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Characterization of Intuitionistic Fuzzy Collineations in Intuitionistic Fuzzy Projective Planes

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Abstract

In this paper, the intuitionistic fuzzy counterparts of the collineations defined in classical projective planes are defined in intuitionistic fuzzy projective planes. The properties of the intuitionistic fuzzy projective plane left invariant under the intuitionistic fuzzy collineations are characterized depending on the base point, base line, membership degrees, and the non-membership degrees of the intuitionistic fuzzy projective plane.

1. Introduction

The fuzzy concept was first proposed by Zadeh in 1965 [1], and many scientists have contributed to this field. Projective planes have been fuzzified by Kuijken et al., see [2]. Also a fuzzy group corresponding to the fuzzy projective geometry was created, so that through these fuzzy projective geometries a relationship between fuzzy vector spaces and fuzzy groups was obtained by Kuijken, Maldeghem and Kerre in 1999 [3]. The fuzzy projective plane collinations were described by Kuijken and Maldeghem in 2003 [4].
As a generalization of Zadeh's Fuzzy Sets, Intuitionistic Fuzzy Set which is characterized by a membership function and a nonmembership function was proposed by Atannassov [5]. In 2009, a new model of intuitionistic fuzzy projective geometry was constructed by Ghassan [6] and it is seen that this new intuitionistic fuzzy projective plane is closely related to the fibered projective geometry. And also in different algebraic structures many theories were introduced. By Sharma, a relation relating to the Intuitionistic fuzzy subgroup of a group with its homomorphic image by using the properties of their (α,β)–cut sets was determined [7]. By developing and holding to some properties of Atanassov's intuitionistic fuzzy relations, some connections of their properties with lattice operations were introduced in 2012 [8]. Pradhan and Pal introduced the set of all linear transformations L(V) defined over an intuitionistic fuzzy vector space V not form an vector space and the concept of the inverse of an intuitionistic fuzzy linear transformation (IFLT) in 2012 [9]. In 2015, Bayar and Ekmekci showed that intuitionistic fuzzy versions of some classical configurations in projective plane are valid in the intuitionistic fuzzy projective plane with base Desarguesian or Pappian plane [10]. In intuitionistic fuzzy projective plane, the conditions to the intuitionistic fuzzy versions of Menelaus and Ceva 6-figures have been determined by Akca et. al. [11]. In 2021, by constructing a homomorphism between intuitionistic fuzzy abstract algebras, intuitionistic fuzzy congruence relations were examined and also some isomorphism theorems on intuitionistic abstract algebras were introduced by Cuvalcioglu and Tarsuslu [12]. In the fuzzy and intuitionistic fuzzy projective planes, Altintas and Bayar introduced the fuzzy counterparts and the intuitionistic fuzzy counterparts of the central collineations defined in the classical projective planes and showed some properties of central fuzzy and intuitionistic fuzzy collineations [13]. This paper is an extention of examinations by Altintas et. al. [14] on the

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Fuzzy Collineations of Fuzzy Projective Planes to intuitionistic fuzzy projective planes. In [14], we introduced fuzzy versions of some classical properties related to collineations of projective plane by collineations in fuzzy projective planes by using the base point, the base line and the membership degrees of fuzzy projective plane.
The aim of this study is to define the intuitionistic fuzzy equivalents of collineations defined in classical projective planes in intuitionistic fuzzy projective planes such that every point and every line in the base plane possess the degree of membership and the degree of non-membership and to prove the properties that are invariant under the intuitionistic fuzzy collineations in intuitionistic fuzzy projective planes.

Definition 1.1. *A fuzzy set λ of a set X is a function $\lambda : X \to [0,1] : x \to \lambda(x)$. The number $\lambda(x)$is called the degree of membership of the point x in λ. The intersection $\lambda \wedge \mu$ of the two fuzzy sets λ and μ on X is given by the fuzzy set*
$\lambda \wedge \mu : X \to [0,1] : \lambda(x) \wedge \mu(x)$,
where $\wedge$ denotes the minimum operator and also $\vee$ denotes the maximum operator [1].

