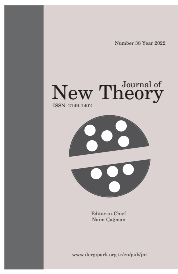
Journal of
New Theory
ISSN: 2149-1402

38 (2022) 1-13
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Another View on Picture Fuzzy Soft Sets and Their Product Operations with Soft Decision-Making

Samet Memiş[1]

Article History
Received: 15 Dec 2021
Accepted: 29 Mar 2022
Published: 31 Mar 2022
10.53570/jnt.1037280
Research Article

Abstract — Cuong [Picture Fuzzy Sets, Journal of Computer Science and Cybernetics 30 (4) (2014) 409–420] has introduced the concept of picture fuzzy soft sets (*pfs*-sets) relying on his definition and operations of picture fuzzy sets (*pf*-sets), in which there exist some inconsistencies. Yang et al. [Adjustable Soft Discernibility Matrix Based on Picture Fuzzy Soft Sets and Its Applications in Decision Making, Journal of Intelligent & Fuzzy Systems 29 (4) (2015) 1711–1722] have claimed that they have introduced the concept of *pfs*-sets with the inconsistencies in Cuong's definition of *pf*-sets. Therefore, this study redefines the concept of *pfs*-sets to deal with the inconsistencies therein. Moreover, it investigates some of the properties of *pfs*-sets and their product operations and proposes a soft decision-making method via *pfs*-sets. Finally, *pfs*-sets, their product operations, and the proposed method are discussed for further research.

Keywords — *Fuzzy sets, intuitionistic fuzzy sets, picture fuzzy sets, soft sets, picture fuzzy soft sets*

Mathematics Subject Classification (2020) — 03E72, 03E99

1. Introduction

Various uncertainties may occur in real-world problems. Classical mathematical tools are inadequate in modelling such uncertainties. To overcome this problem, introducing of new mathematical tools are needed. One of the well-known mathematical tool to model uncertainty is fuzzy sets [1]. In a short time, it has been applied to pure mathematics such as algebra, topology, and mathematical analysis and computer science such as machine learning, image processing, and artificial intelligence [2]. Shortly after the introducing of fuzzy sets, intuitionistic fuzzy sets [3] have been proposed as an extension of fuzzy sets to model further uncertainty than fuzzy uncertainty. An element of a considered fuzzy set has a membership degree denoted by $\mu(x)$ while those of a considered intuitionistic fuzzy set has the membership and non-membership degrees denoted by $\mu(x)$ and $\nu(x)$ such that $\mu(x) + \nu(x) \leq 1$, respectively. A intuitionistic fuzzy set represents as a fuzzy set if $\mu(x)+\nu(x) = 1$, whose the membership and non-membership degrees are equal to $\mu(x)$ and $1 - \mu(x)$, respectively. Moreover, the indeterminacy degrees of fuzzy sets and intuitionistic fuzzy sets are equal to 0 and $1 - (\mu(x) + \nu(x))$, respectively.

One of the other state-of-the-art mathematical tools is soft sets defined by Molodstov [4] in 1999 to parameterise the alternative set for the considered problems without employing the specific membership functions. Due to its ease of implementation, it has been applied to a great variety of fields such as algebra [5–7], topology [8–10], decision-making [11–15], and machine learning [16–18]. After that, the hybrid structures of fuzzy sets and soft sets are studied, and fuzzy soft sets [19, 20], fuzzy parameterized

[1]samettmemis@gmail.com (Corresponding Author)
[1]Department of Computer Engineering, Faculty of Engineering and Natural Sciences, İstanbul Rumeli University, İstanbul, Turkey

soft sets [21], and fuzzy parameterized fuzzy soft sets [22] are introduced to model problems containing fuzzy parameters or alternatives.

In the real world, many more problems and uncertainties are encountered that fuzzy sets and intuitionistic fuzzy sets can not model. For example, let us consider a voting process for an election. The electorate's decisions in the process may separate into three types: yes, no, and abstain. To deal with this problem, Cuong [23] has introduced the concept of picture fuzzy sets (*pf*-sets). The membership, neutral membership, and non-membership degrees are denoted by $\mu(x)$, $\eta(x)$, and $\nu(x)$, respectively, for a *pf*-set such that $\mu(x)+\eta(x)+\nu(x) \leq 1$. In the Cuong's definition, the indeterminacy degree is denoted by $1-(\mu(x)+\eta(x)+\nu(x))$ for a *pf*-set. In the same study [23], Cuong has put forward the concept of picture fuzzy soft sets (*pfs*-sets) to model problems containing picture fuzzy alternatives and investigate some of their properties. However, the investigation is so limited, and Cuong's definitions and operations of *pf*-sets and *pfs*-sets have theoretical inconsistencies.

Recently, *pfs*-sets have been redefined [24] relying on definition of Cuong's *pf*-sets without mentioning the definition of Cuong's *pfs*-sets. Therefore, the concepts of *pfs*-sets in [24] inherit from the inconsistencies [23]. To overcome the problem therein, Memiş [25], has been redefined the concept of *pf*-sets, in which $\mu(x)+\nu(x) \leq 1$ and $\mu(x)+\eta(x)+\nu(x) \leq 2$, improved their operations, and investigated their properties extensively. In this study, the main goal is that *pfs*-sets are redefined relying on the definition of *pf*-sets in [25] to deal with the inconsistencies of definition and operations in *pfs*-sets [24] and to ensure their consistency.

In Section 2 of the present study, we present concepts of fuzzy sets, intuitionistic fuzzy sets, *pf*-sets, and basic operations of *pf*-sets. In Section 3, we present the counter-examples provided in [25] related to Cuong's definitions and operations and motivation of the redefining of *pfs*-sets. In Section 4, we redefine the concept of *pfs*-sets, investigate and revise some of its basic operations, and define the product operations of *pfs*-sets. In Section 5, we propose a soft decision-making method rely on the concept of *pfs*-sets and compare its ranking orders with those in [24]. Finally, we discuss *pfs*-sets, their product operations, and the proposed soft decision-making method and provide conclusive remarks for further research.

2. Preliminaries

This section provides the concepts of fuzzy sets [1], intuitionistic fuzzy sets [3], and picture fuzzy sets (*pf*-sets) [23, 25] and some of *pf*-sets' operations and properties provided in [25] by considering the notations used throughout this paper.

In the present paper, let E be a parameter set, $F(E)$ be the set of all fuzzy sets over E, and $\mu \in F(E)$. Here, a fuzzy set is denoted by $\{^{\mu(x)}x : x \in E\}$ instead of $\{(x, \mu(x)) : x \in E\}$.

Definition 2.1. [3] Let κ be a function from E to $[0,1] \times [0,1]$. Then, the set $\{(x, f(x)) : x \in E\}$, being the graphic of κ is called an intuitionistic fuzzy set (*if*-set) over E.

Here, for all $x \in E$, $\kappa(x) = (\mu(x), \nu(x))$ such that $\mu(x)+\nu(x) \leq 1$. Moreover, μ and ν are called the membership function and non-membership function, respectively, and $\pi(x) = 1-(\mu(x)+\nu(x))$ is called the degree of indeterminacy of the element $x \in E$. For brevity, we represent an intuitionistic fuzzy set over E with $\kappa = \left\{^{\mu(x)}_{\nu(x)}x : x \in E\right\}$ instead of $\kappa = \{(x, \mu(x), \nu(x)) : x \in E\}$. Obviously, each ordinary fuzzy set can be written as $\left\{^{\mu(x)}_{1-\mu(x)}x : x \in E\right\}$.

