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WRONSKI DETERMINANT OF TRIGONOMETRIC SYSTEM

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Abstract. In this paper, we calculate the Wronskian of the trigonometric system

$$\cos\lambda_1 x, \sin\lambda_1 x, \cos\lambda_2 x, \sin\lambda_2 x, \ldots, \cos\lambda_n x, \sin\lambda_n x$$

and prove that this system is linearly independent when $\lambda_k \neq 0$ and $\lambda_k^2 \neq \lambda_l^2$ for $k \neq l$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, $n \in \mathbb{N}$ are constants and x is a complex variable. By using it, we evaluate the determinant below

$$\begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ \lambda_1 & 0 & \lambda_2 & 0 & \cdots & \lambda_n & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 & \cdots & 0 & \lambda_n \\ \lambda_1^2 & 0 & \lambda_2^2 & 0 & \cdots & \lambda_n^2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \lambda_1^{n-2} & 0 & \lambda_2^{n-2} & \cdots & 0 & \lambda_n^{n-2} \\ \lambda_1^{n-1} & 0 & \lambda_2^{n-1} & 0 & \cdots & \lambda_n^{n-1} & 0 \\ 0 & \lambda_1^{n-1} & 0 & \lambda_2^{n-1} & \cdots & 0 & \lambda_n^{n-1} \end{vmatrix}.$$

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1. Introduction

Wronski determinant (or shortly Wronskian) of n functions $f_1, f_2, \ldots, f_n$ from an interval I to the complex numbers $\mathbb{C}$ is the following determinant

$$\begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

and denoted by $W[f_1, f_2, \ldots, f_n](x)$. This determinant is important for the linear independence of the functions $f_1, f_2, \ldots, f_n$. If these functions are linearly dependent, then the Wronskian is 0 for each x in the interval I, i.e. the functions $f_1, f_2, \ldots, f_n$ is linearly independent if there exists $x_0 \in I$ with $W[f_1, f_2, \ldots, f_n](x_0) \neq 0$ [2]. Wronskian is directly related to the ordinary differential equations (see the following lemma).

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Lemma 1.1. *(Abel's identity [2]) Assume that* $f_1, f_2, \ldots, f_n$ *are* n *solutions of a homogeneous linear differential equation*

$$y^{(n)} + a_1(x)\, y^{(n-1)} + \cdots + a_{n-1}(x)\, y' + a_n(x)\, y = 0 \tag{1.1}$$

on an interval I*, then the relation*

$$W[f_1, f_2, \ldots, f_n](x) = W[f_1, f_2, \ldots, f_n](x_0)\, e^{-\int_{x_0}^{x} a_1(t)dt}$$

holds for each $x_0 \in I$*.*

Corollary 1.2. *Let* $f_1, f_2, \ldots, f_n$ *be* n *solutions of the equation (1.1). Then they are linearly independent if and only if the corresponding Wronskian is not zero at any point in* I*.*

Corollary 1.3. *The Wronskian of* $f_1, f_2, \ldots, f_n$ *is a constant if and only if the coefficient function* a_1 *of the corresponding linear differential equation is the zero function.*

In 2006, O. Christensen and K. Christensen gave a criterion for linear independence of some trigonometric functions [3]. The criterion is that if $\{\lambda_k\}_{k=1}^{n}$ and $\{\mu_l\}_{l=1}^{m}$ are the sets of real numbers such that $\mu_l \neq 0$ for each l and such that $|\lambda_l| \neq |\lambda_j|$ and $|\mu_l| \neq |\mu_j|$ when $l \neq j$, then the set of functions

$$\{\cos(\lambda_k x)\}_{k=1}^{n} \cup \{\sin(\mu_l x)\}_{l=1}^{m}$$

is linearly independent on any interval I. By the above criterion and Corollary1.2, the Wronskian of the set $\{\cos(\lambda_k x)\}_{k=1}^{n} \cup \{\sin(\mu_l x)\}_{l=1}^{m}$ is not zero under same conditions. However, it is too hard to calculate the Wronskian of the set $\{\cos(\lambda_k x)\}_{k=1}^{n} \cup \{\sin(\mu_l x)\}_{l=1}^{m}$. So, we calculate that Wronskian under light conditions. We assume that $n = m$ and $\lambda_k = \mu_k$ for each k. Note that the numbers λ_k are not only in reals but also in complexes.

