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Invariants of Immersions on n-Dimensional Affine Manifold

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Highlights

- A system of generators of affine invariant functions a vector field for the affine groups is given.
- Rigidity and uniqueness theorems for immersions in affine geometry were obtained.
- Rigidity and uniqueness theorems for immersions are given in terms of affine invariants of immersions.

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Abstract

Main results: The system of Christoffel symbols of the connection of an immersion $\xi: J \to R^n$ of an n-dimensional manifold J in the n-dimensional linear space R^n is a system of generators of the differential field of all $Aff(n)$-invariant differential rational functions of ξ, where $Aff(n)$ is the group of all affine transformations of R^n. A similar result have obtained for the subgroup $SAff(n)$ of Aff(n) generated by all unimodular linear transformations and parallel translations of R^n. Rigidity and uniqueness theorems for immersions $\xi: J \to R^n$ in geometries of groups $Aff(n)$ and $SAff(n)$ were obtained. These theorems are given in terms of the affine connection and the volume form of immersions.

1. INTRODUCTION

Let $GL(n)$ be the group of all non-degenerate linear transformations. Denote by $SL(n)$ the subgroup of all $g \in GL(n)$ such that det $g = 1$.

Let $Aff(n)$ be a group of all affine transformations $f: \mathbb{R}^n \to \mathbb{R}^n, f(x) = Ax + b, x \in \mathbb{R}^n$ with $b \in \mathbb{R}^n$ and for all $A \in GL(n)$.

Denote by $SAff(n)$ the subgroup of $Aff(n)$ such that $f \in Aff(n), f(x) = Ax + b, x \in \mathbb{R}^n$ with $b \in \mathbb{R}^n$ and for all $A \in SL(n)$.

The Bonnet's fundamental rigidity and uniqueness theorem for hypersurface immersions in the geometry of the special Euclidean group $SM(n)$ is known in [1, 2]. An analogue of Bonnet's fundamental rigidity and uniqueness theorem for hypersurfaces in the geometry of the group $SAff$ (n) was given in [2-4]. For surfaces in the geometry of the group $SL(3)$ it is given in [4] and for surfaces in the geometry of the group $SAff$ (3) it is given in [3].

Two analogues of the rigidity and uniqueness theorems for immersions of an n dimensional manifold in an n-dimensional Euclidean space were obtained. The first analogue is given for the Euclidean group $M(n)$ in [5-7].

Another analogue of the rigidity and uniqueness theorem for vector fields in a Euclidean geometry is given in [8]. Note that in this book and papers mentioned below in Introduction, the term "vector field" is used for any map $\xi: J \to R^n$ of an open subset $J \subset R^n$. The vector field can be also named "n-parametric surface".

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The theorem in the book [8] is essentially the rigidity and uniqueness theorem for a system of three orthonormal vector fields in the geometry of the orthogonal group $O(3)$. For the Euclidean group $M(n)$ and the special Euclidean group $SM(n)$, other forms of rigidity theorems for vector fields have given in [9]. In the paper, it is obtained also that the system of coefficients of the Riemannian metric of an parametric surface is a system of generators of the differential field of all $M(n)$-invariant differential rational functions of an n-parametric surface.

Therefore, for the geometry of the n-dimensional pseudo-Euclidean group of index p, (it will denoted by $M(n,p)$), the rigidity and uniqueness theorems for immersions of an n-dimensional manifold were given in [10].

Investigations of the problem of $Aff(n)$-equivalence and $SAff(n)$-equivalence of immersions (vector fields), $Aff\ (n)$-invariant and $SAff\ (n)$-invariant immersions (vector fields) and $Aff\ (n)$-invariants and $SAff\ (n)$-invariants of immersions (vector fields) play an important and critical role in science, technology, engineering, mathematics, mathematical physics and computer vision and pattern recognition, etc. (see some references [11-15]).

The problem of description of the general form of all invariant polynomial vector fields for a compact Lie groups is intensively studied in the bifurcation theory [16-18]. The problem of equivalence of smooth vector fields and the problem of a description of complete systems of invariants of polynomial vector fields are investigated in the theory of differential equations [19,20].

The structure of the paper is organized as follows. In section 2, for a vector field $\xi(u)$ on an open subset J of $\mathbb{R}^n$, we describe a system of generators of the differential field of all G-invariant differential rational functions of $\xi(u)$ for groups $G = Aff(n), SAff(n)$ (Theorems 1 and 2).

In section 3, for an n-dimensional connected manifold M, using results of Section 2, we obtain the following results:

(1) The rigidity theorem for the connection on M induced by the immersion $\xi: M \longrightarrow \mathbb{R}^n$ (Theorem 3) and some consequences of this theorem (Corollaries 3 and 4). By Corollary 3, Theorem 2 means that the system of Christoffel symbols of the connection on M induced by the immersion $\xi: M \longrightarrow \mathbb{R}^n$ is a system of generators of the differential field of all $Aff\ (n)$-invariant differential rational functions of $\xi(u)$.

(2) The rigidity theorem for the connection and the volume form on M induced by an immersion $\xi: M \longrightarrow \mathbb{R}^n$ (Theorem 4).

In section 4, for an n-dimensional connected, simply connected manifold M, we prove the existence theorem for a connection on M (Theorem 5).

2. GENERATING SYSTEMS OF AFFINE INVARIANT DIFFERENTIAL RATIONAL FUNCTIONS OF A VECTOR FIELD

Let J be an open subset of $\mathbb{R}^n$. Throughout this paper, we will take a vector field $\xi(u)$ such that $\xi: J \to \mathbb{R}^n$ is a C^∞-mapping. Here a C^∞-mapping ξ is called to be an n-parametric surface (J-vector field, for shortness) in $\mathbb{R}^n$.