Definition 2.2. [25] Let κ be a function from E to $[0,1] \times [0,1] \times [0,1]$. Then, the set $\{(x, f(x)) : x \in E\}$, being the graphic of κ is called a picture fuzzy set (*pf*-set) over E.

Here, for all $x \in E$, $\kappa(x) = (\mu(x), \eta(x), \nu(x))$ such that $0 \leq \mu(x)+\nu(x) \leq 1$ and $0 \leq \mu(x)+\eta(x)+\nu(x) \leq 2$. We denote a *pf*-set over E by $\kappa = \left\{\left\langle\begin{smallmatrix}\mu(x)\\ \eta(x)\\ \nu(x)\end{smallmatrix}\right\rangle x : x \in E\right\}$ instead of $\kappa = \{(x, \mu(x), \eta(x), \nu(x)) : x \in E\}$ for brevity.

Moreover, μ, η, and ν are called the membership function, neutral membership function, and non-membership function, respectively,

Note 2.3. Indeterminacy-membership of the element $x \in E$ in a pf-set over E must be defined by $\pi(x) = 1 - (\mu(x) + \nu(x))$ in order to that a pf-set can model a real-world problem and has theoretical consistency.

Manifestly, each ordinary fuzzy set can be written as $\left\{ \left\langle \begin{smallmatrix} \mu(x) \\ 1 \\ 1-\mu(x) \end{smallmatrix} \right\rangle x : x \in E \right\}$ and each intuitionistic fuzzy set can be written as $\left\{ \left\langle \begin{smallmatrix} \mu(x) \\ 1 \\ \nu(x) \end{smallmatrix} \right\rangle x : x \in E \right\}$.

In the present paper, the set of all the pf-sets over E is denoted by $PF(E)$ and $\kappa \in PF(E)$. In $PF(E)$, since the graph(κ) and κ have generated each other uniquely, the notations are interchangeable. Therefore, we represent a pf-set graph(κ) with κ as long as it causes no confusion.

Example 2.4. Let $E = \{x_1, x_2, x_3, x_4\}$. Then,

$$\kappa_1 = \left\{ \left\langle \begin{smallmatrix} 0.6 \\ 0.4 \\ 0.2 \end{smallmatrix} \right\rangle x_1, \left\langle \begin{smallmatrix} 0.3 \\ 0 \\ 0.4 \end{smallmatrix} \right\rangle x_2, \left\langle \begin{smallmatrix} 0.7 \\ 1 \\ 0.2 \end{smallmatrix} \right\rangle x_3, \left\langle \begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix} \right\rangle x_4 \right\}$$

and

$$\kappa_2 = \left\{ \left\langle \begin{smallmatrix} 0.2 \\ 0.7 \\ 0.1 \end{smallmatrix} \right\rangle x_1, \left\langle \begin{smallmatrix} 0.1 \\ 0 \\ 0.9 \end{smallmatrix} \right\rangle x_2, \left\langle \begin{smallmatrix} 0.2 \\ 0.8 \\ 0.3 \end{smallmatrix} \right\rangle x_3, \left\langle \begin{smallmatrix} 0.8 \\ 0 \\ 1 \end{smallmatrix} \right\rangle x_4 \right\}$$

are two pf-sets over E.

Definition 2.5. [25] Let $\kappa \in PF(E)$. For all $x \in E$, if $\mu(x) = \lambda$, $\eta(x) = \varepsilon$, and $\nu(x) = \omega$, then κ is called $(\lambda, \varepsilon, \omega)$-$pf$-set and is denoted by $\left\langle \begin{smallmatrix} \lambda \\ \varepsilon \\ \omega \end{smallmatrix} \right\rangle E$.

Definition 2.6. [25] Let $\kappa \in PF(E)$. For all $x \in E$, if $\mu(x) = 0$, $\eta(x) = 1$, and $\nu(x) = 1$, then κ is called empty pf-set and is denoted by $\left\langle \begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right\rangle E$ or 0_E.

Definition 2.7. [25]Let $\kappa \in PF(E)$. For all $x \in E$, if $\mu(x) = 1$, $\eta(x) = 0$, and $\nu(x) = 0$, then κ is called universal pf-set and is denoted by $\left\langle \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\rangle E$ or 1_E.

Definition 2.8. [25] Let $\kappa_1, \kappa_2 \in PF(E)$. For all $x \in E$, if $\mu_1(x) \leq \mu_2(x)$, $\eta_1(x) \geq \eta_2(x)$, and $\nu_1(x) \geq \nu_2(x)$, then κ_1 is called a subset of κ_2 and is denoted by $\kappa_1 \tilde{\subseteq} \kappa_2$.

Definition 2.9. [25] Let $\kappa_1, \kappa_2 \in PF(E)$. For all $x \in E$, if $\mu_1(x) = \mu_2(x)$, $\eta_1(x) = \eta_2(x)$, and $\nu_1(x) = \nu_2(x)$, then κ_1 and κ_2 are called equal pf-sets and is denoted by $\kappa_1 = \kappa_2$.

Definition 2.10. [25] Let $\kappa_1, \kappa_2 \in PF(E)$. If $\kappa_1 \tilde{\subseteq} \kappa_2$ and $\kappa_1 \neq \kappa_2$, then κ_1 is called a proper subset of κ_2 and is denoted by $\kappa_1 \tilde{\subsetneq} \kappa_2$.

Definition 2.11. [25] Let $\kappa_1, \kappa_2, \kappa_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \min\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called union of κ_1 and κ_2 and is denoted by $\kappa_3 = \kappa_1 \tilde{\cup} \kappa_2$.

Definition 2.12. [25] Let $\kappa_1, \kappa_2, \kappa_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \min\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \max\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called intersection of κ_1 and κ_2 and is denoted by $\kappa_3 = \kappa_1 \tilde{\cap} \kappa_2$.

Definition 2.13. [25] Let $\kappa_1, \kappa_2 \in PF(E)$. For all $x \in E$, if $\mu_2(x) = \nu_1(x)$, $\eta_2(x) = 1 - \eta_1(x)$, and $\nu_2(x) = \mu_1(x)$, then κ_2 is called complement of κ_1 and is denoted by $\kappa_2 = \kappa_1^{\tilde{c}}$.

Definition 2.14. [25] Let $\kappa_1, \kappa_2, \kappa_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \min\{\mu_1(x), \nu_2(x)\}$, $\eta_3(x) = \max\{\eta_1(x), 1 - \eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x), \mu_2(x)\}$, then κ_3 is called difference between κ_1 and κ_2, and is denoted by $\kappa_3 = \kappa_1 \tilde{\setminus} \kappa_2$.

Definition 2.15. [25] Let $\kappa_1, \kappa_2, \kappa_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\min\{\mu_1(x), \nu_2(x)\}, \min\{\mu_2(x), \nu_1(x)\}\}$, $\eta_3(x) = \min\{\max\{\eta_1(x), 1 - \eta_2(x)\}, \max\{\eta_2(x), 1 - \eta_1(x)\}\}$, and $\nu_3(x) = \min\{\max\{\nu_1(x), \mu_2(x)\}, \max\{\nu_2(x), \mu_1(x)\}\}$, then κ_3 is called symmetric difference between κ_1 and κ_2, and is denoted by $\kappa_3 = \kappa_1 \tilde{\triangle} \kappa_2$.

3. Motivations of the Redefining of Picture Fuzzy Soft Sets

This section presents the definition and basic operations of picture fuzzy sets and the counter examples for the Cuong's definition provided in [23] and [25], respectively, considering the notations used across the present paper.