2. Calculation of the Wronskian of Trigonometric System

The Wronskian of the functions

$$\cos\lambda_1 x, \sin\lambda_1 x, \cos\lambda_2 x, \sin\lambda_2 x, \ldots, \cos\lambda_n x, \sin\lambda_n x \tag{2.1}$$

is an $2n \times 2n$ determinant as follows:

$$W = \tag{2.2}$$

$$\begin{vmatrix} \cos\lambda_1 x & \sin\lambda_1 x & \cos\lambda_2 x & \sin\lambda_2 x & \cdots & \cos\lambda_n x & \sin\lambda_n x \\ -\lambda_1 \sin\lambda_1 x & \lambda_1 \cos\lambda_1 x & -\lambda_2 \sin\lambda_2 x & \lambda_2 \cos\lambda_2 x & \cdots & -\lambda_n \sin\lambda_n x & \lambda_n \cos\lambda_n x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1}\lambda_1^{2n-2}\cos\lambda_1 x & (-1)^{n-1}\lambda_1^{2n-2}\sin\lambda_1 x & (-1)^{n-1}\lambda_2^{2n-2}\cos\lambda_2 x & (-1)^{n-1}\lambda_2^{2n-2}\sin\lambda_2 x & \cdots & (-1)^{n-1}\lambda_n^{2n-2}\cos\lambda_n x & (-1)^{n-1}\lambda_n^{2n-2}\sin\lambda_n x \\ (-1)^{n}\lambda_1^{2n-1}\sin\lambda_1 x & (-1)^{n-1}\lambda_1^{2n-1}\cos\lambda_1 x & (-1)^{n}\lambda_2^{2n-1}\sin\lambda_2 x & (-1)^{n-1}\lambda_2^{2n-1}\cos\lambda_2 x & \cdots & (-1)^{n}\lambda_n^{2n-1}\sin\lambda_n x & (-1)^{n-1}\lambda_n^{2n-1}\cos\lambda_n x \end{vmatrix}.$$

The functions in (2.1) are $2n$ solutions of the homogeneous differential equation generated by the following characteristic polynomial of degree $2n$

$$\left(t^2 + \lambda_1^2\right)\left(t^2 + \lambda_2^2\right)\ldots\left(t^2 + \lambda_n^2\right) = 0.$$

The above polynomial has not any odd power of t, particularly $2n - 1$. Consequently, the corresponding linear differential equation does not include the expression y^{2n-1}. By Corollary1.3, the Wronskian of the system (2.1) is independent of the variable x, i.e. in order to calculate the determinant in (2.2), we can write an arbitrary complex number instead of x. First, we calculate the determinant in (2.2) for the values

$$\lambda_k = \lambda_{k,0} = 8k - 7, \quad k = \overline{1, n} \tag{2.3}$$

and

$$x = \frac{\pi}{4}.$$

Then, we have

$$W =$$

$$\begin{vmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & \cdots & \cos\frac{(8n-7)\pi}{4} & \sin\frac{(8n-7)\pi}{4} \\ -\lambda_{1,0}\sin\frac{\pi}{4} & \lambda_{1,0}\cos\frac{\pi}{4} & \cdots & -\lambda_{n,0}\sin\frac{(8n-7)\pi}{4} & \lambda_{n,0}\cos\frac{(8n-7)\pi}{4} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1}\lambda_{1,0}^{2n-2}\cos\frac{\pi}{4} & (-1)^{n-1}\lambda_{1,0}^{2n-2}\sin\frac{\pi}{4} & \cdots & (-1)^{n-1}\lambda_{n,0}^{2n-2}\cos\frac{(8n-7)\pi}{4} & (-1)^{n-1}\lambda_{n,0}^{2n-2}\sin\frac{(8n-7)\pi}{4} \\ (-1)^{n}\lambda_{1,0}^{2n-1}\sin\frac{\pi}{4} & (-1)^{n-1}\lambda_{1,0}^{2n-1}\cos\frac{\pi}{4} & \cdots & (-1)^{n}\lambda_{n,0}^{2n-1}\sin\frac{(8n-7)\pi}{4} & (-1)^{n-1}\lambda_{n,0}^{2n-1}\cos\frac{(8n-7)\pi}{4} \end{vmatrix} =$$