Denote the set of all non-negative integers by $\mathbb{N}_0$. For $\alpha_i \in \mathbb{N}_0$ for $i = 1,2,\dots,n$, we put

$$\xi^{(\alpha_1,\alpha_2,\dots,\alpha_n)} = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial u_1^{\alpha_1}\partial u_2^{\alpha_2}\dots\partial u_n^{\alpha_n}}\xi(u). \text{ It is clear that } \xi(0,0,\dots,0) = \xi(u).$$

Throughout this paper, we will take the real numbers $\mathbb{R}$ to be ground field. The ring of differential polynomials of $f\left(\xi, \xi^{(1,0,\dots,0)}, \xi^{(0,1,\dots,0)}, \dots, \xi^{(\alpha_1,\alpha_2,\dots,\alpha_n)}\right)$ in a finite number of partial derivatives of ξ with real coefficients is denoted $\mathbb{R}\{\xi\}$. This being case, we denote $f\left(\xi, \xi^{(1,0,\dots,0)}, \xi^{(0,1,\dots,0)}, \dots, \xi^{(\alpha_1,\alpha_2,\dots,\alpha_n)}\right)$ by $f\{\xi\}$. Therefore, $\mathbb{R}\{\xi\}$ is a differential $\mathbb{R}$-algebra and an integral domain. In this case, denote its quotient field by $\mathbb{R}\langle\xi\rangle$. Then $\mathbb{R}\langle\xi\rangle$ is a differential field and its an element f is a differential rational function of ξ. This being case, denote it by $f\langle\xi\rangle$.

This definitions can be generalized as follow: Let $\xi_1, \xi_2, \dots, \xi_k$ be k-tuple vector fields defined on the same subset J in $\mathbb{R}^n$. In this case, denote a differential polynomial and a differential rational function of $\xi_1, \xi_2, \dots, \xi_k$ by $f\{\xi_1, \xi_2, \dots, \xi_k\}$ and $f\langle\xi_1, \xi_2, \dots, \xi_k\rangle$, resp. Their ring of all differential polynomials and field of all differential rational functions is denoted by $\mathbb{R}\{\xi_1, \xi_2, \dots, \xi_k\}$ and $\mathbb{R}\langle\xi_1, \xi_2, \dots, \xi_k\rangle$, resp.

Let G be one of the groups $Aff(n)$ or $SAff\ (n)$.

Definition 1. A differential G-invariant function is a real-valued function $f: J^k \to \mathbb{R}$ which satisfied $f\langle F\xi_1, F\xi_2, \dots, F\xi_k)\rangle = f\langle\xi_1, \xi_2, \dots, \xi_k\rangle$ for all $F \in G$ and $\xi_1, \xi_2, \dots, \xi_k$ are J - vector fields in $\mathbb{R}^n$.

It is easy to see that given a J - vector field $\xi(u)$, then every affine transformation F transforms $\xi(u)$ into a new J - vector field $F\xi(u)$.

In this paper, we are interested in the set

$$\mathbb{R}\langle\xi_1, \xi_2, \dots, \xi_k\rangle^G = \{f \in \mathbb{R}\langle\xi_1, \xi_2, \dots, \xi_k\rangle \mid f \text{ is a } G\text{-invariant function }\}$$

of all functions which are invariant under the action of G. This set is a differential subfield of $\mathbb{R}\langle\xi_1, \xi_2, \dots, \xi_k\rangle$. We call $\mathbb{R}\langle\xi_1, \xi_2, \dots, \xi_k\rangle^G$ the set of all G-invariant differential rational functions of $\xi_1, \xi_2, \dots, \xi_k$.

Now we will find a set of generators for $\mathbb{R}\langle\xi_1, \xi_2, \dots, \xi_k\rangle^G$ which is one of the fundamental problems of invariant theory.

We will consider element $a_r \in \mathbb{R}^n$ in the form $a_r = \begin{pmatrix} a_{r1} \\ a_{r2} \\ \vdots \\ a_{rn} \end{pmatrix}$ for all $r = 1,2, \dots, n$. For $a_r \in \mathbb{R}^n$, denote the determinant of the matrix (a_{ij}) by $[a_1 a_2 \dots a_n]$.

Hence applying a_k to elements $a_r = \xi^{(\alpha_{r1},\alpha_{r2},\dots,\alpha_{rn})}$ for all $r = 1,2, \dots, n$, we obtain the determinant $\left[\xi^{(\alpha_{11},\alpha_{12},\dots,\alpha_{1n})} \xi^{(\alpha_{21},\alpha_{22},\dots,\alpha_{2n})} \dots \xi^{(\alpha_{n1},\alpha_{n2},\dots,\alpha_{nn})}\right]$.

In the case, we put
$\Omega = \left[\frac{\partial\xi}{\partial u_1} \frac{\partial\xi}{\partial u_2} \cdots \frac{\partial\xi}{\partial u_n}\right]$ and $\Omega_{ij}^k = \left[\frac{\partial\xi}{\partial u_1} \cdots \frac{\partial\xi}{\partial u_{k-1}} \frac{\partial^2\xi}{\partial u_i \partial u_j} \frac{\partial\xi}{\partial u_{k+1}} \cdots \frac{\partial\xi}{\partial u_n}\right]$ for all $i, j, k = 1,2, \dots, n$.

Theorem 1. The system

$$S = \left\{\Omega, \Omega_{ij}^k; i, j, k = 1,2, \dots, n\right\} \tag{1}$$

is a set of generators of $R\langle\xi\rangle^{SAff(n)}$.

Proof. Firstly, we give some lemmas for the proof of the theorem.

Let $\mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle$ and $\mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^G$ be the differential field of all differential rational functions and the differential field of all G-invariant differential rational functions of $\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}$, resp.

Lemma 1. $\mathbb{R}\langle\xi\rangle^{SAff(n)} = \mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{SAff(n)} = \mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{SL(n)}$.

Proof. The proof is similar to the proof of Lemma 1 in [9].

Lemma 2. Let $f\in\mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{SL(n)}$. Then there exist $SL(n)$-invariant differential polynomials f_1, f_2 such that $f = f_1/f_2$.

Proof. The proof is similar to the proof of lemma in [21].

Lemma 3. The system

$$W=\left\{\left[\xi^{(\alpha_{11},\alpha_{12},\dots,\alpha_{1n})}\xi^{(\alpha_{21},\alpha_{22},\dots,\alpha_{2n})}\dots\xi^{(\alpha_{n1},\alpha_{n2},\dots,\alpha_{nn})}\right],\sum_{j=1}^{n}\alpha_{ij}\geq 1, 1\leqslant i\leqslant n\right\} \quad (2)$$

is a set of generators of $\mathbb{R}\{\xi\}^{SAff(n)}$ as an $\mathbb{R}$-algebra.