Definition 3.1. [23] Let κ be a function from E to $[0,1]\times[0,1]\times[0,1]$. Then, the set $\{(x, f(x)) : x \in E\}$, being the graphic of κ is called a picture fuzzy set (*pf*-set) over E.

In this section, the set of all the *pf*-sets over E according to Cuong's definition is denoted by $PF_C(E)$ and $\kappa \in PF_C(E)$.

Definition 3.2. [23] Let $\kappa_1, \kappa_2 \in PF_C(E)$. For all $x \in E$, if $\mu_1(x) \leq \mu_2(x)$, $\eta_1(x) \leq \eta_2(x)$, and $\nu_1(x) \geq \nu_2(x)$, then κ_1 is called a subset of κ_2 and is denoted by $\kappa_1 \tilde{\subseteq} \kappa_2$.

Definition 3.3. [23] Let $\kappa_1, \kappa_2 \in PF_C(E)$. If $\kappa_1 \tilde{\subseteq} \kappa_2$ and $\kappa_2 \tilde{\subseteq} \kappa_1$, then κ_1 and κ_2 are called equal *pf*-sets and is denoted by $\kappa_1 = \kappa_2$.

Definition 3.4. [23] Let $\kappa_1, \kappa_2, \kappa_3 \in PF_C(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \min\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called union of κ_1 and κ_2, and is denoted by $\kappa_3 = \kappa_1 \tilde{\cup} \kappa_2$.

Definition 3.5. [23] Let $\kappa_1, \kappa_2, \kappa_3 \in PF_C(E)$. For all $x \in E$, if $\mu_3(x) = \min\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called intersection of κ_1 and κ_2, and is denoted by $\kappa_3 = \kappa_1 \tilde{\cap} \kappa_2$.

Definition 3.6. [23] Let $\kappa_1, \kappa_2 \in PF_C(E)$. For all $x \in E$, if $\mu_2(x) = \nu_1(x)$, $\eta_2(x) = \eta_1(x)$, and $\nu_2(x) = \mu_1(x)$, then κ_2 is called complement of κ_1 and is denoted by $\kappa_2 = \kappa_1^{\tilde{c}}$.

Memiş [25] have provided the following several counter-examples related to definition and operations of *pf*-sets in [23]. According to Definition 3.2, the definitions of empty and universal *pf*-sets should be as in Definition 3.7 and Definition 3.8, respectively, to be held the following conditions [25]:

- Empty *pf*-set over E is a subset of all the *pf*-set over E.
- All *pf*-sets over E are the subset of universal *pf*-set over E.

Definition 3.7. [25] Let $\kappa \in PF_C(E)$. For all $x \in E$, if $\mu(x) = 0$, $\eta(x) = 0$, and $\nu(x) = 1$, then κ is called empty *pf*-set and is denoted by $\left\langle \begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix} \right\rangle E_C$ or 0_{E_C}.

Definition 3.8. [25] Let $\kappa \in PF_C(E)$. For all $x \in E$, if $\mu(x) = 1$, $\eta(x) = 1$, and $\nu(x) = 0$, then κ is called empty *pf*-set and is denoted by $\left\langle \begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix} \right\rangle E_C$ or 1_{E_C}.

Example 3.9. [25] There is a contradiction in Definition 3.8 since $1 + 1 + 0 \nleq 1$, i.e., $1_{E_C} \notin PF_C(E)$. On the other hand, even if $1_{E_C} \in PF_C(E)$, $(1_{E_C})^{\tilde{c}} \neq 0_{E_C}$.

Example 3.10. [25] Let $\kappa \in PF_C(E)$ such that $\kappa = \left\{ \left\langle \begin{smallmatrix} 0.1 \\ 0.2 \\ 0.3 \end{smallmatrix} \right\rangle x \right\}$. Then, $\kappa \tilde{\cup} 0_E \neq \kappa$ and $\kappa \tilde{\cup} 1_{E_C} \neq 1_{E_C}$.

To deal with the aforesaid inconsistencies in Example 3.9 and 3.10, the concept of *pf*-sets and their operations have been redefined by Memiş [25].

Secondly, the definitions of picture fuzzy soft sets (*pfs*-sets) provided in [23, 24] considering the notations used across the present paper.

Definition 3.11. [23] Let E be the set of parameters and $A \subseteq E$ set. A pair (F, A) is called *pfs*-set over U, where F is a mapping given by $F : A \to PF_C(U)$.

Definition 3.12. [24] Let U be the initial universe set and E a set of parameters. By *pfs*-set over U we mean a pair $\langle F, A\rangle$, where $A \subseteq E$ and F is a mapping given by $F : A \to PF_C(U)$.

Cuong [23] has defined the concept of *pfs*-sets relying on his own definition and operations of *pf*-sets. Therefore, the aforementioned inconsistencies have transferred to his concept of *pfs*-sets. Moreover, Yang et. al. [24] have claimed that they have introduced the concept of *pfs*-sets while Cuong has defined the concept of *pfs*-sets in [23]. Although the *pfs*-sets have been redefined in [24], the inconsistencies mentioned above has also transferred to the concept of *pfs*-sets due to it based on the definition and operations of *pf*-sets in [23].

Therefore, the concept of *pfs*-sets should be redefined to overcome the inconsistencies in the concept of *pfs*-sets and their operations.

4. Picture Fuzzy Soft Sets, Some of Their Properties, and Their Product Operations

In this section, we redefine the concepts of *pfs*-sets and investigate some of their properties according to new definition herein by considering the notations used throughout the present paper.

Definition 4.1. Let U be a universal set, E be a parameter set, and f is a function from E to $PF(U)$. Then the set $\{(x, f_A(x)) : x \in E\}$,being the graphic of f, is called a picture fuzzy soft set (*pfs*-set) parameterized via E over U (or briefly over U).

Example 4.2. Let $E = \{x_1, x_2, x_3, x_4\}$ be a parameter set and $U = \{u_1, u_2, u_3, u_4\}$ be a universal set. Then,

$$f = \left\{ \left(x_1, \left\{ \left\langle \begin{smallmatrix} 0.4 \\ 0.1 \\ 0.9 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0 \\ 0.7 \\ 0.3 \end{smallmatrix} \right\rangle u_4 \right\} \right), \left(x_2, \left\{ \left\langle \begin{smallmatrix} 1 \\ 0.2 \\ 0 \end{smallmatrix} \right\rangle u_2 \right\} \right), (x_3, 0_U), \left(\left\langle \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\rangle x_4, 1_U \right) \right\}$$

is a *pfs*-set over U.

Note 4.3. We do not display the element $(x, 0_U)$ in a *pfs*-set where 0_U is empty *pf*-set over U.

Henceforth, the set of all the *pfs*-sets over U is denoted by $PFS(U)$. In $PFS(U)$, the notations graph(f) and f are interchangeable since they have generated each other uniquely. Thus, a *pfs*-set graph(f) is denoted by f as long as it leads no confusion.

Definition 4.4. Let $f \in PFS(U)$. If for all $x \in E$, $f(x) = \left\langle \begin{smallmatrix} \lambda \\ \varepsilon \\ \omega \end{smallmatrix} \right\rangle U$, then f is called $(\lambda, \varepsilon, \omega)$-*pfs*-set and is denoted by $\left(E, \left\langle \begin{smallmatrix} \lambda \\ \varepsilon \\ \omega \end{smallmatrix} \right\rangle U \right)$.

Definition 4.5. Let $f \in PFS(U)$ and f be $(\lambda, \varepsilon, \omega)$-*pfs*-set. If $\lambda = 0$, $\varepsilon = 1$, and $\omega = 1$, then f is called empty *pfs*-set and is denoted by $\left(E, \left\langle \begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right\rangle U \right)$ or briefly $\tilde{0}$.