$$\begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \cdots & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\lambda_{1,0}\frac{\sqrt{2}}{2} & \lambda_{1,0}\frac{\sqrt{2}}{2} & \cdots & -\lambda_{n,0}\frac{\sqrt{2}}{2} & \lambda_{n,0}\frac{\sqrt{2}}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1}\lambda_{1,0}^{2n-2}\frac{\sqrt{2}}{2} & (-1)^{n-1}\lambda_{1,0}^{2n-2}\frac{\sqrt{2}}{2} & \cdots & (-1)^{n-1}\lambda_{n,0}^{2n-2}\frac{\sqrt{2}}{2} & (-1)^{n-1}\lambda_{n,0}^{2n-2}\frac{\sqrt{2}}{2} \\ (-1)^{n}\lambda_{1,0}^{2n-1}\frac{\sqrt{2}}{2} & (-1)^{n-1}\lambda_{1,0}^{2n-1}\frac{\sqrt{2}}{2} & \cdots & (-1)^{n}\lambda_{n,0}^{2n-1}\frac{\sqrt{2}}{2} & (-1)^{n-1}\lambda_{n,0}^{2n-1}\frac{\sqrt{2}}{2} \end{vmatrix} =$$

$$\frac{1}{2^n}\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ -\lambda_{1,0} & \lambda_{1,0} & \cdots & -\lambda_{n,0} & \lambda_{n,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1}\lambda_{1,0}^{2n-2} & (-1)^{n-1}\lambda_{1,0}^{2n-2} & \cdots & (-1)^{n-1}\lambda_{n,0}^{2n-2} & (-1)^{n-1}\lambda_{n,0}^{2n-2} \\ (-1)^{n}\lambda_{1,0}^{2n-1} & (-1)^{n-1}\lambda_{1,0}^{2n-1} & \cdots & (-1)^{n}\lambda_{n,0}^{2n-1} & (-1)^{n-1}\lambda_{n,0}^{2n-1} \end{vmatrix} =$$

$$\frac{1}{(2i)^n}\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ -i\lambda_{1,0} & i\lambda_{1,0} & \cdots & -i\lambda_{n,0} & i\lambda_{n,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-i\lambda_{1,0})^{2n-2} & (i\lambda_{1,0})^{2n-2} & \cdots & (-i\lambda_{n,0})^{2n-2} & (i\lambda_{n,0})^{2n-2} \\ (-i\lambda_{1,0})^{2n-1} & (i\lambda_{1,0})^{2n-1} & \cdots & (-i\lambda_{n,0})^{2n-1} & (i\lambda_{n,0})^{2n-1} \end{vmatrix}.$$

The last determinant is the Vandermonde determinant of the complex numbers $-i\lambda_{1,0}, i\lambda_{1,0}, -i\lambda_{2,0}, i\lambda_{2,0}, \ldots, -i\lambda_{n,0}, i\lambda_{n,0}$. Then, we have

$$W = \frac{1}{(2i)^n} i^{\frac{(2n-1)2n}{2}} \times$$

$$\begin{aligned}
&\times (\lambda_{n,0}+\lambda_{1,0})(\lambda_{n,0}-\lambda_{1,0})(\lambda_{n,0}+\lambda_{2,0})(\lambda_{n,0}-\lambda_{2,0})\ldots(\lambda_{n,0}+\lambda_{n-1,0})(\lambda_{n,0}-\lambda_{n-1,0})(\lambda_{n,0}+\lambda_{n,0})\\
&\times (-\lambda_{n,0}+\lambda_{1,0})(-\lambda_{n,0}-\lambda_{1,0})(-\lambda_{n,0}+\lambda_{2,0})(-\lambda_{n,0}-\lambda_{2,0})\ldots(-\lambda_{n,0}+\lambda_{n-1,0})(-\lambda_{n,0}-\lambda_{n-1,0})\\
&\times (\lambda_{n-1,0}+\lambda_{1,0})(\lambda_{n-1,0}-\lambda_{1,0})(\lambda_{n-1,0}+\lambda_{2,0})(\lambda_{n-1,0}-\lambda_{2,0})\ldots(\lambda_{n-1,0}+\lambda_{n-2,0})(\lambda_{n-1,0}-\lambda_{n-2,0})(\lambda_{n-1,0}+\lambda_{n-1,0})\\
&\times (-\lambda_{n-1,0}+\lambda_{1,0})(-\lambda_{n-1,0}-\lambda_{1,0})(-\lambda_{n-1,0}+\lambda_{2,0})(-\lambda_{n-1,0}-\lambda_{2,0})\ldots(-\lambda_{n-1,0}+\lambda_{n-2,0})(-\lambda_{n-1,0}-\lambda_{n-2,0})\\
&\cdots\cdots\\
&\times (\lambda_{2,0}+\lambda_{1,0})(\lambda_{2,0}-\lambda_{1,0})(\lambda_{2,0}+\lambda_{2,0})\\
&\times (-\lambda_{2,0}+\lambda_{1,0})(-\lambda_{2,0}-\lambda_{1,0})\\
&\times (\lambda_{1,0}+\lambda_{1,0})
\end{aligned}$$

$$= \prod_{j=1}^{n} \lambda_{j,0} \times \prod_{1 \le l < k \le n} \left(\lambda_{l,0}^2 - \lambda_{k,0}^2\right)^2.$$