Definition 1.2. *[5] Let X be a nonempty fixed set. An intuitionistic fuzzy set A on X is an object having the form $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ where the function $\lambda : X \to I$ and $\mu : X \to I$ denote the degree of membership (namely, $\lambda(x)$) and the degree of nonmembership (namely, $\mu(x)$) of each element $x \in X$ to the set A, respectively $0 \leq \lambda(x) + \mu(x) \leq 1$ for each $x \in X$. An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ can be written in the $A = \{\langle x, \lambda, \mu\rangle : x \in X\}$, or simply $A = \langle \lambda, \mu\rangle$. Let $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ and $B = \{\langle x, \delta(x), \gamma(x)\rangle : x \in X\}$ be an intuitionistic fuzzy sets on X. Then,*
- ***(a)** $\bar{A} = \{\langle x, \mu(x), \lambda(x)\rangle : x \in X\}$ (the complement of A).*
- ***(b)** $A \cap B = \{\langle x, \lambda(x) \wedge \delta(x), \mu(x) \vee \gamma(x)\rangle : x \in X\}$ (the meet of A and B).*
- ***(c)** $A \cup B = \{\langle x, \lambda(x) \vee \delta(x), \mu(x) \wedge \gamma(x)\rangle : x \in X\}$ (the join of A and B).*
- ***(d)** $A \subseteq B \Leftrightarrow \lambda(x) \leq \delta(x)$ and $\mu(x) \geq \gamma(x)$ for each $x \in X$.*
- ***(e)** $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.*
- ***(f)** $\tilde{1} = \{\langle x, 1, 0\rangle : x \in X\}, \tilde{0} = \{\langle x, 0, 1\rangle : x \in X\}$.*

Definition 1.3. *Let $A = (\lambda, \mu)$ be an intuitionistic fuzzy set of a classical vector space V over F. For any $x, y \in V$ and $\alpha, \beta \in F$, if it satisfy $\lambda_A(\alpha x + \beta y) \geq min\{\lambda_A(x), \lambda_A(y)\}$ and $\mu_A(\alpha x + \beta y) \leq max\{\mu_A(x), \mu_A(y)\}$, then A is called an intuitionistic fuzzy subspace of V. Let V_n denotes the set of all $n-$tuples $(\langle x_{1\lambda}, x_{1\mu}\rangle, \langle x_{2\lambda}, x_{2\mu}\rangle, ..., \langle x_{n\lambda}, x_{n\mu}\rangle)$ over F. An element of V_n is called an intuitionistic fuzzy vector (IFV) of dimension n, where $x_{i\lambda}$ and $x_{i\mu}$ are the membership and non-membership values of the component x_i [9].*

Definition 1.4. *An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ on $n-$dimensional projective space S is an intuitionistic fuzzy $n-$dimensional projective space on S if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$ and $\mu(p) \leq \mu(q) \vee \mu(r)$,for any three collinear points p, q, r of A we denoted $[A, S]$. The projective space S is called the base projective space of $[A, S]$ if $[A, S]$ is an intuitionistic fuzzy point, line, plane, ... , we use base point, base line, base plane, ..., respectively [6].*

Definition 1.5. *Let $\langle \lambda, \mu\rangle$ be an intuitionistic fuzzy projective space and let U be a subspace of $\mathscr{P}$. Then (λ_U, U) is called a fuzzy subspace of $(\lambda, \mathscr{P})$ if $\lambda_U(x) \leq \lambda(x)$ and $\mu_U(x) \geq \mu(x)$ for $x \in U$, and $\lambda_U(x) = 0, \mu_U(x) = 1$ for $x \notin U$.*

Definition 1.6. *Let (λ, μ) be an intuitionistic fuzzy projective space of dimension n. Then there are constants $a_i, b_i \in [0,1]$, $i = 0, 1, ..., n$, with $a_i + b_i \leq 1$, and a chain of subspaces $(U_i)_{0 \leq i \leq n}$ with $U_i \leq U_{i+1}$ and $dimU_i = i$, such that*

$$\begin{array}{rcl} (\lambda, \mu) & : & \mathscr{P} \to [0,1] \times [0,1] \\ & & \bar{u} \to (a_0, b_0) \quad for\, \bar{u} \in U_0 \\ & & \bar{u} \to (a_i, b_i) \quad for\ \bar{u} \in U_i \setminus U_{i-1}, i = 1, 2, ..., n \end{array}$$

Definition 1.7. *[10] Consider the projective plane $\mathscr{P} = (\mathscr{N}, \mathscr{D}, \circ)$. Suppose $a \in \mathscr{N}$ and $\alpha, \beta \in [0,1]$. The IF-point (a, α, β) is the following intuitionistic fuzzy set on the point set $\mathscr{N}$ of $\mathscr{P}$:*

$$(a, \alpha, \beta) : \mathscr{N} \to [0,1] : \begin{array}{l} a \to \alpha, \ a \to \beta \\ x \to 0, \ x \in \mathscr{N} \setminus \{a\} \end{array}$$

The point $a \in \mathscr{N}$ is called the base point of the IF-point (a, α, β). An IF-line (L, α, β) with base line L is defined in a similar way. The IF- lines (L, α, β) and (M, σ, ω) intersect in the unique IF-point $(L \cap M, \alpha \wedge \sigma, \beta \vee \omega)$. The IF-points (a, α, β) and(b, σ, ω) span the unique IF-line $(\langle a, b\rangle, \alpha \wedge \sigma, \beta \vee \omega)$.