Proof. The proof is obtained from [22] and Lemmas 2 and 9.

Remark 1. Similar proofs of Lemma 3 are given in [9, 10].

Lemma 4. The system W in Lemma 3 is a set of generators of $\mathbb{R}\{\xi\}^{\mathrm{SAff}(n)}$ as a field.

Proof. The proof is obvious from Lemmas 2,3 and 9.

Let $\mathbb{R}\{S\}$ and $\mathbb{R}\{S,\omega^{-1}\}$ be the $\mathbb{R}$-subalgebras of $\mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{SL(n)}$. From Lemmas 4 and 9 for a proof of the theorem, it is enough to prove that $W\in\mathbb{R}\{S,\omega^{-1}\}$. Now, let

$$A=\left[\xi^{(\alpha_{11},\alpha_{12},\dots,\alpha_{1n})}\xi^{(\alpha_{21},\alpha_{22},\dots,\alpha_{2n})}\dots\xi^{(\alpha_{n1},\alpha_{n2},\dots,\alpha_{nn})}\right]. \quad (3)$$

Let $s(A)$ be the number of elements of the set

$$\left\{\xi^{(\alpha_{11},\alpha_{12},\dots,\alpha_{1n})},\xi^{(\alpha_{21},\alpha_{22},\dots,\alpha_{2n})},\dots,\xi^{(\alpha_{n1},\alpha_{n2},\dots,\alpha_{nn})}\right\}\setminus\left\{\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\}.$$

We set $r(A)=\max_{1\leq i\leq n}\sum_{j=1}^{n}\alpha_{ij}$.

Lemma 5. Let A be a differential polynomial of the form (3), where $s(A)\geq 2$. Then A is a polynomial of Ω^{-1} and differential polynomials B of the form (3), where $s(B)<s(A)$ and $r(B)\leq r(A)$.

Proof. By $s(A)\geq 2$, there exists $k\in\{1,2,\dots,n\}$, such that

$\frac{\partial \xi}{\partial u_k} \notin \{\xi^{(\alpha_{11},\alpha_{12},\dots,\alpha_{1n})}, \xi^{(\alpha_{21},\alpha_{22},\dots,\alpha_{2n})}, \dots, \xi^{(\alpha_{n1},\alpha_{n2},\dots,\alpha_{nn})}\}$.

In [23], we put

$$x_1 = \xi^{(\alpha_{11},\alpha_{12},\dots,\alpha_{1n})}, x_2 = \xi^{(\alpha_{21},\alpha_{22},\dots,\alpha_{2n})}, \dots, x_n = \xi^{(\alpha_{n1},\alpha_{n2},\dots,\alpha_{nn})}, x_0 = \frac{\partial \xi}{\partial u_k},$$
$$y_2 = \frac{\partial \xi}{\partial u_1}, \dots, y_k = \frac{\partial \xi}{\partial u_{k-1}}, y_{k+1} = \frac{\partial \xi}{\partial u_{k+1}}, \dots, y_n = \frac{\partial \xi}{\partial u_n}.$$

Then

$$\begin{aligned}
&\left[\xi^{(\alpha_{11},\alpha_{12},\dots,\alpha_{1n})}\xi^{(\alpha_{21},\alpha_{22},\dots,\alpha_{2n})}\dots\xi^{(\alpha_{n1},\alpha_{n2},\dots,\alpha_{nn})}\right] \times \\
&\left[\frac{\partial \xi}{\partial u_k}\frac{\partial \xi}{\partial u_1}\dots\frac{\partial \xi}{\partial u_{k-1}}\frac{\partial \xi}{\partial u_{k+1}}\dots\frac{\partial \xi}{\partial u_n}\right] - \\
&\left[\frac{\partial \xi}{\partial u_k}\xi^{(\alpha_{21},\alpha_{22},\dots,\alpha_{2n})}\dots\xi^{(\alpha_{n1},\alpha_{n2},\dots,\alpha_{nn})}\right] \times \\
&\left[\xi^{(\alpha_{11},\alpha_{12},\dots,\alpha_{1n})}\frac{\partial \xi}{\partial u_1}\dots\frac{\partial \xi}{\partial u_{k-1}}\frac{\partial \xi}{\partial u_{k+1}}\dots\frac{\partial \xi}{\partial u_n}\right] - \cdots \\
&-\left[\xi^{(\alpha_{11},\alpha_{12},\dots,\alpha_{1n})}\dots\xi^{(\alpha_{n-11},\alpha_{n-12},\dots,\alpha_{n-1n})}\frac{\partial \xi}{\partial u_k}\right] \times \\
&\left[\xi^{(\alpha_{n1},\alpha_{n2},\dots,\alpha_{nn})}\frac{\partial \xi}{\partial u_1}\dots\frac{\partial \xi}{\partial u_{k-1}}\frac{\partial \xi}{\partial u_{k+1}}\dots\frac{\partial \xi}{\partial u_n}\right] = 0
\end{aligned} \quad (4)$$

Put

$$v_0 = \left[\frac{\partial \xi}{\partial u_k}\frac{\partial \xi}{\partial u_1}\cdots\frac{\partial \xi}{\partial u_{k-1}}\frac{\partial \xi}{\partial u_{k+1}}\cdots\frac{\partial \xi}{\partial u_n}\right],$$
$$v_t = \left[\xi^{(\alpha_{t1},\alpha_{t2},\cdots,\alpha_{tn})}\frac{\partial \xi}{\partial u_1}\cdots\frac{\partial \xi}{\partial u_{k-1}}\frac{\partial \xi}{\partial u_{k+1}}\cdots\frac{\partial \xi}{\partial u_n}\right],$$
$$h_t = \left[\xi^{(\alpha_{11},\alpha_{12},\cdots,\alpha_{1n})}\dots\xi^{(\alpha_{t-11},\alpha_{t-12},\cdots,\alpha_{t-1n})}\frac{\partial \xi}{\partial u_k}\xi^{(\alpha_{l+11},\alpha_{t+12},\cdots,\alpha_{t+1n})}\dots\xi^{(\alpha_{n1},\alpha_{n2},\cdots,\alpha_{nn})}\right].$$