We consider the determinant in (2.2) as an analytic function of the variables $\lambda_1, \lambda_2, \ldots, \lambda_n, x$ from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ and denote it by

$$F\left(\lambda_1, \lambda_2, \ldots, \lambda_n; x\right).$$

Note that the function F is independent of the variable x and it satisfies the following relation

$$F\left(\lambda_{1,0}, \lambda_{2,0}, \ldots, \lambda_{n,0}; x\right) = \prod_{j=1}^{n} \lambda_{j,0} \times \prod_{1\le l<k\le n} \left(\lambda_{l,0}^2 - \lambda_{k,0}^2\right)^2, \tag{2.4}$$

for the numbers $\lambda_{1,0}, \lambda_{2,0}, \ldots, \lambda_{n,0}$ in the equality (2.3).

Now, we define a new analytic function $G : \mathbb{C}^{n+1} \to \mathbb{C}$ as follows

$$G\left(\lambda_1, \lambda_2, \ldots, \lambda_n; x\right) = \prod_{j=1}^{n} \lambda_j \times \prod_{1\le l<k\le n} \left(\lambda_l^2 - \lambda_k^2\right)^2.$$

We can obtain the following relations by (2.2) and (2.3) for $m \in \mathbb{N}$:

$$\prod_{j=1}^{n} \lambda_{j,0} \times \prod_{1\le l<k\le n} \left(\lambda_{l,0}^2 - \lambda_{k,0}^2\right)^2 = F\left(\lambda_{1,0}, \lambda_{2,0}, \ldots, \lambda_{n,0}; \frac{\pi}{4}\right) =$$

$$F\left(\lambda_{1,0}, \lambda_{2,0}, \ldots, \lambda_{n,0}; \frac{\pi}{4m}\right) =$$

$$\begin{vmatrix} \cos\frac{\pi}{4m} & \sin\frac{\pi}{4m} & \cdots & \cos\frac{(8n-7)\pi}{4m} & \sin\frac{(8n-7)\pi}{4m} \\ -\lambda_{1,0}\sin\frac{\pi}{4m} & \lambda_{1,0}\cos\frac{\pi}{4m} & \cdots & -\lambda_{n,0}\sin\frac{(8n-7)\pi}{4m} & \lambda_{n,0}\cos\frac{(8n-7)\pi}{4m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1}\lambda_{1,0}^{2n-2}\cos\frac{\pi}{4m} & (-1)^{n-1}\lambda_{1,0}^{2n-2}\sin\frac{\pi}{4m} & \cdots & (-1)^{n-1}\lambda_{n,0}^{2n-2}\cos\frac{(8n-7)\pi}{4m} & (-1)^{n-1}\lambda_{n,0}^{2n-2}\sin\frac{(8n-7)\pi}{4m} \\ (-1)^{n}\lambda_{1,0}^{2n-1}\sin\frac{\pi}{4m} & (-1)^{n-1}\lambda_{1,0}^{2n-1}\cos\frac{\pi}{4m} & \cdots & (-1)^{n}\lambda_{n,0}^{2n-1}\sin\frac{(8n-7)\pi}{4m} & (-1)^{n-1}\lambda_{n,0}^{2n-1}\cos\frac{(8n-7)\pi}{4m} \end{vmatrix} =$$

