{\delta\zeta(1-\alpha)}$$

which gives solution is bounded.

Theorem 4.2. Every solution of Equation (1.1) is unbounded if $\alpha > 1$.

Proof.

Let $\{w_n\}_{n=-2}^{\infty}$ be a solution of Equation (1.1). From Equation (1.1) we note that

$$w_{n+1} = \alpha w_n + \frac{\beta + \gamma w_{n-2}}{\delta + \zeta w_{n-2}} > \alpha w_n, \quad \text{for all} \quad n \geq 2$$

We can write the right hand side as

$$z_{n+1} = \alpha z_n \ \Rightarrow \ z_n = \alpha^n z_0$$

and this equation is unstable since $\alpha > 1$ and $\lim_{n\to\infty} z_n = \infty$. Then, $\{w_n\}_{n=-2}^{\infty}$ is unbounded from above where we have used ratio test.

5. Periodicity of the Solution

The existence of periodic solutions of Equation (1.1) is deeply investigated in this section. The next theorem confirms that our equation has periodic solutions of prime period two under necessary conditions.

Theorem 5.1. Equation (1.1) has a period two solution if and only if

$$(1+\alpha)^2\zeta^2(\delta+\alpha\delta+\gamma)^2 - 4\alpha(1+\alpha)\zeta^2[\delta(\delta+\alpha\delta+\gamma)+\alpha\beta\zeta] > 0 \tag{5.1}$$

Proof.

We claim that there exists a period two solution

$$\ldots, p, q, p, q, \ldots$$

of Equation (1.1), and we prove that condition (5.1) holds.

From Equation (1.1), we have

$$p = \alpha q + \frac{\beta + \gamma q}{\delta + \zeta q}$$

and

$$q = \alpha p + \frac{\beta + \gamma p}{\delta + \zeta p}$$

Therefore,

$$\delta p + \zeta pq = \alpha\delta q + \alpha\zeta q^2 + \beta + \gamma q \tag{5.2}$$

and

$$\delta q + \zeta pq = \alpha\delta p + \alpha\zeta p^2 + \beta + \gamma p \tag{5.3}$$

Subtracting (5.3) from (5.2) we get

$$\delta(p-q) = -\alpha\delta(p-q) - \alpha\zeta(p^2-q^2) - \gamma(p-q)$$

So,

$$\delta = -\alpha\delta - \alpha\zeta(p+q) - \gamma$$

Indeed $p \neq q$, it gives that

$$p + q = \frac{-(\delta + \alpha\delta + \gamma)}{\alpha\zeta} \tag{5.4}$$

Again, adding (5.2) and (5.3) gives

$$\delta(p+q) + 2\zeta pq = \alpha\delta(p+q) + \alpha\zeta(p^2+q^2) + 2\beta + \gamma(p+q)$$

$$2\zeta pq = \alpha\zeta(p^2+q^2) + (\alpha\delta + \gamma - \delta)(p+q) + 2\beta \tag{5.5}$$

By using this relation

$$p^2 + q^2 = (p+q)^2 - 2pq \quad \text{for all } \; p, q \in R$$

and from (5.4) ,(5.5) yields

$$2\zeta pq = \alpha\zeta((p+q)^2 - 2pq) + \frac{(\alpha\delta + \gamma - \delta)(-\delta - \alpha\delta - \gamma)}{\alpha\zeta} + 2\beta$$

that is

$$2\zeta(1+\alpha)pq = \alpha\zeta\frac{(\delta + \alpha\delta + \gamma)^2}{\alpha^2\zeta^2} + \frac{(\alpha\delta + \gamma - \delta)(-\delta - \alpha\delta - \gamma)}{\alpha\zeta} + 2\beta$$

And

$$2\alpha\zeta^2(1+\alpha)pq = 2\delta^2 + 2\alpha\delta^2 + 2\delta\gamma + 2\alpha\beta\zeta$$

$$\alpha\zeta^2(1+\alpha)pq = \delta(\delta + \alpha\delta + \gamma) + \alpha\beta\zeta$$