Then $s(v_0) = 0, r(v_0) \le r(A)$ and $s(v_t) \le 1, r(h_t) \le r(A)$ for all $t = 1,2,\dots,n$. Using Equation (4), we get $A = v_1h_1(v_0)^{-1} + \cdots + v_nh_n(v_0)^{-1}$. Since $v_0 = (-1)^{k-1}\Omega$, we have $A = (-1)^{k-1}\Omega^{-1}(v_1h_1 + \cdots + v_nh_n)$. By $s(A) \ge 2$, the number of non-zero elements v_jh_j is $s(A) \ge 2$. For h_j such that $v_jh_j \ne 0$, we have $s(h_j) < s(A)$. Therefore A is a polynomial of the system Ω^{-1}, v_j, h_j, with $s(v_j) = 1, r(v_j) \le r(A), s(h_j) < s(A), r(h_j) \le r(A)$.

Lemma 6. Let A be a differential polynomial of the form (3), where $s(A) \ge 2$. Then A is a polynomial of Ω, Ω^{-1} and differential polynomials B of the form

$\left[\frac{\partial \xi}{\partial u_1}\cdots\frac{\partial \xi}{\partial u_{k-1}}\xi^{(\alpha_1,\alpha_2,\cdots,\alpha_n)}\frac{\partial \xi}{\partial u_{k+1}}\cdots\frac{\partial \xi}{\partial u_n}\right]$, where $\sum_{i=1}^{n}\alpha_i \le r(A)$.

Proof. Using Lemma 6 and induction on $s(A)$, we obtain that every differential polynomial A of the form (3), where $s(A) \ge 2$, is a polynomial of Ω^{-1} and differential polynomials B of the form (3), where $s(B) \le$

1 and $r(B) \leq r(A)$. Every non-zero differential polynomial B of the form (3), where $s(B) = 0$, is equal to Ω. Every differential polynomial B of the form (3), where $s(B) = 1$, has the following form

$\left[\frac{\partial \xi}{\partial u_1} \cdots \frac{\partial \xi}{\partial u_{k-1}} \xi^{(\alpha_1, \alpha_2, \cdots, \alpha_n)} \frac{\partial \xi}{\partial u_{k+1}} \cdots \frac{\partial \xi}{\partial u_n}\right]$, where $\sum_{i=1}^{n} \alpha_i > 1$.

Lemma 7. Let A be a differential polynomial of the form

$\left[\frac{\partial \xi}{\partial u_1} \cdots \frac{\partial \xi}{\partial u_{k-1}} \xi^{(\alpha_1, \alpha_2, \cdots, \alpha_n)} \frac{\partial \xi}{\partial u_{k+1}} \cdots \frac{\partial \xi}{\partial u_n}\right]$, where $\sum_{i=1}^{n} \alpha_i > 2$. Then A is a differential polynomial of differential polynomials B of the form (3), where $r(B) < r(A)$.

Proof. Assume that A such that $r(A) = \sum_{i=1}^{n} \alpha_i > 2$. Then $\alpha_s > 0$ for some s. Consider the following differential polynomial

$$B_0 = \left[\frac{\partial \xi}{\partial u_1} \cdots \frac{\partial \xi}{\partial u_{k-1}} \xi^{(\alpha_1, \dots, \alpha_{s-1}, \alpha_s - 1, \alpha_{s+1} \cdots, \alpha_n)} \frac{\partial \xi}{\partial u_{k+1}} \cdots \frac{\partial \xi}{\partial u_n}\right].$$

For B_0, we have $r(B_0) = r(A) - 1$. Set

$$B_i = \left[\frac{\partial \xi}{\partial u_1} \cdots \frac{\partial \xi}{\partial u_{i-1}} \frac{\partial}{\partial u_s}\left(\frac{\partial \xi}{\partial u_i}\right) \frac{\partial \xi}{\partial u_{i+1}} \cdots \frac{\partial \xi}{\partial u_{k-1}} \xi^{(\alpha_1, \dots, \alpha_{s-1}, \alpha_s - 1, \alpha_{s+1} \cdots, \alpha_n)} \frac{\partial \xi}{\partial u_{k+1}} \cdots \frac{\partial \xi}{\partial u_n}\right]$$

for $i < k$ and

$$B_i = \left[\frac{\partial \xi}{\partial u_1} \cdots \frac{\partial \xi}{\partial u_{k-1}} \xi^{(\alpha_1, \dots, \alpha_{s-1}, \alpha_s - 1, \alpha_{s+1} \cdots, \alpha_n)} \frac{\partial \xi}{\partial u_{k+1}} \cdots \frac{\partial \xi}{\partial u_{i-1}} \frac{\partial}{\partial u_s}\left(\frac{\partial \xi}{\partial u_i}\right) \frac{\partial \xi}{\partial u_{i+1}} \cdots \frac{\partial \xi}{\partial u_n}\right]$$

for $k < i$. We have the following equation

$$\frac{\partial}{\partial u_s} B_0 = B_1 + \cdots + B_{k-1} + A + B_{k+1} + \cdots + B_n.$$

Hence

$$A = \frac{\partial}{\partial u_s} B_0 - (B_1 + \cdots + B_{k-1} + B_{k+1} + \cdots + B_n). \tag{5}$$

Since $r(B_i) = r(A) - 1$ for all $i = 0,1, \dots, k, k+1, \dots, n$, the Equation (5) implies that A is a differential polynomial of differential polynomials B of the form (3), where $r(B) = r(A) - 1$.

Lemma 8. Let A be a differential polynomial of the form (3), where $s(A) \geq 2$. Then A is a differential polynomial of Ω^{-1} and elements of the system (1).

Proof. It follows from Lemmas 5-7 by induction on $s(A)$ and $r(A)$.

The proof of Theorem 1 is completed by Lemmas 1-4 and Lemma 8 .