$$m^{\frac{(2n-1)2n}{2}} \begin{vmatrix} \cos\frac{\pi}{4m} & \sin\frac{\pi}{4m} & \cdots & \cos\frac{(8n-7)\pi}{4m} & \sin\frac{(8n-7)\pi}{4m} \\ -\frac{\lambda_{1,0}}{m}\sin\frac{\pi}{4m} & \frac{\lambda_{1,0}}{m}\cos\frac{\pi}{4m} & \cdots & -\frac{\lambda_{n,0}}{m}\sin\frac{(8n-7)\pi}{4m} & \frac{\lambda_{n,0}}{m}\cos\frac{(8n-7)\pi}{4m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1}\left(\frac{\lambda_{1,0}}{m}\right)^{2n-2}\cos\frac{\pi}{4m} & (-1)^{n-1}\left(\frac{\lambda_{1,0}}{m}\right)^{2n-2}\sin\frac{\pi}{4m} & \cdots & (-1)^{n-1}\left(\frac{\lambda_{n,0}}{m}\right)^{2n-2}\cos\frac{(8n-7)\pi}{4m} & (-1)^{n-1}\left(\frac{\lambda_{n,0}}{m}\right)^{2n-2}\sin\frac{(8n-7)\pi}{4m} \\ (-1)^{n}\left(\frac{\lambda_{1,0}}{m}\right)^{2n-1}\sin\frac{\pi}{4m} & (-1)^{n-1}\left(\frac{\lambda_{1,0}}{m}\right)^{2n-1}\cos\frac{\pi}{4m} & \cdots & (-1)^{n}\left(\frac{\lambda_{n,0}}{m}\right)^{2n-1}\sin\frac{(8n-7)\pi}{4m} & (-1)^{n-1}\left(\frac{\lambda_{n,0}}{m}\right)^{2n-1}\cos\frac{(8n-7)\pi}{4m} \end{vmatrix} =$$

$$m^{\frac{(2n-1)2n}{2}} F\left(\frac{\lambda_{1,0}}{m}, \frac{\lambda_{2,0}}{m}, \ldots, \frac{\lambda_{n,0}}{m}; \frac{\pi}{4}\right).$$

By dividing both sides of the last equality by $m^{\frac{(2n-1)2n}{2}}$, we have

$$F\left(\frac{\lambda_{1,0}}{m}, \frac{\lambda_{2,0}}{m}, \ldots, \frac{\lambda_{n,0}}{m}; \frac{\pi}{4}\right) = \frac{1}{m^{\frac{(2n-1)2n}{2}}} \prod_{j=1}^{n} \lambda_{j,0} \times \prod_{1\le l<k\le n} \left(\lambda_{l,0}^2 - \lambda_{k,0}^2\right)^2 =$$

$$\prod_{j=1}^{n} \frac{\lambda_{j,0}}{m} \times \prod_{1\le l<k\le n} \left(\left(\frac{\lambda_{l,0}}{m}\right)^2 - \left(\frac{\lambda_{k,0}}{m}\right)^2\right)^2.$$

Then, we have shown that the functions F and G are equal to each other on the vectors $\left(\frac{\lambda_{1,0}}{m}, \frac{\lambda_{2,0}}{m}, \ldots, \frac{\lambda_{n,0}}{m}; x\right)$, where $m \in \mathbb{N}$.

Now, we fix the numbers $\lambda_2 = \frac{\lambda_{2,0}}{m}, \lambda_3 = \frac{\lambda_{3,0}}{m}, \ldots, \lambda_n = \frac{\lambda_{n,0}}{m}$ for a fixed $m \in \mathbb{N}$ (except for λ_1). Then, F and G are two univariate entire functions of the variable λ_1, and they are equal to each other on the set

$\left\{\lambda_1 = \frac{\lambda_{1,0}}{m} | m \in \mathbb{N}\right\}$. By the identity theorem for analytic functions, see [1], F and G are equal to each other on the whole $\mathbb{C}$ as two univariate functions of the complex variable λ_1, in other words, F and G are equal to each other on the set

$$\left\{\left(\lambda_1, \frac{\lambda_{2,0}}{m}, \frac{\lambda_{3,0}}{m}, \ldots, \frac{\lambda_{n,0}}{m}; x\right) | x, \lambda_1 \in \mathbb{C}, m \in \mathbb{N}\right\}.$$

Similarly, we fix a complex number λ_1 and $\lambda_3 = \frac{\lambda_{3,0}}{m}, \lambda_4 = \frac{\lambda_{4,0}}{m}, \ldots, \lambda_n = \frac{\lambda_{n,0}}{m}$ for a fixed $m \in \mathbb{N}$ (except for λ_2). Then, F and G are two univariate entire functions of the variable λ_2. By a similar method, F and G are equal to each other on the whole $\mathbb{C}$ as two univariate functions of the complex variable λ_2, Hence, F and G are equal to each other on the set

$$\left\{\left(\lambda_1, \lambda_2, \frac{\lambda_{3,0}}{m}, \frac{\lambda_{4,0}}{m}, \ldots, \frac{\lambda_{n,0}}{m}; x\right) | x, \lambda_1, \lambda_2 \in \mathbb{C}, m \in \mathbb{N}\right\}.$$