Definition 4.6. Let $f \in PFS(U)$ and f be $(\lambda, \varepsilon, \omega)$-*pfs*-set. If $\lambda = 1$, $\varepsilon = 0$, and $\omega = 0$, then f is called universal *pfs*-set and is denoted by $\left(E, \left\langle \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\rangle U \right)$ or briefly $\tilde{1}$.

Definition 4.7. Let $f, f_1 \in PFS(U)$ and $A \subseteq E$. Then, A_{f_1}-restriction of f, denoted by f_{Af_1}, is defined by

$$f_{Af_1}(x) := \begin{cases} f(x), & x \in A \\ f_1(x), & x \in E \setminus A \end{cases}$$

Briefly, if $f_1 = \tilde{0}$, then f_A can be employed instead of f_{Af_1}. It is clear that

$$f_A(x) := \begin{cases} f(x), & x \in A \\ \tilde{0}, & x \in E \setminus A \end{cases}$$

Example 4.8. Let us consider the *pfs*-set f provided in Example 4.2, $A = \{x_1, x_3\}$, and $f_1 \in PFS(U)$ such that

$$f_1 = \left\{ (x_1, 1_U), \left(x_4, \left\{ \left\langle \begin{matrix} 0.2 \\ 0.5 \\ 0.4 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0.6 \\ 0.3 \\ 0.2 \end{matrix} \right\rangle u_4 \right\} \right) \right\}$$

Then,

$$f_{Af_1} = \left\{ \left(x_1, \left\{ \left\langle \begin{matrix} 0.4 \\ 0.1 \\ 0.9 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0 \\ 0.7 \\ 0.3 \end{matrix} \right\rangle u_4 \right\} \right), \left(x_4, \left\{ \left\langle \begin{matrix} 0.2 \\ 0.5 \\ 0.4 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0.6 \\ 0.3 \\ 0.2 \end{matrix} \right\rangle u_4 \right\} \right) \right\}$$

Definition 4.9. 4.10 Let $f_1, f_2 \in PFS(U)$. If for all $x \in E$, $f_1(x) \tilde{\subseteq} f_2(x)$, then f_1 is called a subset of f_2 and is denoted by $f_1 \tilde{\subseteq} f_2$.

Proposition 4.10. Let $f, f_1, f_2, f_3 \in PFS(U)$. Then,

i. $f \tilde{\subseteq} \tilde{1}$

ii. $\tilde{0} \tilde{\subseteq} f$

iii. $f \tilde{\subseteq} f$

iv. $\left[f_1 \tilde{\subseteq} f_2 \wedge f_2 \tilde{\subseteq} f_3 \right] \Rightarrow f_1 \tilde{\subseteq} f_3$

Remark 4.11. $f_1 \tilde{\subseteq} f_2$ does not mean that every element of f_1 is an element of f_2. For instance, let $E = \{x_1, x_2\}$ be parameter set, $U = \{u_1, u_2\}$ be a universal set,

$$f_1 = \left\{ \left(x_1, \left\{ \left\langle \begin{matrix} 0.3 \\ 0.8 \\ 0.2 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0.8 \\ 0.6 \\ 0.1 \end{matrix} \right\rangle u_2 \right\} \right), \left(x_2, \left\{ \left\langle \begin{matrix} 0.3 \\ 0.6 \\ 0.7 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0.2 \\ 0.6 \\ 0.8 \end{matrix} \right\rangle u_2 \right\} \right) \right\}$$

and

$$f_2 = \left\{ \left(x_1, \left\{ \left\langle \begin{matrix} 0.8 \\ 0.6 \\ 0.1 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0.9 \\ 0.3 \\ 0.1 \end{matrix} \right\rangle u_2 \right\} \right), \left(x_2, \left\{ \left\langle \begin{matrix} 0.5 \\ 0.3 \\ 0.1 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0.3 \\ 0.3 \\ 0.2 \end{matrix} \right\rangle u_2 \right\} \right) \right\}$$

Thus, $f_1 \tilde{\subseteq} f_2$ because $f_1(x) \tilde{\subseteq} f_2(x)$ for all $x \in E$. However, $f_1 \not\subseteq f_2$ since $\left(x_1, \left\{ \left\langle \begin{matrix} 0.3 \\ 0.8 \\ 0.2 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0.8 \\ 0.6 \\ 0.1 \end{matrix} \right\rangle u_2 \right\} \right) \notin f_2$ while $\left(x_1, \left\{ \left\langle \begin{matrix} 0.3 \\ 0.8 \\ 0.2 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0.8 \\ 0.6 \\ 0.1 \end{matrix} \right\rangle u_2 \right\} \right) \in f_1$, where the notation $\subseteq$ indicates classic inclusion relation.

Definition 4.12. Let $f_1, f_2 \in PFS(U)$. If for all $x \in E$, $f_1(x) = f_2(x)$, then f_1 and f_2 are called equal *pfs*-sets and is denoted by $f_1 = f_2$.

Proposition 4.13. Let $f_1, f_2, f_3 \in PF(E)$. Then,

i. $\left[f_1 \tilde{\subseteq} f_2 \wedge f_2 \tilde{\subseteq} f_1 \right] \Leftrightarrow f_1 = f_2$

ii. $[f_1 = f_2 \wedge f_2 = f_3] \Rightarrow f_1 = f_3$

Definition 4.14. Let $f_1, f_2 \in PFS(U)$. If $f_1 \tilde{\subseteq} f_2$ and $f_1 \neq f_2$, then f_1 is called a proper subset of f_2 and is denoted by $f_1 \tilde{\subsetneq} f_2$

Definition 4.15. Let $f_1, f_2, f_3 \in PFS(U)$. If for all $x \in E$, $f_3(x) = f_1(x) \tilde{\cup} f_2(x)$, then f_3 is called union of f_1 and f_2 and is denoted by $f_3 = f_1 \tilde{\cup} f_2$.

Proposition 4.16. Let $f, f_1, f_2, f_3 \in PFS(U)$. Then,

i. $f \tilde{\cup} f = f$

ii. $f \tilde{\cup} \tilde{1} = \tilde{1}$

iii. $f \tilde{\cup} \tilde{0} = f$

iv. $f_1 \tilde{\cup} f_2 = f_2 \tilde{\cup} f_1$

v. $f_1 \tilde{\cup} (f_2 \tilde{\cup} f_3) = (f_1 \tilde{\cup} f_2) \tilde{\cup} f_3$

vi. $f_1 \tilde{\subseteq} f_2 \Rightarrow f_1 \tilde{\cup} f_2 = f_2$

Definition 4.17. Let $f_1, f_2, f_3 \in PFS(U)$. If for all $x \in E$, $f_3(x) = f_1(x) \tilde{\cap} f_2(x)$, then f_3 is called intersection of f_1 and f_2 and is denoted by $f_3 = f_1 \tilde{\cap} f_2$.