Definition 2. A differential rational $Aff(n)$-relative invariant function is a real-valued function $f: J^k \to \mathbb{R}$ which satisfied $f\langle F\xi_1, F\xi_2, \dots, F\xi_k\rangle = (\det\ F)^m f\langle \xi_1, \xi_2, \dots, \xi_k\rangle$ for all $F \in G, \xi_1, \xi_2, \dots, \xi_k$ are J - vector fields in $\mathbb{R}^n$ and $m \in \mathbb{N}_0$. The number m is called weight of f and it is denoted by $W(f)$.

Theorem 2. The system

$$\left\{\frac{\Omega_{ij}^{s}}{\Omega}; i,j,s = 1,2,\dots,n\right\} \tag{6}$$

is a set of generators of $\mathbb{R}\langle\xi\rangle^{Aff(n)}$.

Proof. Firstly, we give the following lemmas for the proof of the theorem.

Lemma 9. $\mathbb{R}\langle\xi\rangle^{SAff(n)} = \mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{SAff(n)} = \mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{SL(n)}$.

Proof. The proof is similar to the proof of Lemma 1 in [9].

The following lemma is similar to Lemma 1.

Lemma 10. $\mathbb{R}\langle\xi\rangle^{Aff(n)} = \mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{Aff(n)} = \mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{GL(n)}$.

Lemma 11. Let $f \in \mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{GL(n)}$. Then there exist $GL(n)$-relative invariant differential polynomials f_1, f_2 such that $f = f_1/f_2$ and $W(f_1) = W(f_2)$.

Proof. The proof is similar to the proof of lemma in [21]. Let $f \in \mathbb{R}\left\langle\frac{\partial\xi}{\partial u_1},\dots,\frac{\partial\xi}{\partial u_n}\right\rangle^{GL(n)}$. By Lemma 11 , there exist $GL(n)$-relative invariant differential polynomials f_1, f_2 of $\xi(u)$ such that $f = f_1/f_2$ and $W(f_1) = W(f_2)$. Since differential polynomials f_1, f_2 are $GL(n)$-relative invariant, they are $SL(n)$-invariant. Then, by Lemma 3 and 8, there exist polynomials

$$h_1\{\Omega, \Omega_{ij}^{k}; i,j,k = 1,2,\dots,n\}, h_2\{\Omega, \Omega_{ij}^{k}; i,j,k = 1,2,\dots,n\}$$

of elements of the system (1) such that

$$f_1 = \frac{h_1\{\Omega, \Omega_{ij}^{k}; i,j,k = 1,2,\dots,n\}}{\Omega^{r_1}}, f_2 = \frac{h_2\{\Omega, \Omega_{ij}^{k}; i,j,k = 1,2,\dots,n\}}{\Omega^{r_2}}$$

for some $r_1, r_2 \in \mathbb{N}_0$. Since f_1, f_2, Ω are $GL(n)$-relative invariant differential polynomials, $W(\Omega) = 1$ and $W(f_1) = W(f_2)$, we have

$W(f_1) = W(h_1) - r_1, W(f_2) = W(h_2) - r_2$. These imply the following equations

$$f_1 = \frac{\Omega^{W(h_1)} h_1\left\{1, \frac{\Omega_{ij}^{k}}{\Omega}; i,j,k = 1,2,\dots,n\right\}}{\Omega^{r_1}} = \Omega^{W(h_1)-r_1} h_1\left\{1, \frac{\Omega_{ij}^{k}}{\Omega}; i,j,k = 1,2,\dots,n\right\},$$

$$f_2 = \frac{\Omega^{W(h_2)} h_2\left\{1, \frac{\Omega_{ij}^{k}}{\Omega}; i,j,k = 1,2,\dots,n\right\}}{\Omega^{r_2}} = \Omega^{W(h_2)-r_2} h_1\left\{1, \frac{\Omega_{ij}^{k}}{\Omega}; i,j,k = 1,2,\dots,n\right\}.$$

Hence we have $f = f_1/f_2 = \dfrac{h_1\left\{1, \frac{\Omega_{ij}^k}{\Omega}; i,j,k=1,2,\dots,n\right\}}{h_2\left\{1, \frac{\Omega_{ij}^k}{\Omega}; i,j,k=1,2,\dots,n\right\}}$.

So, the proof of the theorem is completed.

Corollary 1. The system

$$\left\{\Omega, \frac{\Omega_{ij}^k}{\Omega}; i,j,k = 1,2,\dots,n\right\} \tag{7}$$

is a set of generators of $\mathbb{R}\langle\xi\rangle^{SAff(n)}$.

Proof. It follows from Theorem 1.

3. GENERATING SYSTEMS OF AFFINE INVARIANT DIFFERENTIAL RATIONAL FUNCTIONS FOR IMMERSIONS OF A MANIFOLD AND AFFINE EQUIVALENCE PROBLEMS FOR TWO AFFINE IMMERSIONS

Now we give some basic definitions.

Let M be a connected C^∞-manifold of dimension dim $M = n$, and $\xi: M \to \mathbb{R}^n$ a C^∞-immersion, i.e. a differentiable mapping of rank n. For simplicity, we use the term "M-immersion".

A chart on M is a pair (ϕ, U) where U is an open subset of M and ϕ is a homeomorphism of U with an open subset $\phi(U)$ of $\mathbb{R}^n$. U is called a coordinate neighbourhood and $\phi(U)$ its coordinate space.

Let $\Lambda = \{(\phi_\alpha, U_\alpha,), \alpha \in A\}$ a collection of charts of M. Then we can be given an n-form on M by $\left[\frac{\partial\xi^{(u)}}{\partial u_1}\frac{\partial\xi^{(u)}}{\partial u_2}\dots\frac{\partial\xi^{(u)}}{\partial u_n}\right] du_1 \wedge \dots \wedge du_n$, where $\xi^{(u)}$ is a representation of ξ in the local coordinates of U_α.

Then, n-form on M is called the volume form induced by an M-immersion and denoted it by $\omega(\xi)$.

Proposition 1. Let ξ be an M-immersion. Then $\omega(\xi) \neq 0$ for all $p \in M$.

Proof. For a similar proof, see [10].

Corollary 2. Let M be a C^∞-manifold of dim $M = n$. If an M-immersion of manifold M exists, then M is an orientable C^∞-manifold.