By continuing the above method, we prove that the functions F and G are equal to each other on $\mathbb{C}^{n+1}$. Finally, we obtain that the Wronskian of the trigonometric system $\cos \lambda_1 x, \sin \lambda_1 x, \cos \lambda_2 x, \sin \lambda_2 x, \ldots, \cos \lambda_n x, \sin \lambda_n x$ is the following product

$$\prod_{j=1}^{n} \lambda_j \times \prod_{1 \le l < k \le n} \left(\lambda_l^2 - \lambda_k^2\right)^2.$$

As an application, we calculate the Wronskian of the system

$$1, \cos \lambda_1 x, \sin \lambda_1 x, \cos \lambda_2 x, \sin \lambda_2 x, \ldots, \cos \lambda_n x, \sin \lambda_n x$$

as follows:

$$\begin{vmatrix} 1 & \cos \lambda_1 x & \sin \lambda_1 x & \cdots & \cos \lambda_n x & \sin \lambda_n x \\ 0 & -\lambda_1 \sin \lambda_1 x & \lambda_1 \cos \lambda_1 x & \cdots & -\lambda_n \sin \lambda_n x & \lambda_n \cos \lambda_n x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{n-1} \lambda_1^{2n-2} \cos \lambda_1 x & (-1)^{n-1} \lambda_1^{2n-2} \sin \lambda_1 x & \cdots & (-1)^{n-1} \lambda_n^{2n-2} \cos \lambda_n x & (-1)^{n-1} \lambda_n^{2n-2} \sin \lambda_n x \\ 0 & (-1)^{n} \lambda_1^{2n-1} \sin \lambda_1 x & (-1)^{n-1} \lambda_1^{2n-1} \cos \lambda_1 x & \cdots & (-1)^{n} \lambda_n^{2n-1} \sin \lambda_n x & (-1)^{n-1} \lambda_n^{2n-1} \cos \lambda_n x \\ 0 & (-1)^{n} \lambda_1^{2n} \cos \lambda_1 x & (-1)^{n} \lambda_1^{2n} \sin \lambda_1 x & \cdots & (-1)^{n} \lambda_n^{2n} \cos \lambda_n x & (-1)^{n} \lambda_n^{2n} \sin \lambda_n x \end{vmatrix} =$$

$$\begin{vmatrix} -\lambda_1 \sin \lambda_1 x & \lambda_1 \cos \lambda_1 x & \cdots & -\lambda_n \sin \lambda_n x & \lambda_n \cos \lambda_n x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1} \lambda_1^{2n-2} \cos \lambda_1 x & (-1)^{n-1} \lambda_1^{2n-2} \sin \lambda_1 x & \cdots & (-1)^{n-1} \lambda_n^{2n-2} \cos \lambda_n x & (-1)^{n-1} \lambda_n^{2n-2} \sin \lambda_n x \\ (-1)^{n} \lambda_1^{2n-1} \sin \lambda_1 x & (-1)^{n-1} \lambda_1^{2n-1} \cos \lambda_1 x & \cdots & (-1)^{n} \lambda_n^{2n-1} \sin \lambda_n x & (-1)^{n-1} \lambda_n^{2n-1} \cos \lambda_n x \\ (-1)^{n} \lambda_1^{2n} \cos \lambda_1 x & (-1)^{n} \lambda_1^{2n} \sin \lambda_1 x & \cdots & (-1)^{n} \lambda_n^{2n} \cos \lambda_n x & (-1)^{n} \lambda_n^{2n} \sin \lambda_n x \end{vmatrix} =$$

$$\begin{vmatrix} \lambda_1 \cos \lambda_1 x & \lambda_1 \sin \lambda_1 x & \cdots & \lambda_n \cos \lambda_n x & \lambda_n \sin \lambda_n x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1} \lambda_1^{2n-2} \sin \lambda_1 x & (-1)^{n} \lambda_1^{2n-2} \cos \lambda_1 x & \cdots & (-1)^{n-1} \lambda_n^{2n-2} \sin \lambda_n x & (-1)^{n} \lambda_n^{2n-2} \cos \lambda_n x \\ (-1)^{n-1} \lambda_1^{2n-1} \cos \lambda_1 x & (-1)^{n-1} \lambda_1^{2n-1} \sin \lambda_1 x & \cdots & (-1)^{n-1} \lambda_n^{2n-1} \cos \lambda_n x & (-1)^{n-1} \lambda_n^{2n-1} \sin \lambda_n x \\ (-1)^{n} \lambda_1^{2n} \sin \lambda_1 x & (-1)^{n-1} \lambda_1^{2n} \cos \lambda_1 x & \cdots & (-1)^{n} \lambda_n^{2n} \sin \lambda_n x & (-1)^{n-1} \lambda_n^{2n} \cos \lambda_n x \end{vmatrix} =$$