Proposition 4.18. Let $f, f_1, f_2, f_3 \in PFS(U)$. Then,

i. $f \tilde{\cap} f = f$

ii. $f \tilde{\cap} \tilde{1} = f$

iii. $f \tilde{\cap} \tilde{0} = \tilde{0}$

iv. $f_1 \tilde{\cap} f_2 = f_2 \tilde{\cap} f_1$

v. $f_1 \tilde{\cap} (f_2 \tilde{\cap} f_3) = (f_1 \tilde{\cap} f_2) \tilde{\cap} f_3$

vi. $f_1 \tilde{\subseteq} f_2 \Rightarrow f_1 \tilde{\cap} f_2 = f_1$

Proposition 4.19. Let $f_1, f_2, f_3 \in PFS(U)$. Then,

i. $f_1 \tilde{\cup} (f_2 \tilde{\cap} f_3) = (f_1 \tilde{\cup} f_2) \tilde{\cap} (f_1 \tilde{\cup} f_3)$

ii. $f_1 \tilde{\cap} (f_2 \tilde{\cup} f_3) = (f_1 \tilde{\cap} f_2) \tilde{\cup} (f_1 \tilde{\cup} f_3)$

PROOF. *i.* Let $f_1, f_2, f_3 \in PFS(U)$. Then,

$$
\begin{aligned}
f_1 \tilde{\cup} (f_2 \tilde{\cap} f_3) &= \{(x, f_1(x)) : x \in E\} \tilde{\cup} \{(x, f_2(x) \tilde{\cap} f_3(x)) : x \in E\} \\
&= \{(x, f_1(x) \tilde{\cup} (f_2(x) \tilde{\cap} f_3(x))) : x \in E\} \\
&= \{(x, (f_1(x) \tilde{\cup} f_2(x)) \tilde{\cap} (f_1(x) \tilde{\cup} f_3(x))) : x \in E\} \\
&= \{(x, (f_1(x) \tilde{\cup} f_2(x))) : x \in E\} \tilde{\cap} \{(x, (f_1(x) \tilde{\cup} f_3(x))) : x \in E\} \\
&= (f_1 \tilde{\cup} f_2) \tilde{\cap} (f_1 \tilde{\cup} f_3)
\end{aligned}
$$

□

Definition 4.20. Let $f_1, f_2 \in PFS(U)$. If $f_1 \tilde{\cap} f_2 = \tilde{0}$, then f_1 and f_2 are called disjoint *pfs*-sets.

Definition 4.21. Let $f_1, f_2 \in PFS(U)$. If for all $x \in E$, $f_2(x) = f_1^{\tilde{c}}(x)$, then f_2 is called complement of f_1 and is denoted by $f_2 = f_1^{\tilde{c}}$.

Proposition 4.22. Let $f, f_1, f_2 \in PFS(U)$. Then,

i. $\left(f^{\tilde{c}}\right)^{\tilde{c}} = f$

ii. $\tilde{0}^{\tilde{c}} = \tilde{1}$

iii. $f_1 \tilde{\subseteq} f_2 \Rightarrow f_2^{\tilde{c}} \tilde{\subseteq} f_1^{\tilde{c}}$

Definition 4.23. Let $f_1, f_2, f_3 \in PFS(U)$. If for all $x \in E$, $f_3(x) = f_1(x) \tilde{\setminus} f_2(x)$, then f_3 is called difference between f_1 and f_2 and is denoted by $f_3 = f_1 \tilde{\setminus} f_2$.

Proposition 4.24. Let $f, f_1, f_2 \in PFS(U)$. Then,

i. $f \tilde{\setminus} \tilde{0} = f$

ii. $f \tilde{\setminus} \tilde{1} = \tilde{0}$

iii. $f_1 \tilde{\setminus} f_2 = f_1 \tilde{\cap} f_2^{\tilde{c}}$

Remark 4.25. It must be noted that the difference is non-commutative and non-associative. For example, Let $f_1 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.2\\0\\0.3\end{matrix}\right\rangle u\right\}\right)\right\}$, $f_2 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.3\\0\\0.1\end{matrix}\right\rangle u\right\}\right)\right\}$, and $f_3 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.4\\0.1\\0.6\end{matrix}\right\rangle u\right\}\right)\right\}$. Then,

i. $\left[f_1\tilde{\setminus}f_2 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.1\\1\\0.3\end{matrix}\right\rangle u\right\}\right)\right\} \wedge f_2\tilde{\setminus}f_1 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.3\\1\\0.2\end{matrix}\right\rangle u\right\}\right)\right\}\right] \Rightarrow f_1\tilde{\setminus}f_2 \neq f_2\tilde{\setminus}f_1$

ii. $\left[f_1\tilde{\setminus}\left(f_2\tilde{\setminus}f_3\right) = \left\{\left(x, \left\{\left\langle\begin{matrix}0.2\\0.1\\0.3\end{matrix}\right\rangle u\right\}\right)\right\} \wedge \left(f_1\tilde{\setminus}f_2\right)\tilde{\setminus}f_3 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.1\\1\\0.4\end{matrix}\right\rangle u\right\}\right)\right\}\right] \Rightarrow f_1\tilde{\setminus}\left(f_2\tilde{\setminus}f_3\right) \neq \left(f_1\tilde{\setminus}f_2\right)\tilde{\setminus}f_3$

Proposition 4.26. Let $f_1, f_2 \in PF(E)$. Then, the following De Morgan's Laws are valid.

i. $(f_1\tilde{\cup}f_2)^{\tilde{c}} = f_1^{\tilde{c}}\tilde{\cap}f_2^{\tilde{c}}$

ii. $(f_1\tilde{\cap}f_2)^{\tilde{c}} = f_1^{\tilde{c}}\tilde{\cup}f_2^{\tilde{c}}$

PROOF. *i.* Let $f_1, f_2 \in PFS(U)$. Then,

$$\begin{aligned}
(f_1\tilde{\cup}f_2)^{\tilde{c}} &= \left(\left\{\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x, f_1\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\} \tilde{\cup} \left\{\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x, f_2\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\}\right)^{\tilde{c}} \\
&= \left(\left\{\left(\left\langle\begin{matrix}\max\{\mu_1(x),\mu_2(x)\}\\\min\{\eta_1(x),\eta_2(x)\}\\\min\{\nu_1(x),\nu_2(x)\}\end{matrix}\right\rangle x, f_1\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right)\tilde{\cup}f_2\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\}\right)^{\tilde{c}} \\
&= \left\{\left(\left\langle\begin{matrix}\min\{\nu_1(x),\nu_2(x)\}\\1-\min\{\eta_1(x),\eta_2(x)\}\\\max\{\mu_1(x),\mu_2(x)\}\end{matrix}\right\rangle x, f_1^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right)\tilde{\cap}f_2^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\} \\
&= \left\{\left(\left\langle\begin{matrix}\min\{\nu_1(x),\nu_2(x)\}\\\max\{1-\eta_1(x),1-\eta_2(x)\}\\\max\{\mu_1(x),\mu_2(x)\}\end{matrix}\right\rangle x, f_1^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right)\tilde{\cap}f_2^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\} \\
&= \left\{\left\langle\begin{matrix}\nu_1(x)\\1-\eta_1(x)\\\mu_1(x)\end{matrix}\right\rangle x, f_1^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right) : x \in E\right\} \tilde{\cap} \left\{\left(\left\langle\begin{matrix}\nu_2(x)\\1-\eta_2(x)\\\mu_2(x)\end{matrix}\right\rangle x, f_2^{\tilde{c}}\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\} \\
&= \left(\left\{\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x, f_1\left(\left\langle\begin{matrix}\mu_1(x)\\\eta_1(x)\\\nu_1(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\}\right)^{\tilde{c}} \tilde{\cap} \left(\left\{\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x, f_2\left(\left\langle\begin{matrix}\mu_2(x)\\\eta_2(x)\\\nu_2(x)\end{matrix}\right\rangle x\right)\right) : x \in E\right\}\right)^{\tilde{c}} \\
&= f_1^{\tilde{c}}\tilde{\cap}f_2^{\tilde{c}}
\end{aligned}$$

□

Definition 4.27. Let $f_1, f_2, f_3 \in PFS(U)$. If for all $x \in E$, $f_3(x) = f_1(x)\tilde{\triangle}f_2(x)$, then f_3 is called symmetric difference between f_1 and f_2 and is denoted by $f_3 = f_1\tilde{\triangle}f_2$.