Proof. Using [24] and Proposition 1, the proof is completed.

Remark 2. There is an orientable C^∞-manifold M of dim $M = n$ without M-immersions in $\mathbb{R}^n$. (See [24]).

Let ξ and $u_1, u_2, \dots, u_n$ be an M-immersion in $\mathbb{R}^n$ and a coordinate system, resp. Let us write $\partial_i = \frac{\partial}{\partial u_i}$ for the corresponding vector fields. Then,

$$\partial u_i\, \partial u_j \xi^{(u)} = \sum_{k=1}^n \Gamma_{ij}^k\{\xi\}\, \partial u_k \xi^{(u)},\ \ i,j = 1,2,\dots,n \tag{8}$$

where the functions $\Gamma_{ij}^k\{\xi\}$ is called the Christoffel symbols of the M-immersion ξ on a chart of M and u is an element of a chart of M.

Let $G(\xi) = \{\Gamma_{ij}^k\{\xi\}; i,j,k = 1,2,\dots,n\}$ be the system of Christoffel symbols of a connection on M and denote this connection by $\nabla(\xi)$.

Proposition 2. Let ξ be an M-immersion in $\mathbb{R}^n$. Then

$$\Gamma_{ij}^k\{\xi\} = \frac{[\partial_1\xi^{(u)} \dots \partial_{k-1}\xi^{(u)}\, \partial_i(\partial_j\xi^{(u)})\, \partial_{k+1}\xi^{(u)} \dots \partial_n\xi^{(u)}]}{[\partial_1\xi^{(u)}\, \partial_2\xi^{(u)} \dots \partial_n\xi^{(u)}]} \tag{9}$$

for all $i,j,k = 1,2,\dots,n$. Proof The proof is obtained from the system (8).

Corollary 3. Let ξ be an M-immersion in $\mathbb{R}^n$. Then the system $G(\xi) = \{\Gamma_{ij}^k(\xi); i,j,k = 1,2,\dots,n\}$ is a set of generators of $\mathbb{R}\langle\xi\rangle^{Aff(n)}$.

Proof. The proof is obtained from Theorem 2 and Proposition 2.

It is easy to see that given an M - immersion $\xi(u)$, then every affine transformation F transforms $\xi(u)$ into a new M-immersion $F\xi(u)$.

Definition 3. Let ξ and η be two M-immersions. Then these immersions are called $Aff\ (n)$-equivalent if there is $F \in Aff(n)$ such that $\eta(p) = F\xi(p)$ for all $p \in M$, and denote it by $\xi \overset{Aff(n)}{\sim} \eta$.

Theorem 3. Let ξ and η be two M-immersions. Then $\xi \overset{Aff(n)}{\sim} \eta$ if and only if $\nabla(\xi) = \nabla(\eta)$.

Proof. $\Rightarrow$: $\xi \overset{Aff(n)}{\sim} \eta$. Then, by Proposition 2, since coefficients $\Gamma_{ij}^k\{\xi\}$ of $\nabla(\xi)$ is Aff (n)-invariant, we have $\nabla(\xi) = \nabla(\eta)$.

$\Leftarrow$: Conversely, assume that $\nabla(\xi) = \nabla(\eta)$. Then $\Gamma_{ij}^k\{\xi(u)\} = \Gamma_{ij}^k\{\eta(u)\}$ holds for all $i,j,s = 1,2,\dots,n$ and for all elements u of a chart of M. We put the matrices

$A(\xi) = \|\partial u_1\xi \dots \partial u_n\xi\|$, $\partial u_i A(\xi) = \|\partial u_i(\partial u_1\xi) \dots \partial u_i(\partial u_n\xi)\|$, where $\partial u_1\xi$ is a column-vector and for all elements u of a chart of M. The Equation (9) implies

$$A(\xi)^{-1}\, \partial u_i A(\xi) = \left\|\Gamma_{ij}^k\{\xi\}\right\|_{i,j,k=1,\dots,n}$$

Since $\Gamma_{ij}^k\{\xi(u)\} = \Gamma_{ij}^k\{\eta(u)\}$ for all $i,j,k = 1,2,\dots,n$, we get

$$A(\xi(u))^{-1}\, \partial u_i A(\xi(u)) = A(\eta(u))^{-1}\, \partial u_i A(\eta(u))$$

for all $i = 1,2,\dots,n$ and for all elements u of a chart of M.

The equation $A(\xi)^{-1}\, \partial u_i A(\xi) = A(\eta)^{-1}\, \partial u_i A(\eta)$ implies

$$\begin{aligned}\partial u_i(A(\eta(u))A(\xi(u))^{-1}) &= (\partial u_i A(\eta(u)))A(\xi(u))^{-1} + A(\eta(u))\, \partial u_i(A(\xi(u))^{-1}) = \\ &(\partial u_i A(\eta(u)))A(\xi(u))^{-1} - A(\eta(u))A(\xi(u))^{-1}(\partial u_i A(\xi(u)))A(\xi(u))^{-1} = \\ &A(\eta(u))(A(\eta(u))^{-1}\, \partial u_i A(\eta(u)) - A(\xi(u))^{-1}\, \partial u_i A(\xi(u)))A(\xi(u))^{-1} = 0\end{aligned}$$

for all elements u of a chart of M. From the last equality, we get $A(\eta(u))A(\xi(u))^{-1}$ is not depend on the element u of a chart of M. Since M is a connected immersion, it is obvious that $A(\eta(p))A(\xi(p))^{-1}$ does not depend on $p \in M$.

Let $F = A(\eta(u))A(\xi(u))^{-1}$. Since $\det A(\xi(u)) \neq 0$ and $\det A(\eta(u)) \neq 0$ for all $u \in M$, we have $\det F \neq 0$ and $A(\eta(p)) = FA(\xi(p))$ for all $p \in M$. The equality $A(\eta(u)) = FA(\xi(u))$ implies $\partial u_i \eta(u) = F\,\partial u_i \xi(u)$ for all $i = 1,2,\dots,n$ and for all elements u of a chart of M. Then there is $b \in \mathbb{R}^n$ such that $\eta(u) = F\xi(u) + b$ for all for all elements u of a chart of M. Since M is connected immersion, we see that b does not depend on $\alpha \in A$. Remark 3 By the definition of the complete systems of invariants [9], this theorem means that $G(\xi)$ is a complete systems of affine invariants of the immersion ξ. Moreover, every Aff (n)-invariant of an immersion ξ is a function of elements of $G(\xi)$.