Definition 1.8. *Suppose $\mathscr{P}$ is a projective plane $\mathscr{P} = (\mathscr{N}, \mathscr{D}, \circ)$. The intuitionistic fuzzy set $Z = \langle \lambda, \mu\rangle$ on $\mathscr{N} \cup \mathscr{D}$ is an intuitionistic fuzzy projective plane on $\mathscr{P}$ if :*

$$\textbf{(1)}\ \lambda(L) \geq \lambda(p) \wedge \lambda(q) \text{ and } \mu(L) \leq \mu(p) \vee \mu(q); \forall p, q : \langle p, q\rangle = L$$

$$\textbf{(2)} \lambda(p) \geq \lambda(L) \wedge \lambda(M) \text{ and } \mu(p) \leq \mu(L) \vee \mu(M); \forall L, M : L \cap M = p.$$

The intuitionistic fuzzy projective plane can be considered as an ordinary projective plane, where to every point (and only to points) one (and only one) degrees of membership and nonmembership are assigned [6].

Now here, the intuitionistic fuzzy counterparts of the theorems and proofs related to the fuzzy linear maps in Abdulhalikov's works [15] are given by using the intuitionistic fuzzy linear maps definition.

Definition 1.9. *Let V and W be two vector spaces over the same field F and T be is a linear map from V to W. Suppose that (V,λ_V,μ_V) and (W,λ_W,μ_W) be intuitionistic fuzzy vector spaces on F. For all $x\in V$, if*

$$\lambda_W(T(x))\geq\lambda_V(x) \text{ and } \mu_W(T(x))\leq\mu_V(x)$$

is satisfied such that $0\leq\lambda_V+\mu_V\leq 1$ and $0\leq\lambda_W+\mu_W\leq 1$, T is called as an intuitionistic fuzzy linear maps from the intuitionistic fuzzy vector space (V,λ_V,μ_V) to the intuitionistic fuzzy vector space (W,λ_W,μ_W).

Definition 1.10. *If T which is a zero linear map from V to W, is an intuitionistic fuzzy linear map defined between the intuitionistic fuzzy vector spaces (V,λ_V,μ_V) and (W,λ_W,μ_W), then T is called as an intuitionistic fuzzy zero linear map.*

2. Intuitionistic collineations of intuitionistic fuzzy projective plane

In this paper, our aim is to investigate intuitionistic collineations of intuitionistic fuzzy projective planes. Compared to isomorphisms, collineations of projective plane and fuzzy isomorphisms, fuzzy collineations of fuzzy projective plane have the advantages and properties. In projective planes, a collineation is a point-to-point and line-to-line transformation that preserves the relation of incidence. Thus it transforms ranges into ranges, pencils into pencils, quadrangles into quadrangles, and so on. Clearly, it is a self-dual concept, the inverse of a collineation, and the product of two collineations is again a collineation [4]. Our aim is now to define the intuitionistic fuzzy counterparts of homomorphism and isomorphism defined in vector spaces in intuitionistic fuzzy projective planes and to apply theorems about properties of collineations in projective plane to intuitionistic fuzzy projective plane. Furthermore, we will show that each collineation can be uniquely extended to a fuzzy projective collineation.
The definitions of homomorphism, isomorphism and collineation in projective planes can be adopted to fuzzy projective planes as follows:

Definition 2.1. *Let $[\mathscr{P},\lambda_{\mathscr{P}},\mu_{\mathscr{P}}]$ and $[\mathscr{P}',\lambda_{\mathscr{P}'},\mu_{\mathscr{P}'}]$ be two intuitionistic fuzzy projective planes with base planes $\mathscr{P}=(\mathscr{N},\mathscr{D},\circ)$, $\mathscr{P}'=\left(\mathscr{N}',\mathscr{D}',\circ'\right)$, respectively. Suppose that f be a homomorphism of a projective plane $\mathscr{P}$ into a projective plane P'P'.$\bar{f}$ is called as* **intuitionistic fuzzy homomorphism** *from $[\mathscr{P},\lambda_{\mathscr{P}},\mu_{\mathscr{P}}]$ into $[\mathscr{P}',\lambda_{\mathscr{P}'},\mu_{\mathscr{P}'}]$ if $\bar{f}(p,\alpha,\beta)=(f(p),\alpha',\beta')$ for all $(p,\alpha,\beta)\in[\mathscr{P},\lambda_{\mathscr{P}},\mu_{\mathscr{P}}]$ where $\lambda_{\mathscr{P}}(p)=\alpha$, $\mu_{\mathscr{P}}(p)=\beta$, $\lambda_{\mathscr{P}'}(f(p))=\alpha'$, $\mu_{\mathscr{P}'}(f(p))=\beta'$ and $\alpha\leq\alpha'$, $\beta\geq\beta'$. If f is an isomorphism of $\mathscr{P}$ into $\mathscr{P}'$ and $\alpha=\alpha',\beta=\beta'$, then $\bar{f}$ is called as* **intuitionistic fuzzy isomorphism** *between the intuitionistic fuzzy projective planes $[\mathscr{P},\lambda_{\mathscr{P}},\mu_{\mathscr{P}}]$ and $[\mathscr{P}',\lambda_{\mathscr{P}'},\mu_{\mathscr{P}'}]$. Also if $\mathscr{P}=\mathscr{P}'$, this $\bar{f}$ intuitionistic fuzzy isomorphism is called as* **intuitionistic fuzzy collineation**.

Theorem 2.2. *Let $\bar{f}:[\mathscr{P},\lambda,\mu]\to[\mathscr{P}',\lambda',\mu']$ is intuitionistic fuzzy isomorphism, the following holds:*
(i) *For any pair of intuitionistic fuzzy points (p_1,α_1,β_1) and (p_2,α_2,β_2), $p_1\neq p_2$ in $[\mathscr{P},\lambda,\mu]$,*

$$\bar{f}(\langle(p_1,\alpha_1,\beta_1),(p_2,\alpha_2,\beta_2)\rangle)=\langle\bar{f}(p_1,\alpha_1,\beta_1),\bar{f}(p_2,\alpha_2,\beta_2)\rangle.$$

(ii) *For any pair of intuitionistic fuzzy lines (L,γ_1,σ_1) and (M,γ_2,σ_2), $L\neq M$ in $[\mathscr{P},\lambda,\mu]$,*

$$\bar{f}((L,\gamma_1,\sigma_1)\cap(M,\gamma_2,\sigma_2))=\bar{f}(L,\gamma_1,\sigma_1)\cap\bar{f}(M,\gamma_2,\sigma_2).$$

(iii) *For any intuitionistic fuzzy point (p,α,β) and intuitionistic fuzzy line (L,γ,σ) in $[\mathscr{P},\lambda,\mu]$, if p is not on L, then the intuitionistic fuzzy point $\bar{f}(p,\alpha,\beta)$ is not on $\bar{f}(L,\gamma,\sigma)$ in $[\mathscr{P}',\lambda',\mu']$.*

Proof **(i)** Let $\bar{f}$ be an intuitionistic fuzzy isomorphism between $[\mathscr{P},\lambda,\mu]$ and $[\mathscr{P}',\lambda',\mu']$. The intuitionistic fuzzy line spanned by the intuitionistic fuzzy points (p_1,α_1,β_1) and (p_2,α_2,β_2) with distinct base points p_1, p_2 is $\langle(p_1,\alpha_1,\beta_1),(p_2,\alpha_2,\beta_2)\rangle=(\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)$. Since f is isomorphism between the base projective planes $\mathscr{P}$ and $\mathscr{P}'$, $f(p_1)\neq f(p_2)$. So $\bar{f}(p_1,\alpha_1,\beta_1)\neq\bar{f}(p_2,\alpha_2,\beta_2)$. Using the definitions of $\bar{f}$ and f,

$$\begin{aligned}\bar{f}(\langle(p_1,\alpha_1,\beta_1),(p_2,\alpha_2,\beta_2)\rangle)&=(f(\langle p_1,p_2\rangle),\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)=(\langle f(p_1),f(p_2)\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)\\&=\langle(f(p_1),\alpha_1,\beta_1),(f(p_2),\alpha_2,\beta_2)\rangle=\langle\bar{f}(p_1,\alpha_1,\beta_1),\bar{f}(p_2,\alpha_2,\beta_2)\rangle\end{aligned}$$

is obtained.