Proposition 4.28. Let $f, f_1, f_2 \in PFS(U)$. Then,

i. $f\tilde{\triangle}\tilde{0} = f$

ii. $f\tilde{\triangle}\tilde{1} = f^{\tilde{c}}$

iii. $f_1\tilde{\triangle}f_2 = f_2\tilde{\triangle}f_1$

iv. $f_1\tilde{\triangle}f_2 = (f_1\tilde{\setminus}f_2)\tilde{\cup}(f_2\tilde{\setminus}f_1)$

Remark 4.29. It must be noted that the symmetric difference is non-associative. Let us consider the *pfs*-sets f_1, f_2, and f_3 provided in Remark 4.25.

Since $f_1\tilde{\triangle}\left(f_2\tilde{\triangle}f_3\right) = \left\{\left(x, \left\{\left\langle\begin{matrix}0.3\\0.1\\0.3\end{matrix}\right\rangle u\right\}\right)\right\}$ and $\left(f_1\tilde{\triangle}f_2\right)\tilde{\triangle}f_3 = \left\{\left(x, \left\{\left\langle\begin{matrix}0.3\\0.1\\0.4\end{matrix}\right\rangle u\right\}\right)\right\}$, then $f_1\tilde{\triangle}\left(f_2\tilde{\triangle}f_3\right) \neq \left(f_1\tilde{\triangle}f_2\right)\tilde{\triangle}f_3$.

We secondly present the AND, OR, ANDNOT, and ORNOT-products of *pfs*-sets and their examples.

Definition 4.30. Let $f_1 \in PFS_{E_1}(U)$, $f_2 \in PFS_{E_2}(U)$, and $f_3 \in PFS_{E_1 \times E_2}(U)$. For all $x \in E_1$ and $y \in E_2$, if

$$f_3((x,y)) := f_1(x) \tilde{\cap} f_2(y)$$

then f_3 is called AND-product of f_1 and f_2 and is denoted by $f_1 \wedge f_2$.

Definition 4.31. Let $f_1 \in PFS_{E_1}(U)$, $f_2 \in PFS_{E_2}(U)$, and $f_3 \in PFS_{E_1 \times E_2}(U)$. For all $x \in E_1$ and $y \in E_2$, if

$$f_3((x,y)) := f_1(x) \tilde{\cup} f_2(y)$$

then f_3 is called OR-product of f_1 and f_2 and is denoted by $f_1 \vee f_2$.

Definition 4.32. Let $f_1 \in PFS_{E_1}(U)$, $f_2 \in PFS_{E_2}(U)$, and $f_3 \in PFS_{E_1 \times E_2}(U)$. For all $x \in E_1$ and $y \in E_2$, if

$$f_3((x,y)) := f_1(x) \tilde{\cap} f_2^{\tilde{c}}(y)$$

then f_3 is called ANDNOT-product of f_1 and f_2 and is denoted by $f_1 \overline{\wedge} f_2$.

Definition 4.33. Let $f_1 \in PFS_{E_1}(U)$, $f_2 \in PFS_{E_2}(U)$, and $f_3 \in PFS_{E_1 \times E_2}(U)$. For all $x \in E_1$ and $y \in E_2$, if

$$f_3((x,y)) := f_1(x) \tilde{\cup} f_2^{\tilde{c}}(y)$$

then f_3 is called ORNOT-product of f_1 and f_2 and is denoted by $f_1 \underline{\vee} f_2$.

Example 4.34. Let us consider the *pfs*-sets f_1 and f_2 provided in Remark 4.11. Then,

$$f_1 \wedge f_2 = \left\{ \left((x_1,x_1), \left\{ \left\langle\begin{smallmatrix}0.3\\0.8\\0.2\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.8\\0.6\\0.1\end{smallmatrix}\right\rangle u_2 \right\} \right), \left((x_1,x_2), \left\{ \left\langle\begin{smallmatrix}0.3\\0.8\\0.2\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.3\\0.6\\0.1\end{smallmatrix}\right\rangle u_2 \right\} \right), \right.$$
$$\left. \left((x_2,x_1), \left\{ \left\langle\begin{smallmatrix}0.3\\0.6\\0.7\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.2\\0.6\\0.1\end{smallmatrix}\right\rangle u_2 \right\} \right), \left((x_2,x_2), \left\{ \left\langle\begin{smallmatrix}0.3\\0.6\\0.7\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.2\\0.6\\0.2\end{smallmatrix}\right\rangle u_2 \right\} \right) \right\}$$

$$f_1 \vee f_2 = \left\{ \left((x_1,x_1), \left\{ \left\langle\begin{smallmatrix}0.8\\0.6\\0.1\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.9\\0.3\\0.1\end{smallmatrix}\right\rangle u_2 \right\} \right), \left((x_1,x_2), \left\{ \left\langle\begin{smallmatrix}0.5\\0.3\\0.1\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.8\\0.3\\0.1\end{smallmatrix}\right\rangle u_2 \right\} \right), \right.$$
$$\left. \left((x_2,x_1), \left\{ \left\langle\begin{smallmatrix}0.8\\0.6\\0.1\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.9\\0.3\\0.1\end{smallmatrix}\right\rangle u_2 \right\} \right), \left((x_2,x_2), \left\{ \left\langle\begin{smallmatrix}0.5\\0.3\\0.1\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.3\\0.3\\0.2\end{smallmatrix}\right\rangle u_2 \right\} \right) \right\}$$

$$f_1 \overline{\wedge} f_2 \left\{ \left((x_1,x_1), \left\{ \left\langle\begin{smallmatrix}0.1\\0.8\\0.8\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.1\\0.6\\0.9\end{smallmatrix}\right\rangle u_2 \right\} \right), \left((x_1,x_2), \left\{ \left\langle\begin{smallmatrix}0.1\\0.8\\0.5\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.1\\0.7\\0.3\end{smallmatrix}\right\rangle u_2 \right\} \right), \right.$$
$$\left. \left((x_2,x_1), \left\{ \left\langle\begin{smallmatrix}0.1\\0.6\\0.8\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.1\\0.8\\0.9\end{smallmatrix}\right\rangle u_2 \right\} \right), \left((x_2,x_2), \left\{ \left\langle\begin{smallmatrix}0.1\\0.7\\0.7\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.1\\0.7\\0.8\end{smallmatrix}\right\rangle u_2 \right\} \right) \right\}$$

$$f_1 \underline{\vee} f_2 \left\{ \left((x_1,x_1), \left\{ \left\langle\begin{smallmatrix}0.3\\0.4\\0.2\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.8\\0.4\\0.1\end{smallmatrix}\right\rangle u_2 \right\} \right), \left((x_1,x_2), \left\{ \left\langle\begin{smallmatrix}0.3\\0.7\\0.2\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.8\\0.6\\0.1\end{smallmatrix}\right\rangle u_2 \right\} \right), \right.$$
$$\left. \left((x_2,x_1), \left\{ \left\langle\begin{smallmatrix}0.3\\0.4\\0.7\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.2\\0.4\\0.8\end{smallmatrix}\right\rangle u_2 \right\} \right), \left((x_2,x_2), \left\{ \left\langle\begin{smallmatrix}0.3\\0.6\\0.5\end{smallmatrix}\right\rangle u_1, \left\langle\begin{smallmatrix}0.2\\0.6\\0.3\end{smallmatrix}\right\rangle u_2 \right\} \right) \right\}$$

5. A Soft Decision-Making Method Based on *pfs*-Sets and Its Comparison

This section proposes a soft decision-making method via *pfs*-sets. Its algorithm steps are as follows:

Proposed Method

Step 1. Construct a *pfs*-set $f = \left\{ \left(x, \left\{ \left\langle \begin{smallmatrix} \mu(x) \\ \eta(x) \\ \nu(x) \end{smallmatrix} \right\rangle u \right\} \right) : x \in E \right\}$ over U.