Theorem 4. Let ξ and η be two M-immersions. Then

$\xi \overset{SAff(n)}{\sim} \eta$ if and only if $\nabla(\xi) = \nabla(\eta)$ and $\omega(\xi) = \omega(\eta)$.

Proof. $\Rightarrow$: $\xi \overset{SAff(n)}{\sim} \eta$. Since coefficients $\Gamma_{ij}^k\{\xi\}$ and $\omega(\xi)$ are $SAff(n)$-invariant, we have $\nabla(\xi) = \nabla(\eta)$ and $\omega(\xi) = \omega(\eta)$.

$\Leftarrow$: Conversely, assume that $\nabla(\xi) = \nabla(\eta)$ and $\omega(\xi) = \omega(\eta)$. From the equality $\nabla(\xi) = \nabla(\eta)$, we obtain $\xi \overset{Aff(n)}{\sim} \eta$. Since $\xi \overset{Aff(n)}{\sim} \eta$, there are $F \in GL(n)$ and $b \in \mathbb{R}^n$ such that $\eta(p) = F\xi(p) + b$ for all $p \in M$. Using this equality and $\omega(\xi) = \omega(\eta)$ in local coordinates, we get

$[\partial u_1 \eta\, \partial u_2\, \eta \dots \partial u_n \eta] = [\partial u_1 F\xi\, \partial u_2 F\xi \dots \partial u_n F\xi] = \det F\, [\partial u_1 \xi\, \partial u_2 \xi \dots \partial u_n \xi]$

Since $[\partial u_1 \xi\, \partial u_2 \xi \dots \partial u_n \xi] \neq 0$ for all $p \in M$, we obtain $\det F = 1$. That is $\xi \overset{SAff(n)}{\sim} \eta$.

Remark 4. This theorem means that every $SAff\ (n)$-invariant of an immersion ξ is a function of elements of $G(\xi)$ and the function $[\partial u_1 \xi\, \partial u_2 \xi \dots \partial u_n \xi]$.

4. RELATIONS BETWEEN THE TORSION-FREE TENSOR AND RIEMANNIAN CURVATURE TENSOR OF AN IMMERSION

Let M be a connected C^∞-manifold of dimension $\dim M = n$, and $\xi: M \to \mathbb{R}^n$ a C^∞ immersion, i.e. a differentiable mapping of rank n.

Let $\Lambda = \{(\phi_\alpha, U_\alpha,), \alpha \in A\}$ a collection of charts of M.

Then we can be given an $(n \times n)$-matrix C^∞-function $\xi^{(u)}(p)$ by $\left\|\xi_1^{(u)}(p) \dots \xi_n^{(u)}(p)\right\|$, where $\xi^{(u)}$ is a representation of ξ in the local coordinates $u = (u_1, \dots, u_n)$ of U_α and $\xi_i^{(u)}$ for all $i = 1,2,\dots,n$ is a column matrix form of $\xi^{(u)}$.

The following definition is taken from [25]:

Definition 4. A collection of an $(n \times n)$-matrix C^∞-function

$$\xi^{(u)}(a) = \left\|\xi_1^{(u)}(a)\xi_2^{(u)}(a) \dots \xi_n^{(u)}(a)\right\|$$

on M will be called a covariant tensor field of rank 1 if it is transformed according to law

$$\xi_i^{(v)} = \sum_{s=1}^{n} \frac{\partial u_s}{\partial v_i} \xi_s^{(u)}$$

when passing from one chart to another; here $v_1, \dots, v_n$ and $u_1, \dots, u_n$ are, respectively, "old" and "new" coordinates in the intersection of the charts. Let $R(\gamma)$ and $T(\gamma)$ be the Riemannian curvature tensor and the torsion tensor of a connection γ on M, resp.

Theorem 5. Let M be a simply connected C^∞-manifold and γ be a connection on M such that $R(\gamma) = 0$ and $T(\gamma) = 0$. Then there is an M-immersion η in $\mathbb{R}^n$ such that $\nabla(\eta) = \gamma$.

Proof. Let $\gamma^{(u)} = \{\gamma_{ij}^k(u), i, j, k = 1,2, \dots, n\}$ be the expressing of the Christoffel symbols of γ in the local coordinates $u = (u_1, \dots, u_n)$ of U_α. Set $Q_k\{\gamma(u)\} = \|\gamma_{ij}^k(u)\|_{i,j=1,2,\dots,n}$ for $k = 1,2, \dots, n$. For the following system of equations

$$\frac{\partial}{\partial u_k}\xi^{(u)} = \xi^{(u)} Q_k\{\gamma(u)\} \tag{10}$$

where $\mathrm{k} = 1,2, \dots, \mathrm{n}$, for an $(n \times n)$-matrix C^∞-function $\xi^{(u)}(p) = \left\|\xi_1^{(u)}(p)\xi_2^{(u)}(p) \dots \xi_n^{(u)}(p)\right\|$ on M, where $\xi^{(u)}$ is a covariant tensor field of the 1st-rank on M. It is obvious that the form of the system (10) of equations is the same in 'old"' and 'new' coordinates in the intersection of the charts. Since the Riemannian curvature tensor of γ is equal to zero, the following system of equations

$$\frac{\partial}{\partial u_k} Q_l\{\gamma(u)\} - \frac{\partial}{\partial u_l} Q_k\{\gamma(u)\} = [Q_l\{\gamma(u)\}, Q_k\{\gamma(u)\}] \tag{11}$$