Then,

$$pq = \frac{\delta(\delta + \alpha\delta + \gamma) + \alpha\beta\zeta}{\alpha\zeta^2(1+\alpha)} \tag{5.6}$$

From Equations (5.4) and (5.6), clearly that p and q are the two distinct roots of the following quadratic equation

$$t^2 - (p+q)t + pq = 0$$

So,

$$t^2 + \frac{(\delta + \alpha\delta + \gamma)}{\alpha\zeta}t + \left(\frac{\delta(\delta + \alpha\delta + \gamma) + \alpha\beta\zeta}{\alpha\zeta^2(1+\alpha)}\right) = 0$$

that is,

$$\alpha\zeta^2(1+\alpha)t^2+(1+\alpha)\zeta(\delta+\alpha\delta+\gamma)t+\left[\delta(\delta+\alpha\delta+\gamma)+\alpha\beta\zeta\right]=0 \tag{5.7}$$

and so

$$(1+\alpha)^2\zeta^2(\delta+\alpha\delta+\gamma)^2>4\alpha\zeta^2(1+\alpha)[\delta(\delta+\alpha\delta+\gamma)+\alpha\beta\zeta]$$

or

$$(1+\alpha)^2\zeta^2(\delta+\alpha\delta+\gamma)^2-4\alpha\zeta^2(1+\alpha)[\delta(\delta+\alpha\delta+\gamma)+\alpha\beta\zeta]>0$$

Therefore, condition (5.1) holds.

On the contrary side, assume that condition (5.1) is true. We shall prove that Equation (1.1) has a prime period two solution.

Suppose

$$p=\frac{-(1+\alpha)\zeta(\delta+\alpha\delta+\gamma)+\lambda}{2\alpha\zeta(1+\alpha)}$$

and

$$q=\frac{-(1+\alpha)\zeta(\delta+\alpha\delta+\gamma)-\lambda}{2\alpha\zeta(1+\alpha)}$$

where $\lambda=\sqrt{(1+\alpha)^2\zeta^2(\delta+\alpha\delta+\gamma)^2-4\alpha\zeta^2(1+\alpha)\left[\delta(\delta+\alpha\delta+\gamma)+\alpha\beta\zeta\right]}$

We see from condition (5.1) that

$$(1+\alpha)^2\zeta^2(\delta+\alpha\delta+\gamma)^2-4\alpha\zeta^2(1+\alpha)[\delta(\delta+\alpha\delta+\gamma)+\alpha\beta\zeta]>0$$

which equivalents to

$$(1+\alpha)^2\zeta^2(\delta+\alpha\delta+\gamma)^2>4\alpha\zeta^2(1+\alpha)[\delta(\delta+\alpha\delta+\gamma)+\alpha\beta\zeta]$$

Therefore p and q are distinct real numbers.

Put

$$w_{-2}=p,\quad w_{-1}=q\quad\text{and}\quad w_0=p$$

We wish to get that

$$w_1=w_{-1}=q\quad\text{and}\quad w_2=w_0=p$$

From Equation (1.1) we have that

$$w_1=\alpha p+\frac{\beta+\gamma p}{\delta+\zeta p}=\frac{-\alpha(1+\alpha)\zeta(\delta+\alpha\delta+\gamma)+\lambda}{2\alpha\zeta(1+\alpha)}+\frac{\beta+\frac{-(1+\alpha)\gamma\zeta(\delta+\alpha\delta+\gamma)+\lambda}{2\alpha\zeta(1+\alpha)}}{\delta+\frac{-(1+\alpha)\zeta^2(\delta+\alpha\delta+\gamma)+\lambda}{2\alpha\zeta(1+\alpha)}}$$