(ii) Let $\bar{f}$ be an intuitionistic fuzzy isomorphism between $[\mathscr{P},\lambda,\mu]$ and $[\mathscr{P}',\lambda',\mu']$. The intersection point of the intuitionistic fuzzy lines (L,γ_1,σ_1) and (M,γ_2,σ_2) with distinct base lines L, M is $(L,\gamma_1,\sigma_1)\cap(M,\gamma_2,\sigma_2)=(L\cap M,\gamma_1\wedge\gamma_2,\sigma_1\vee\sigma_2)$. Since f is isomorphism between the projective planes $\mathscr{P}$ and $\mathscr{P}'$, $f(L)\neq f(M)$. So $\bar{f}(L,\gamma_1,\sigma_1)\neq\bar{f}(M,\gamma_2,\sigma_2)$. Using the definition of $\bar{f}$ and f

$$\begin{aligned}\bar{f}((L,\gamma_1,\sigma_1)\cap(M,\gamma_2,\sigma_2))&=(f(L\cap M),\gamma_1\wedge\gamma_2,\sigma_1\vee\sigma_2)=(f(L)\cap f(M),\gamma_1\wedge\gamma_2,\sigma_1\vee\sigma_2)\\&=((f(L),\gamma_1,\sigma_1)\cap(f(M),\gamma_2,\sigma_2))=\bar{f}(L,\gamma_1,\sigma_1)\cap\bar{f}(M,\gamma_2,\sigma_2)\end{aligned}$$

(iii) Suppose that the intuitionistic fuzzy point $\bar{f}((p,\alpha,\beta))$ is on the intuitionistic fuzzy line $\bar{f}((L,\gamma,\sigma))$ when the base point p is not on the base line L. Then the intuitionistic fuzzy point (p,α,β) is not on the intuitionistic fuzzy line (L,γ,σ). From definitions of f and $\bar{f}$, $\bar{f}((p,\alpha,\beta)) = (f(p),\alpha,\beta)$ and $\bar{f}((L,\gamma,\sigma)) = (f(L),\gamma,\sigma)$. Since the intuitionistic fuzzy point $\bar{f}((p,\alpha,\beta))$ is on the intuitionistic fuzzy line $\bar{f}((L,\gamma,\sigma))$ and f is isomorphism, $f(p) \circ f(L)$ and $p \circ L$ are obtained. This contradicts the hypothesis.
From now on, we considered the intuitionistic fuzzy projective plane $[\mathscr{P},\lambda,\mu]$ with base plane $\mathscr{P}$ and (λ,μ) in the following form:

$$\begin{array}{rcccl} (\lambda,\mu) & : & PG(V) & \to & [0,1]\times[0,1] \\ & & q & \to & (a_0,b_0) \\ & & p & \to & (a_1,b_1), p \in L\setminus\{q\} \\ & & p & \to & (a_2,b_2), p \in \mathscr{P}\setminus\{L\} \end{array}$$

where L is a projective line of $\mathscr{P}$ contains q and $a_0 \geq a_1 \geq a_2$, $b_0 \leq b_1 \leq b_2$, $0 \leq a_i + b_i \leq 1,\ i = 0,\ 1,\ 2$.

The intuitionistic fuzzy point (q,a_0,b_0) and the intuitionistic fuzzy line (L,a_1,b_1) are called as the base point, the base line of the intuitionistic fuzzy projective plane $[\mathscr{P},\lambda,\mu]$, respectively. The invariant properties under any intuitionistic fuzzy collineation in $[\mathscr{P},\lambda,\mu]$ depending on the base line, the base point, the membership degrees and nonmembership degrees of $[\mathscr{P},\lambda,\mu]$ are investigated in detail with the following theorems.

Theorem 2.3. *Suppose that $\bar{f}$ is an intuitionistic fuzzy collineation of $[\mathscr{P},\lambda,\mu]$ defined by the collineation f of the base plane $\mathscr{P}$. Then,*

(i) *If $a_0 \neq a_1 \neq a_2$, then the intuitionistic fuzzy collineation $\bar{f}$ leaves invariant the base point and the base line of $[\mathscr{P},\lambda,\mu]$.*

(ii) *If $a_0 \neq a_1 = a_2$, then the base point is invariant and the base line turns into a line passing through the base point under the intuitionistic fuzzy collineation $\bar{f}$.*