for $l, k = 1,2, \dots, n$ holds, where $[Q_l\{\gamma\}, Q_k\{\gamma\}]$ denotes $Q_l\{\gamma\}Q_k\{\gamma\} - Q_l\{\gamma\}Q_k\{\gamma\}$. Let $p_0 \in U_\mu$. By (11) and according to the theory of linear differential equations, there exist a neighborhood $V \subset U_\mu$ of the point p_0 and an $(n \times n)$-matrix C^∞-function $\xi^{(u)}(p)$ on V such that $\det\left(\xi^{(u)}(p)\right) \neq 0$ for all $p \in V$, and $\xi^{(u)}(p)$ is a solution of (10) on V. Using connectedness and simply connectedness of the manifold M, according to the theory of linear differential equations on manifolds [26], we see that the unique an $(n \times n)$-matrix C^∞-function $\xi(p) = \|\xi_1(p)\xi_2(p) \dots \xi_n(p)\|$ on M exists such that $\xi(p)$ is a covariant tensor field of the 1st-rank on M and,$\xi(p)$ is a solution of (10) on U_α for every $\alpha \in A$, $\det(\xi(p)) \neq 0$ for all $p \in M$ and $\xi(p) = \xi^{(u)}(p)$ for all $p \in V$. Now we consider the solution $\xi(p)$.

By $Q_k\{\gamma(u)\} = \|\gamma_{lj}^k(u)\|_{l,j=1,2,\dots,n}$ and (10), we obtain

$$\frac{\partial}{\partial u_i}\xi_j^{(u)}(p) = \sum_{i=1}^{n} \gamma_{ij}^k(u)\xi_s^{(u)}(p), \frac{\partial}{\partial u_j}\xi_i^{(u)}(p) = \sum_{i=1}^{n} \gamma_{ji}^k(u)\xi_k^{(u)}(p) \tag{12}$$

in each chart of M with local coordinates $u = (u_1, u_2, \dots, u_n)$. Since the torsion tensor of the connection γ is equal to zero, we have $\gamma_{ij}^k(u) = \gamma_{ji}^k(u)$ for all $i, j, k = 1,2, \dots, n$ and all $u \in U_\mu$. Equation (12) and the equality $\gamma_{ij}^k(u) = \gamma_{ji}^k(u)$ imply $\frac{\partial}{\partial u_i}\xi_j^{(u)}(p) = \frac{\partial}{\partial u_j}\xi_i^{(u)}(p)$ for all $i, j = 1,2, \dots, n$ in each chart of M with local coordinates $u = (u_1, u_2, \dots, u_n)$. Since the $(n \times n)$-matrix C^∞-function

$\xi(p) = \|\xi_1(p)\xi_2(p) \dots \xi_n(p)\|$ on M is a covariant tensor field of the 1 st-rank on M and $\det(\xi(p)) \neq 0$ for all $p \in M$, the last equality implies an existence of a M immersion $\eta(p)$ such that $\frac{\partial}{\partial u_j}\xi^{(u)}(p) = \xi_j^{(u)}(p)$ for all $j = 1,2, \dots, n$ in each chart of M with local coordinates u, where $\eta^{(u)}(p)$ is the expressing of $\eta(p)$ in a local coordinates u. Hence we obtain the following system of equations

$$\frac{\partial}{\partial u_i}\frac{\partial \eta^{(u)}}{\partial u_j} = \sum_{k=1}^{n} \gamma_{ij}^{k}(u)\frac{\partial \eta^{(u)}}{\partial u_k}$$

for $i, j = 1,2, \dots, n$. This means that $\nabla(\eta) = \gamma$.

Between the volume form $\omega(\eta)$ and the connection $\nabla(\eta)$ of the M-immersion η in $\mathbb{R}^n$ there is the following system of equations

$$\frac{\partial}{\partial u_i}\left[\frac{\partial \eta^{(u)}}{\partial u_1}\frac{\partial \eta^{(u)}}{\partial u_2}\cdots\frac{\partial \eta^{(u)}}{\partial u_n}\right] = \sum_{j=1}^{n} \Gamma_{ij}^{j}\left(\eta^{(u)}\right),\ i=1,2,..,n. \quad (13)$$

Corollary 4. Let M be an open connected, simply connected subset of $\mathbb{R}^n$. Let $B(u)\ du_1 \wedge \cdots \wedge du_n$ be a non-zero volume form on M and γ be a connection on M such that $R(\gamma) = 0$ and $T(\gamma) = 0$. Assume that the equation hold:

$$\frac{\partial}{\partial u_i}B(u) = \sum_{j=1}^{n} \gamma_{ij}^{j}(u) \quad (14)$$

for all $i = 1,2, \dots, n$, where $\{\gamma_{ij}^{k}(u)\}$ is the system of Christoffel symbols of γ. Then there is an M-immersion η in $\mathbb{R}^n$ and $a \in \mathbb{R}$ such that $\nabla(\eta) = \gamma$ and

$$B(u) = \left[\frac{\partial \eta(u)}{\partial u_1}\frac{\partial \eta(u)}{\partial u_2}\cdots\frac{\partial \eta(u)}{\partial u_n}\right] + a$$

for all $u \in M$.

Proof. By Theorem 5, there exists an M-immersion of η in $\mathbb{R}^n$ such that $\nabla(\eta) = \gamma$. Using this equation, Equation (13) and Equation (14), we get

$$\frac{\partial}{\partial u_i}B(u) = \sum_{j=1}^{n} \gamma_{ij}^{j}(u) = \sum_{j=1}^{n} \Gamma_{ij}^{j}\left(\eta(u) = \frac{\partial}{\partial u_i}\left[\frac{\partial \eta(u)}{\partial u_1}\cdots\frac{\partial \eta(u)}{\partial u_n}\right]\right.$$

for all $i = 1,2, \dots, n$. Hence

$$\frac{\partial}{\partial u_i}B(u) = \frac{\partial}{\partial u_i}\left[\frac{\partial \eta(u)}{\partial u_1}\frac{\partial \eta(u)}{\partial u_2}\cdots\frac{\partial \eta(u)}{\partial u_n}\right]$$

for all $i = 1,2, \dots, n$. These equations imply an existence of $a \in \mathbb{R}$ such that

$$B(u) = \left[\frac{\partial \eta(u)}{\partial u_1}\frac{\partial \eta(u)}{\partial u_2}\cdots\frac{\partial \eta(u)}{\partial u_n}\right] + a$$

for all $u \in M$.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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