Multiplying the denominator and numerator by $2\alpha\zeta(1+\alpha)$ we get

$$w_1=-\alpha(1+\alpha)\zeta(\delta+\alpha\delta+\gamma)+\lambda+\frac{2\alpha\beta\zeta(1+\alpha)-(1+\alpha)\gamma\zeta(\delta+\alpha\delta+\gamma)+\lambda}{2\alpha\delta\zeta(1+\alpha)-(1+\alpha)\zeta^2(\delta+\alpha\delta+\gamma)+\lambda}$$

We can get by simple computations that

$$w_1=\frac{-(1+\alpha)\zeta(\delta+\alpha\delta+\gamma)-\lambda}{2\alpha\zeta(1+\alpha)}=q$$

Similar way as before, we can conclude that $w_2 = p$. So, by induction we get that

$$w_{2n} = p \quad \text{and} \quad w_{2n+1} = q, \quad \text{for all} \quad n \geq -2$$

Hence, Equation (1.1) has the prime period two solutions

$$\ldots, p, q, p, q, \ldots$$

This completes the proof of theorem.

6. Global Attractivity Results

This section is devoted to investigate the global asymptotic stability of Equation (1.1).

Theorem 6.1. If one of the following statements holds

$$(*) \;\; \gamma\delta \geq \beta\zeta \quad \text{and} \quad \gamma > \delta(1-\alpha), \, \alpha < 1 \tag{6.1}$$

$$(**) \;\; \gamma\delta \leq \beta\zeta \quad \text{and} \quad \alpha < 1 \tag{6.2}$$

then equilibrium point $\overline{w}$ is a global attractor of Equation (1.1).

Proof.
Suppose that a and b are real numbers and assume that $g : (a, b)^2 \longrightarrow (a, b)$ is a function defined as

$$g = \alpha u + \frac{\beta + \gamma v}{\delta + \zeta v}$$

Then, we have

$$\frac{\partial g(u, v)}{\partial u} = \alpha \quad , \quad \frac{\partial g(u, v)}{\partial v} \quad = \quad \frac{\gamma\delta - \beta\zeta}{(\delta + \zeta v)^2}$$

Now, we have to cases to consider:

Case(1): Suppose that (6.1) is true, clearly the function $g(u, v)$ increasing in both u and v.

Let w be a solution of the equation $w = g(w, w)$. Then, we have from Equation (1.1) that

$$w = \alpha w + \frac{\beta + \gamma w}{\delta + \zeta w}$$

or

$$w(1-\alpha) = \frac{\beta + \gamma w}{\delta + \zeta w}$$

then the equation

$$\zeta(1-\alpha) w^2 + (\delta(1-\alpha) - \gamma) w - \beta = 0$$

has a unique positive solution when $\gamma > \delta(1-\alpha)$, $\alpha < 1$ which is

$$w = \frac{(\gamma - \delta + \delta\alpha) + \sqrt{(\gamma - \delta + \delta\alpha)^2 + 4\beta\zeta(1-\alpha)}}{2\zeta(1-\alpha)}$$

Which implies by using Theorem 2.2 that $\overline{w}$ is a global attractor of Equation(1.1).

Case(2): Suppose that (6.2) is true, clearly the function $g(u,v)$ increasing in u and decreasing in v.

Assume that (m,M) be a solution of the system $M = g(M,m)$ and $m = g(m,M)$. Then we have from Equation (1.1) that

$$M = \alpha M + \frac{\beta+\gamma m}{\delta+\zeta m}\ ,\ \ m = \alpha m + \frac{\beta+\gamma M}{\delta+\zeta M}$$

or

$$M(1-\alpha) = \frac{\beta+\gamma m}{\delta+\zeta m}\ ,\ \ m(1-\alpha) = \frac{\beta+\gamma M}{\delta+\zeta M}$$

thus

$$\delta(1-\alpha)M+\zeta(1-\alpha)Mm = \beta+\gamma m\ ,\ \ \delta(1-\alpha)m+\zeta(1-\alpha)mM = \beta+\gamma M$$

implies that

$$(M-m) = \delta(1-\alpha)(M+m)+\gamma = 0$$

Which under the condition $\alpha < 1$ gives

$$M = m$$

Which implies by using Theorem 2.2 that $\overline{w}$ is a global attractor of Equation (1.1). Hence, this completes the proof.