$$\prod_{j=1}^{n} \lambda_j^2 \begin{vmatrix} \cos \lambda_1 x & \sin \lambda_1 x & \cdots & \cos \lambda_n x & \sin \lambda_n x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1} \lambda_1^{2n-3} \sin \lambda_1 x & (-1)^{n} \lambda_1^{2n-3} \cos \lambda_1 x & \cdots & (-1)^{n-1} \lambda_n^{2n-3} \sin \lambda_n x & (-1)^{n} \lambda_n^{2n-3} \cos \lambda_n x \\ (-1)^{n-1} \lambda_1^{2n-2} \cos \lambda_1 x & (-1)^{n-1} \lambda_1^{2n-2} \sin \lambda_1 x & \cdots & (-1)^{n-1} \lambda_n^{2n-2} \cos \lambda_n x & (-1)^{n-1} \lambda_n^{2n-2} \sin \lambda_n x \\ (-1)^{n} \lambda_1^{2n-1} \sin \lambda_1 x & (-1)^{n-1} \lambda_1^{2n-1} \cos \lambda_1 x & \cdots & (-1)^{n} \lambda_n^{2n-1} \sin \lambda_n x & (-1)^{n-1} \lambda_n^{2n-1} \cos \lambda_n x \end{vmatrix} =$$

$$\prod_{j=1}^{n}\lambda_j^3\times\prod_{1\le l<k\le n}\left(\lambda_l^2-\lambda_k^2\right)^2.$$

Corollary 2.1. *Assume that* $\{\lambda_k\}_{k=1}^n$ *is a set of complex numbers such that* $\lambda_k\neq 0$ *and* $\lambda_k^2\neq\lambda_l^2$ *for* $k\neq l$, *then the set of functions*

$$\{1\}\cup\{\cos(\lambda_k x)\}_{k=1}^n\cup\{\sin(\lambda_k x)\}_{k=1}^n$$

is linearly independent on $\mathbb{C}$.

Corollary 2.2. *If we write the value* $x=0$ *in the determinant (2.2), we have the following equality:*

$$\begin{vmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0\\
0 & \lambda_1 & 0 & \lambda_2 & \cdots & 0 & \lambda_n\\
-\lambda_1^2 & 0 & -\lambda_2^2 & 0 & \cdots & -\lambda_n^2 & 0\\
0 & -\lambda_1^3 & 0 & -\lambda_2^3 & \cdots & 0 & -\lambda_n^3\\
\lambda_1^4 & 0 & \lambda_2^4 & 0 & \cdots & \lambda_n^4 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & (-1)^{n-2}\lambda_1^{2n-3} & 0 & (-1)^{n-2}\lambda_2^{2n-3} & \cdots & 0 & (-1)^{n-2}\lambda_n^{2n-3}\\
(-1)^{n-1}\lambda_1^{2n-2} & 0 & (-1)^{n-1}\lambda_2^{2n-2} & 0 & \cdots & (-1)^{n-1}\lambda_n^{2n-2} & 0\\
0 & (-1)^{n-1}\lambda_1^{2n-1} & 0 & (-1)^{n-1}\lambda_2^{2n-1} & \cdots & 0 & (-1)^{n-1}\lambda_n^{2n-1}
\end{vmatrix}$$

$$=\prod_{j=1}^{n}\lambda_j\times\prod_{1\le l<k\le n}\left(\lambda_l^2-\lambda_k^2\right)^2$$

where $\lambda_1,\lambda_2,\ldots,\lambda_n\in\mathbb{C}$. *By taking all the minuses out of the determinant, we obtain the following equality*

$$\begin{vmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0\\
0 & \lambda_1 & 0 & \lambda_2 & \cdots & 0 & \lambda_n\\
\lambda_1^2 & 0 & \lambda_2^2 & 0 & \cdots & \lambda_n^2 & 0\\
0 & \lambda_1^3 & 0 & \lambda_2^3 & \cdots & 0 & \lambda_n^3\\
\lambda_1^4 & 0 & \lambda_2^4 & 0 & \cdots & \lambda_n^4 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \lambda_1^{2n-3} & 0 & \lambda_2^{2n-3} & \cdots & 0 & \lambda_n^{2n-3}\\
\lambda_1^{2n-2} & 0 & \lambda_2^{2n-2} & 0 & \cdots & \lambda_n^{2n-2} & 0\\
0 & \lambda_1^{2n-1} & 0 & \lambda_2^{2n-1} & \cdots & 0 & \lambda_n^{2n-1}
\end{vmatrix}=$$

$$=\prod_{j=1}^{n}\lambda_j\times\prod_{1\le l<k\le n}\left(\lambda_l^2-\lambda_k^2\right)^2.$$