Proof **(i)** Let $a_0 \neq a_1 \neq a_2$.
The image of the base point (q,a_0,b_0) is $\bar{f}(q,a_0,b_0) = (f(q),a_0,b_0)$. Since there is no other point which has membership degree (a_0,b_0) in $[\mathscr{P},\lambda,\mu]$, $(f(q),a_0,b_0)$ must be the base point. So $f(q) = q$, $\bar{f}(q,a_0,b_0) = (q,a_0,b_0)$.
Since $(L,a_1,b_1) = \langle (q,a_0,b_0),(p,a_1,b_1)\rangle \ni p \circ L$, $p \neq q$ and from Theorem 2. 2. i), the base line is

$$\begin{aligned} \bar{f}((L,a_1,b_1)) &= \left\langle \bar{f}(q,a_0,b_0), \bar{f}(p,a_1,b_1)\right\rangle = \langle (f(q),a_0,b_0),(f(p),a_1,b_1)\rangle, (f(q)=q) \\ &= \langle (q,a_0,b_0),(f(p),a_1,b_1)\rangle = (\langle q,f(p)\rangle, a_0 \wedge a_1, b_0 \vee b_1) = (\langle q,f(p)\rangle, a_1, b_1). \end{aligned}$$

Since there is no other line with (a_1,b_1) membership degree, $\bar{f}((L,a_1,b_1)) = (\langle q,f(p_i)\rangle, a_1,b_1) = (L,a_1,b_1)$ is obtained. So the base point and the base line are invariant under the intuitionistic fuzzy collineation $\bar{f}$.
The converse of this proposition is not true. While the base point and the base line are invariant, the membership degrees can be different or equal.

(ii) Let $a_0 \neq a_1 = a_2$.
The image of the base point (q,a_0,b_0) is $\bar{f}(q,a_0,b_0) = (f(q),a_0,b_0)$. Since there is no other line with (a_0,b_0) membership degree in $[\mathscr{P},\lambda,\mu]$, $(f(q),a_0ib_0)$ must be base point. So $f(q) = q$ and $\bar{f}(q,a_0,b_0) = (f(q),a_0,b_0) = (q,a_0,b_0)$. Since $\bar{f}$ is intuitionistic fuzzy isomorphism, $\bar{f}(q,a_0,b_0) \circ \bar{f}(L,a_1,b_1)$. Hence, the base point (q,a_0,b_0) is on $(f(L),a_1,b_1)$.

$$\begin{aligned} \bar{f}(L,a_1,b_1) &= \bar{f}(\langle (q,a_0,b_0),(p,a_1,b_1)\rangle) = \left\langle \bar{f}(q,a_0,b_0), \bar{f}(p,a_1,b_1)\right\rangle = \langle (f(q),a_0,b_0),(f(p),a_1,b_1)\rangle \\ &= \langle (q,a_0,b_0), f(p,a_1,b_1)\rangle = (\langle q,f(p)\rangle, a_0 \wedge a_1, b_0 \vee b_1) = (\langle q,f(p)\rangle, a_1, b_1) \end{aligned}$$

So the base line $f(L) = \langle q,f(p)\rangle$ turns into the line through the base point.
The following theorem states the properties of $\bar{f}$ intuitionistic fuzzy collineation while the base point is invariant.

Theorem 2.4. *Suppose that $\bar{f}$ is an intuitionistic fuzzy collineation of $[\mathscr{P},\lambda,\mu]$ defined by the collineation f of the base plane $\mathscr{P}$ and the base point (q,a_0,b_0) be invariant under the intuitionistic fuzzy collineation $\bar{f}$.*

(i) *If the base line (L,a_1,b_1) is invariant under $\bar{f}$, $[\mathscr{P},\lambda,\mu]$ has at most three membership degrees.*

(ii) *If the base line (L,a_1,b_1) turns into a line other than itself passing through the base point (q,a_0,b_0), there are at most two membership degrees in $[\mathscr{P},\lambda,\mu]$ such that $a_0 \geq a_1 = a_2$ and $b_0 \leq b_1 = b_2$.*

(iii) *The base line (L,a_1,b_1) does not turn into an intuitionistic fuzzy line that does not pass through the base point (q,a_0,b_0) under $\bar{f}$ in $[\mathscr{P},\lambda,\mu]$.*

Proof **(i)** Let the base point (q,a_0,b_0) and the base line (L,a_1,b_1) be invariant under the intuitionistic fuzzy collineation $\bar{f}$. Then $\bar{f}(q,a_0,b_0) = (q,a_0,b_0)$. The image point $\bar{f}(p,a_1,b_1)$ of the intuitionistic fuzzy point (p,a_1,b_1) on the base line (L,a_1,b_1) is $(f(p),a_1,b_1)$ and is on the base line L.

If $a_0 \neq a_1 \neq a_2$ is taken, there are at most three membership degrees in $[\mathscr{P}, \lambda, \mu]$.