7. Numerical Solutions

Here, we consider some numerical examples in order to verify our theoretical results of this paper which provide different types solutions to Equation (1.1).

Example 7.1. Assume that $w_{-2} = 1$, $w_{-1} = 7$, $w_0 = 11$, $\alpha = 0.2$, $\beta = 2$, $\gamma = 5$, $\delta = 3$, and $\zeta = 7$. See Figure 1.

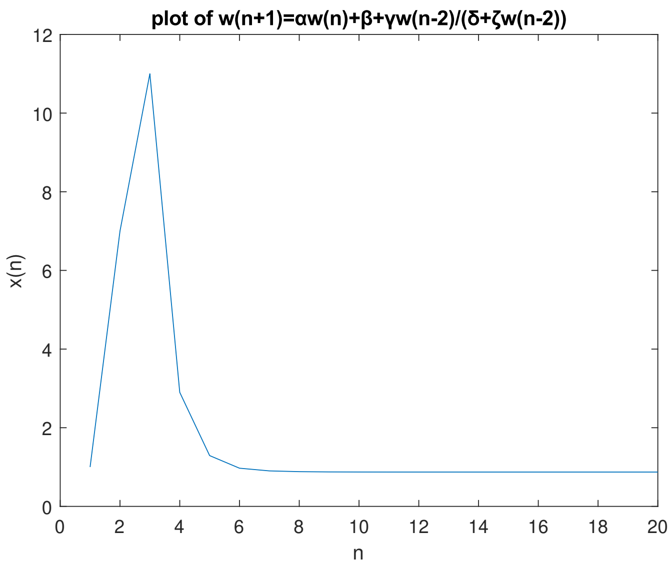

Figure 1. Local Stability of Equation (1.1)

Example 7.2. Suppose that $w_{-2} = 0.5$, $w_{-1} = 3$, $w_0 = 0.9$, $\alpha = 0.1$, $\beta = 3.5$, $\gamma = 5$, $\delta = 0.05$, and $\zeta = 9$. Then, this example demonstrates the global stability behaviour of Equation (1.1). See Figure 2.

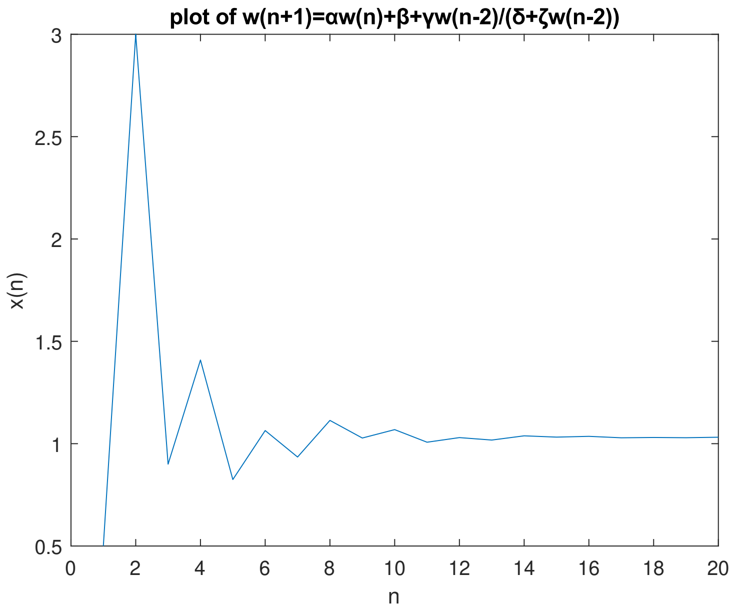

Figure 2. Global Stability of Equation (1.1)

Example 7.3. This example plot the solution when we have $w_{-2} = 0.5$, $w_{-1} = 7$, $w_0 = 3$, $\alpha = 1.2$, $\beta = 4$, $\gamma = 1.5$, $\delta = 0.6$, and $\zeta = 5.3$. See Figure 3.