Step 2. Compute the score values

$$s(u) = \frac{1}{n} \sum_{x \in E} [\mu_u(x) - \eta_u(x)\nu_u(x)], \text{ for all } u \in U$$

such that $\mu_u(x)$, $\eta_u(x)$, and $\nu_u(x)$ denotes the membership, neutral membership, and non-membership degrees of the alternative u according to the parameter x.

Step 3. Obtain the decision set $\{{}^{\hat{s}(u_k)}u_k \mid u_k \in U\}$ such that

$$\hat{s}(u_k) := \begin{cases} \frac{s(u_k) - \min\limits_i\{s(u_i)\}}{\max\limits_i\{s(u_i)\} - \min\limits_i\{s(u_i)\}}, & \max\limits_i\{s(u_i)\} \neq \min\limits_i\{s(u_i)\} \\ 1, & \max\limits_i\{s(u_i)\} = \min\limits_i\{s(u_i)\} \end{cases}$$

Secondly, the section provides the illustrative example in [24] to compare fairly the proposed method with those in [24].

Example 5.1. [24] Suppose that there is an investment firm that wishes to put money into the best option (adapted from [26]). Let us consider the *pfs*-set f, which describes the "attractiveness of projects" being considered for investment by the firm. Assume that there are six alternative projects, i.e., $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ such that $u_1 =$"Project-1", $u_2 =$"Project-2", $u_3 =$"Project-3", $u_4 =$"Project-4" ,$u_5 =$"Project-5", and $u_6 =$"Project-6", and four parameters, i.e., $E = \{x_1, x_2, x_3, x_4\}$ such that $x_1 =$"Risk Analysis", $x_2 =$"Growth Analysis", $x_3 =$"Social-Political Impact Analysis", and $x_4 =$"Environment Analysis", under consideration. The firm evaluates the alternatives according to the parameters and constructs a *pfs*-set f_1 as follows:

$$\begin{aligned} f_1 = \Bigg\{ & \left(x_1, \left\{ \left\langle \begin{smallmatrix} 0.31 \\ 0.22 \\ 0.41 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.12 \\ 0.41 \\ 0.33 \end{smallmatrix} \right\rangle u_2, \left\langle \begin{smallmatrix} 0.23 \\ 0.52 \\ 0.21 \end{smallmatrix} \right\rangle u_3, \left\langle \begin{smallmatrix} 0.45 \\ 0.09 \\ 0.36 \end{smallmatrix} \right\rangle u_4, \left\langle \begin{smallmatrix} 0.57 \\ 0.30 \\ 0.05 \end{smallmatrix} \right\rangle u_5, \left\langle \begin{smallmatrix} 0.44 \\ 0.40 \\ 0.13 \end{smallmatrix} \right\rangle u_6 \right\} \right), \\ & \left(x_2, \left\{ \left\langle \begin{smallmatrix} 0.54 \\ 0.21 \\ 0.15 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.81 \\ 0.11 \\ 0.02 \end{smallmatrix} \right\rangle u_2, \left\langle \begin{smallmatrix} 0.13 \\ 0.48 \\ 0.37 \end{smallmatrix} \right\rangle u_3, \left\langle \begin{smallmatrix} 0.23 \\ 0.59 \\ 0.18 \end{smallmatrix} \right\rangle u_4, \left\langle \begin{smallmatrix} 0.60 \\ 0.23 \\ 0.14 \end{smallmatrix} \right\rangle u_5, \left\langle \begin{smallmatrix} 0.42 \\ 0.36 \\ 0.22 \end{smallmatrix} \right\rangle u_6 \right\} \right), \\ & \left(x_3, \left\{ \left\langle \begin{smallmatrix} 0.60 \\ 0.14 \\ 0.26 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.26 \\ 0.51 \\ 0.20 \end{smallmatrix} \right\rangle u_2, \left\langle \begin{smallmatrix} 0.72 \\ 0.15 \\ 0.03 \end{smallmatrix} \right\rangle u_3, \left\langle \begin{smallmatrix} 0.32 \\ 0.49 \\ 0.15 \end{smallmatrix} \right\rangle u_4, \left\langle \begin{smallmatrix} 0.81 \\ 0.11 \\ 0.06 \end{smallmatrix} \right\rangle u_5, \left\langle \begin{smallmatrix} 0.43 \\ 0.27 \\ 0.13 \end{smallmatrix} \right\rangle u_6 \right\} \right), \\ & \left(x_4, \left\{ \left\langle \begin{smallmatrix} 0.38 \\ 0.21 \\ 0.40 \end{smallmatrix} \right\rangle u_1, \left\langle \begin{smallmatrix} 0.65 \\ 0.15 \\ 0.18 \end{smallmatrix} \right\rangle u_2, \left\langle \begin{smallmatrix} 0.29 \\ 0.58 \\ 0.12 \end{smallmatrix} \right\rangle u_3, \left\langle \begin{smallmatrix} 0.14 \\ 0.32 \\ 0.45 \end{smallmatrix} \right\rangle u_4, \left\langle \begin{smallmatrix} 0.43 \\ 0.18 \\ 0.35 \end{smallmatrix} \right\rangle u_5, \left\langle \begin{smallmatrix} 0.35 \\ 0.29 \\ 0.34 \end{smallmatrix} \right\rangle u_6 \right\} \right) \Bigg\} \end{aligned}$$

Thirdly, the proposed soft decision-making method is applied to the *pfs*-set f_1 and the decision set is as follows:

$$\{{}^{0.3980}\text{Project-1}, {}^{0.4062}\text{Project-2}, {}^{0.2636}\text{Project-3}, {}^{0.1788}\text{Project-4}, {}^{0.5718}\text{Project-5}, {}^{0.3424}\text{Project-6}\}$$

Fourthly, the ranking orders of proposed method and the decision-making method provided in [24] present in Table 1.

Table 1. The ranking orders of the proposed method and literature

Methods	Structures	Ranking Orders
Literature [24]	*pfs*-sets	Project-4 = Project-6 $\prec$ Project-1 = Project-3 $\prec$ Project-2 $\prec$ Project-5
Proposed Method	*pfs*-sets	Project-4 $\prec$ Project-3 $\prec$ Project-6 $\prec$ Project-1 $\prec$ Project-2 $\prec$ Project-5

According to the ranking orders in Table 1, proposed method and the literature is tend to producing the same ranking except for the alternatives Project-1, Project-3, and Project-6. Moreover, they confirm that Project-5 is the most suitable project and Project-4 is not suitable for the firm among the projects.

6. Conclusion

In this paper, we redefined the concept of *pfs*-sets to ensure their theoretical consistency. We then investigated their properties extensively and revised some of their operations. Afterwards, we defined their product operations such as AND, OR, ANDNOT, and ORNOT-products. We then proposed a soft decision-making method based on *pfs*-sets and compared it with the decision-making method provided in [24]. The results manifested that proposed method generate the stable ranking order compared to literature.