(ii) Let the base point (q, a_0, b_0) be invariant and the base line turns into a line other than the base line passing through the base point $[\mathscr{P}, \lambda, \mu]$. Since the base point (q, a_0, b_0) on (L, a_1, b_1) and $\bar{f}$ is an intuitionistic fuzzy isomorphism, the image of the base point (q, a_0, b_0) is also on the image of the base line $(f(L), a_1, b_1)$. Also $L \neq f(L)$ and the line $f(L)$ passes through points of degree of membership (a_2, b_2) not on the base line, the membership degree of $f(L)$ is (a_2, b_2). So, $a_1 = a_2$ and $b_1 = b_2$ are obtained. Consequently, $[\mathscr{P}, \lambda, \mu]$ has at most two membership degrees.

(iii) Since the base point is on the base line, its image is on the image of the base line. However, the being invariant of the base point gives rise to that the image line has to pass through the base point.

Theorem 2.5. *Suppose that $\bar{f}$ is an intuitionistic fuzzy collineation of $[\mathscr{P}, \lambda, \mu]$ defined by the collineation f of the base plane $\mathscr{P}$ and the base point is not invariant and turns into an intuitionistic fuzzy point on the base line under the intuitionistic fuzzy collineation $\bar{f}$.*

***(i)** If the base line is invariant under the intuitionistic fuzzy collineation $\bar{f}$ of $[\mathscr{P}, \lambda, \mu]$, among the membership degrees $0 \leq a_i + b_i \leq 1,\ i = 0, 1, 2,$ there is a relationship $a_0 = a_1 \geq a_2$ and $b_0 = b_1 \leq b_2$.*

***(ii)** If the base point turns into an intuitionistic fuzzy point on the base line other than itself under the intuitionistic fuzzy collineation $\bar{f}$, then there is one membership degree in $[\mathscr{P}, \lambda, \mu]$.*

***(iii)** If the base point of (q, a_0, b_0) turns into any point not on the base line under the collineation f in $\mathscr{P}$, then there is only one membership degree in $[\mathscr{P}, \lambda, \mu]$.*

Proof **(i)** Let the base point q of the intuitionistic fuzzy point (q, a_0, b_0) be not invariant and turn into another point on the base line L, $\bar{f}(q, a_0, b_0) = (f(q), a_1, b_1)$.
Suppose that the base line L is invariant under the collineation f. It is clear that the intuitionistic fuzzy point (q, a_0, b_0) turns into the intuitionistic fuzzy point (p, a_1, b_1) with $p \circ L$, $q \neq p$. Since $\bar{f}$ is intuitionistic fuzzy isomorphism, $a_0 = a_1$ and $b_0 = b_1$. Hence, there are at most two membership degree in $[\mathscr{P}, \lambda, \mu]$.

(ii) Since the intuitionistic fuzzy point (q, a_0, b_0) turns into the intuitionistic fuzzy point (p, a_1, b_1), $p \neq q$ on (L, a_1, b_1). It is clear that $a_0 = a_1$ and $b_0 = b_1$. Next suppose that $f(L) \neq L$. So any intuitionistic fuzzy point different from the base point on the base line L with membership degree (a_1, b_1) turns into any other intuitionistic fuzzy point with membership degree (a_2, b_2). Since $\bar{f}$ is intuitionistic fuzzy isomorphism, then $a_1 = a_2$ and $b_1 = b_2$. Hence, $a_0 = a_1 = a_2$ and $b_0 = b_1 = b_2$.

(iii) Since the base point turns into a point not on the base line, the image of (q, a_0, b_0) under $\bar{f}$ is (p, a_2, b_2) with $p \not{o} L$. It is clearly $a_0 = a_2$ and $b_0 = b_2$. If we use this equality and the conditions $a_0 \geq a_1 \geq a_2$ and $b_0 \leq b_1 \leq b_2$ among the membership degrees in $[\mathscr{P}, \lambda, \mu]$, $a_0 = a_1 = a_2$ and $b_0 = b_1 = b_2$ are obtained.

Corollary If the base point of (q, a_0, b_0) turns into any point not on the base line L of (L, a_1, b_1) under the collineation f, the base line (L, a_1, b_1) with base line L of $[\mathscr{P}, \lambda, \mu]$ is not invariant under the intuitionistic fuzzy collineation $\bar{f}$.

Proof Let the base point q turns into any point not on the base line under the collineation f of $\mathscr{P}$. Since the intuitionistic fuzzy point (q, a_0, b_0) such that the base point q is on the base line L turns into $(f(q), a_2, b_2)$ such that $f(q)$ is not on the base line L, then the base line $L = \langle q, p \rangle$ spanning by the points p and q turns into $f(L) = \langle f(q), f(p) \rangle \neq L$ under the intuitionistic fuzzy collineation $\bar{f}$. Hence, (L, a_1, b_1) is not invariant.