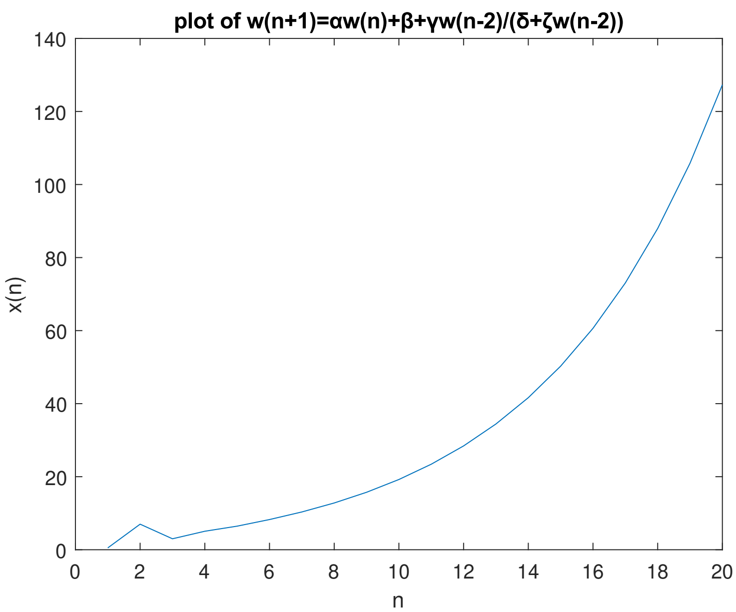

Figure 3. Solution of Equation (1.1)

Example 7.4. Here, we also present the plot of the solution under $w_{-2} = 0.2$, $w_{-1} = 3$, $w_0 = 0.4$, $\alpha = 0.2$, $\beta = 2$, $\gamma = 4$, $\delta = 0.5$, and $\zeta = 4$. See Figure 4.

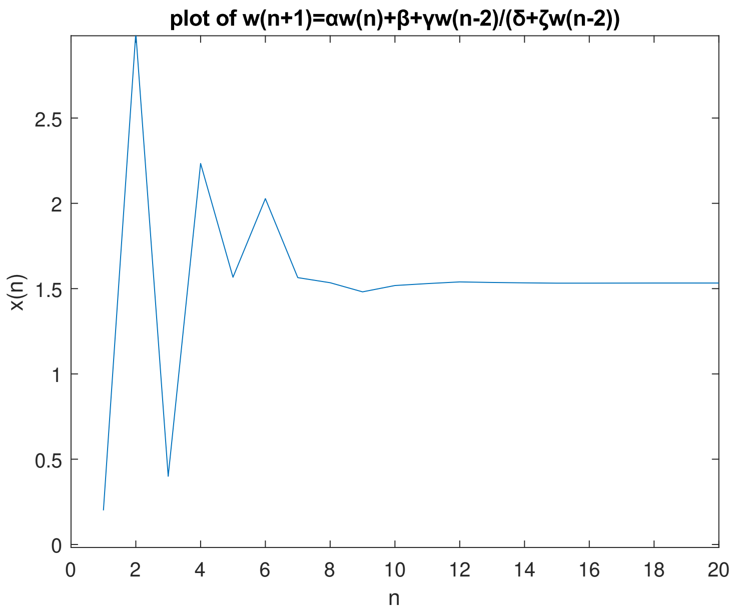

Figure 4. Solution of Equation (1.1)

Example 7.5. Now , we show the behavior of the solution when $w_{-2} = 0.2$, $w_{-1} = 1.5$, $w_0 = 0.2$, $\alpha = 0.1$, $\beta = 6$, $\gamma = 4$, $\delta = 8$, and $\zeta = 11$. See Figure 5.