The concept of *pfs*-sets is a new mathematical tool for modelling the uncertainties. It has not been applied to real-world problems such as image processing and machine learning. To carry out these implementations, the matrix representation of the concept is required. The algebraic operations of picture fuzzy soft matrices (*pfs*-matrices) [27] have been studied, but the concept therein has not been explored substantially. In addition, it has the consistency resulting from definitions provided in [23, 24]. Hence, redefining of *pfs*-matrices is worth studying. On the other hand, applications of *pfs*-matrices to image processing and machine learning are crucial research topics since fuzzy parameterized fuzzy soft matrices, which is a substructure of *pfs*-matrices, are successfully applied to machine learning [28–32].

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8 (1965) 338–353.

[2] H. J. Zimmermann, Fuzzy Set Theory and Its Applications, Springer Science+Business Media, New York, 2011.

[3] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems 20 (1) (1986) 87–96.

[4] D. Molodtsov, *Soft Set Theory-First Results*, Computers and Mathematics with Applications 37 (4-5) (1999) 19–31.

[5] H. Aktaş, N. Çağman, *Soft Sets and Soft Groups*, Information Sciences 177 (13) (2007) 2726–2735.

[6] A. Sezgin, *A New Approach to Semigroup Theory I: Soft Union Semigroups, Ideals and Bi-Ideals*, Algebra Letters 2016 (2016) Article ID 3, 46 pages.

[7] A. Sezgin, N. Çağman, F. Çıtak, *α-Inclusions Applied to Group Theory via Soft Set and Logic*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 68 (1) (2019) 334–352.

[8] N. Çağman, S. Karataş, S. Enginoğlu, *Soft Topology*, Computers and Mathematics with Applications 62 (1) (2011) 351–358.

[9] S. Enginoğlu, N. Çağman, S. Karataş, T. Aydın, *On Soft Topology*, El-Cezerî Journal of Science and Engineering 2 (3) (2015) 23–38.

[10] T. Aydın, S. Enginoğlu, *Some Results on Soft Topological Notions*, Journal of New Results in Science 10 (1) (2021) 65–75.

[11] P. K. Maji, R. Biswas, A. R. Roy, *Soft Set Theory*, Computers and Mathematics with Applications 45 (2003) 555–562.

[12] P. K. Maji, A. R. Roy, R. Biswas, *An Application of Soft Sets in a Decision Making Problem*, Computers and Mathematics with Applications 44 (2002) 1077–1083.

[13] N. Çağman, S. Enginoğlu, *Soft Set Theory and Uni-Int Decision Making*, European Journal of Operational Research 207 (2010) 848–855.

[14] K. Hayat, M. I. Ali, F. Karaaslan, B. Y. Cao, M. H. Shah, *Design Concept Evaluation Using Soft Sets Based on Acceptable and Satisfactory Levels: An Integrated TOPSIS and Shannon Entropy*, Soft Computing 24 (2020) 2229–2263.

[15] K. Hayat, Z. Tariq, E. Lughover, M. F. Aslam, *New Aggregation Operators on Group-Based Generalized Intuitionistic Fuzzy Soft Sets*, Soft Computing 25 (2021) 13353–13364.

[16] M. M. Mushrif, S. Senqupta, A. K. Ray, *Texture Classification Using a Novel, Soft-Set Theory Based Classification Algorithm*, in: P. J. Narayanan, S. K. Nayar, H.-Y. Shum (Eds.), 7th Asian Conference on Computer Vision, Hyderabad, India, 2006, pp. 246–254.

[17] B. Handaga, H. Onn, T. Herawan, *FSSC: An Algorithm for Classifying Numerical Data Using Fuzzy Soft Set Theory*, International Journal of Fuzzy System Applications 3 (4) (2012) 29–46.

[18] I. T. R. Yanto, R. R. Seadudin, S. A. Lashari, Haviluddin, *A Numerical Classification Technique Based on Fuzzy Soft Set Using Hamming Distance*, in: R. Ghazali, M. M. Deris, N. M. Nawi, J. H. Abawajy (Eds.), Third International Conference on Soft Computing and Data Mining, Johor, Malaysia, 2018, pp. 252–260.

[19] P. K. Maji, R. Biswas, A. R. Roy, *Fuzzy Soft Sets*, The Journal of Fuzzy Mathematics 9 (3) (2001) 589–602.

[20] N. Çağman, S. Enginoğlu, F. Çıtak, *Fuzzy Soft Set Theory and Its Applications*, Iranian Journal of Fuzzy Systems 8 (3) (2011) 137–147.

[21] N. Çağman, F. Çıtak, S. Enginoğlu, *FP-Soft Set Theory and Its Applications*, Annals of Fuzzy Mathematics and Informatics 2 (2) (2011) 219–226.

[22] N. Çağman, F. Çıtak, S. Enginoğlu, *Fuzzy Parameterized Fuzzy Soft Set Theory and Its Applications*, Turkish Journal of Fuzzy Systems 1 (1) (2010) 21–35.

[23] B. C. Cuong, *Picture Fuzzy Sets*, Journal of Computer Science and Cybernetics 30 (4) (2014) 409–420.

[24] Y. Yang, C. Liang, S. Ji, T. Liu, *Adjustable Soft Discernibility Matrix Based on Picture Fuzzy Soft Sets and Its Applications in Decision Making*, Journal of Intelligent & Fuzzy Systems 29 (4) (2015) 1711–1722.

[25] S. Memiş, *A Study on Picture Fuzzy Sets*, in: G. Çuvalcıoğlu (Ed.), 7th IFS and Contemporary Mathematics Conference, Mersin, Turkey, 2021, pp. 125–132.

[26] F. Herrera, E. Herrera-Viedma, L. Martnez, *A Fusion Approach for Managing Multi-Granularity Linguistic Term Sets in Decision Making*, Fuzzy Sets and Systems 114 (1) (2000) 43–58.

[27] A. Arikrishnan, S. Sriram, *Algebraic Operations on Picture Fuzzy Soft Matrices*, Advances in Mathematics: Scientific Journal 9 (8) (2020) 6349–6358.

[28] S. Memiş, S. Enginoğlu, *An Application of Fuzzy Parameterized Fuzzy Soft Matrices in Data Classification*, in: M. Kılıç, K. Özkan, M. Karaboyacı, K. Taşdelen, H. Kandemir, A. Beram (Eds.), International Conferences on Science and Technology; Natural Science and Technology ICONST-NST 2019, Prizren, Kosovo, 2019, pp. 68–77.

[29] S. Memiş, S. Enginoğlu, U. Erkan, *A Data Classification Method in Machine Learning Based on Normalised Hamming Pseudo-Similarity of Fuzzy Parameterized Fuzzy Soft Matrices*, Bilge International Journal of Science and Technology Research 3 (Special Issue) (2019) 1–8.

[30] S. Memiş, S. Enginoğlu, U. Erkan, *Numerical Data Classification via Distance-Based Similarity Measures of Fuzzy Parameterized Fuzzy Soft Matrices*, Journal of Applied Mathematics and Computing 9 (2021) 88583–88601.

[31] S. Memiş, S. Enginoğlu, U. Erkan, *A Classification Method in Machine Learning Based on Soft Decision-Making via Fuzzy Parameterized Fuzzy Soft Matrices*, Soft Computing 26 (3) (2022) 1165–1180.

[32] S. Memiş, S. Enginoğlu, U. Erkan, *A New Classification Method Using Soft Decision-Making Based on An Aggregation Operator of Fuzzy Parameterized Fuzzy Soft Matrices*, Turkish Journal of Electrical Engineering & Computer Sciences 30 (3) (2022) 871–890.