By dividing both sides of the last equality by $\prod_{j=1}^{n}\lambda_j$, *we have the following*

$$\begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ \lambda_1^2 & 0 & \lambda_2^2 & 0 & \cdots & \lambda_n^2 & 0 \\ 0 & \lambda_1^2 & 0 & \lambda_2^2 & \cdots & 0 & \lambda_n^2 \\ \lambda_1^4 & 0 & \lambda_2^4 & 0 & \cdots & \lambda_n^4 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \lambda_1^{2n-4} & 0 & \lambda_2^{2n-4} & \cdots & 0 & \lambda_n^{2n-4} \\ \lambda_1^{2n-2} & 0 & \lambda_2^{2n-2} & 0 & \cdots & \lambda_n^{2n-2} & 0 \\ 0 & \lambda_1^{2n-2} & 0 & \lambda_2^{2n-2} & \cdots & 0 & \lambda_n^{2n-2} \end{vmatrix} = \prod_{1\le l<k\le n}\left(\lambda_l^2-\lambda_k^2\right)^2.$$

Finally, we have

$$\begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ \lambda_1 & 0 & \lambda_2 & 0 & \cdots & \lambda_n & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 & \cdots & 0 & \lambda_n \\ \lambda_1^2 & 0 & \lambda_2^2 & 0 & \cdots & \lambda_n^2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \lambda_1^{n-2} & 0 & \lambda_2^{n-2} & \cdots & 0 & \lambda_n^{n-2} \\ \lambda_1^{n-1} & 0 & \lambda_2^{n-1} & 0 & \cdots & \lambda_n^{n-1} & 0 \\ 0 & \lambda_1^{n-1} & 0 & \lambda_2^{n-1} & \cdots & 0 & \lambda_n^{n-1} \end{vmatrix} = \prod_{1\le l<k\le n}\left(\lambda_l-\lambda_k\right)^2.$$

For example, when $n=3$ *and* $n=4$, *we obtain the following relations*

$$\begin{vmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \lambda_1 & 0 & \lambda_2 & 0 & \lambda_3 & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 & 0 & \lambda_3 \\ \lambda_1^2 & 0 & \lambda_2^2 & 0 & \lambda_3^2 & 0 \\ 0 & \lambda_1^2 & 0 & \lambda_2^2 & 0 & \lambda_3^2 \end{vmatrix} = \left(\lambda_1-\lambda_2\right)^2\left(\lambda_1-\lambda_3\right)^2\left(\lambda_2-\lambda_3\right)^2,$$

$$\begin{vmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \lambda_1 & 0 & \lambda_2 & 0 & \lambda_3 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 & 0 & \lambda_3 & 0 & \lambda_4 \\ \lambda_1^2 & 0 & \lambda_2^2 & 0 & \lambda_3^2 & 0 & \lambda_4^2 & 0 \\ 0 & \lambda_1^2 & 0 & \lambda_2^2 & 0 & \lambda_3^2 & 0 & \lambda_4^2 \\ \lambda_1^3 & 0 & \lambda_2^3 & 0 & \lambda_3^3 & 0 & \lambda_4^3 & 0 \\ 0 & \lambda_1^3 & 0 & \lambda_2^3 & 0 & \lambda_3^3 & 0 & \lambda_4^3 \end{vmatrix} =$$

$$= \left(\lambda_1-\lambda_2\right)^2\left(\lambda_1-\lambda_3\right)^2\left(\lambda_1-\lambda_4\right)^2\left(\lambda_2-\lambda_3\right)^2\left(\lambda_2-\lambda_4\right)^2\left(\lambda_3-\lambda_4\right)^2.$$

References

[1] V. L. Ahlfors, *Complex Analysis*, McGraw-Hill Inc., New York, 1979.
[2] W. E. Boyce and R. C. DiPrima, *Elementary differential equations and boundary value problems*, Wiley, New York, 1986.
[3] O. Christensen and K. Christensen, *Linear Independence and Series Expansions in Function Spaces*, Am. Mat. Mon., **113** (2006) no. 7, 611-627.

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