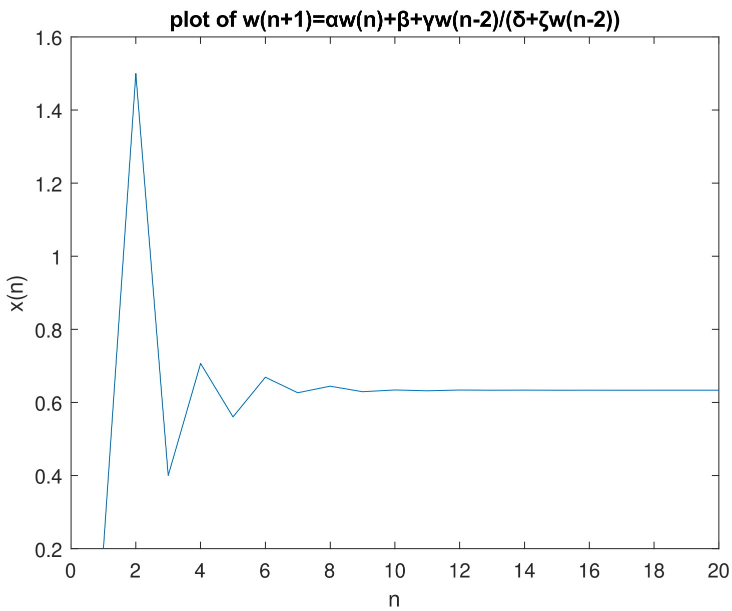

Figure 5. Solution of Equation (1.1)

Example 7.6. In this example , we confirm that our equation has period two when $\alpha = 0.6$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.5$, $\zeta = 0.9$, $w_{-2} = p$, $w_{-1} = q$, $w_0 = q$ (since $p, q = \frac{-(1+\alpha)\zeta(\delta+\alpha\delta+\gamma)\pm\lambda}{2\alpha\zeta(1+\alpha)}$) . See Figure 6.

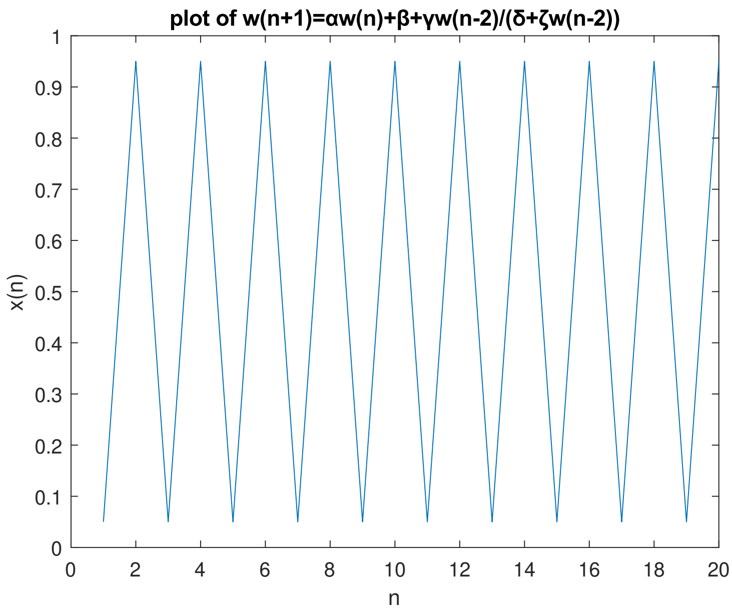

Figure 6. Solution of Period Two

8. Conclusion

In conclusion, we have shown some significant dynamical behaviors of Equation (1.1) such as investigated the local and global stability. Also, we highlighted to the boundedness of the solutions of Equation (1.1) and we established two theorems to show that when the solution of the Equation (1.1) is bounded and unbounded under necessary condition. Furthermore, we have discussed the existence of periodic solutions and obtained that Equation (1.1) has a periodic solutions of period two. Finally, we gave some numerical examples of Equation (1.1) and got some figures to confirm our theoretical results by using Matlab.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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