Theorem 2.6. *Suppose that $\bar{f}$ is an intuitionistic fuzzy collineation of $[\mathscr{P}, \lambda, \mu]$ defined by the collineation f of the base plane $\mathscr{P}$.*

***(i)** If two distinct points p_1 and p_2 in the base plane $\mathscr{P}$ are invariant under the collineation f of $\mathscr{P}$, the intuitionistic fuzzy line spanned by fuzzy points (p_1, α_1, β_1) and (p_2, α_2, β_2) is invariant under the intuitionistic fuzzy collineation $\bar{f}$ of $[\mathscr{P}, \lambda, \mu]$.*

***(ii)** If two distinct lines L_1 and L_2 in the base plane $\mathscr{P}$ are invariant under the collineation f of $\mathscr{P}$ and the intersection point of L_1 and L_2 is not on the base line L in $\mathscr{P}$, then the intersection point of the intuitionistic fuzzy lines (L_1, α_1, β_1) and (L_2, α_2, β_2) is invariant under the intuitionistic fuzzy collineation $\bar{f}$ in $[\mathscr{P}, \lambda, \mu]$.*

***(iii)** Suppose that two distinct lines L_1 and L_2 different from the base line L in the base plane $\mathscr{P}$ are invariant under the collineation f of $\mathscr{P}$ and the intersection point of these lines is on the base line L in $\mathscr{P}$. If the intersection point of the intuitionistic fuzzy lines (L_1, α_1, β_1) and (L_2, α_2, β_2) is invariant under the intuitionistic fuzzy collineation $\bar{f}$ in $[\mathscr{P}, \lambda, \mu]$. There is a relationship $a_0 = a_1 = a_2$, $b_0 = b_1 = b_2$ or $a_1 = a_2$, $b_1 = b_2$ among the membership degrees in $[\mathscr{P}, \lambda, \mu]$.*

Proof

(i) Let the base points p_1 and p_2 in $\mathscr{P}$ of (p_1, α_1, β_1) and (p_2, α_2, β_2) in $[\mathscr{P}, \lambda, \mu]$ be invariant under the collineation f of $\mathscr{P}$. Then by the definition of intuitionistic fuzzy collineation $\bar{f}$ in $[\mathscr{P}, \lambda, \mu]$, $\bar{f}(p_1, \alpha_1, \beta_1) = (f(p_1), \alpha_1, \beta_1) = (p_1, \alpha_1, \beta_1)$ and $\bar{f}(p_2, \alpha_2, \beta_2) = (f(p_2), \alpha_2, \beta_2) = (p_2, \alpha_2, \beta_2)$.
For any pair $((p_1, \alpha_1, \beta_1), (p_2, \alpha_2, \beta_2))$ of fuzzy points, $p_1 \neq p_2$, the fuzzy line $(\langle p_1, p_2 \rangle, \alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2)$ spanned by them, also belongs to the intuitionistic fuzzy projective plane $[\mathscr{P}, \lambda, \mu]$. By using the definition of $\bar{f}$ of $[\mathscr{P}, \lambda, \mu]$ and the remaining invariant of the points p_1 and p_2 under the collineation f in $\mathscr{P}$, the image of the intuitionistic fuzzy line $(\langle (p_1, \alpha_1, \beta_1), (p_2, \alpha_2, \beta_2) \rangle)$ under the intuitionistic fuzzy collineation $\bar{f}$ is

$$(f(\langle p_1,p_2\rangle),\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)=(\langle f(p_1),f(p_2)\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)=(\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2).$$

Hence, the intuitionistic fuzzy line $(\langle p_1,p_2\rangle,\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)$ is invariant under the intuitionistic fuzzy collineation $\bar{f}$.

(ii) Let the base lines L_1 and L_2 in $\mathscr{P}$ of (L_1,α_1,β_1) and (L_2,α_2,β_2) in $[\mathscr{P},\lambda,\mu]$ be invariant under the collineation f of $\mathscr{P}$. Since $L_1\neq L_2\neq L$, the membership degrees $\alpha_i=a_2$, and $\beta_i=b_2$ $i=1,2$. By the definition of $\bar{f}$ and being invariant of the lines L_1 and L_2 under the collineation f in $\mathscr{P}$, $\bar{f}(L_1,a_2,b_2)=(L_1,a_2,b_2)$ and $\bar{f}(L_2,a_2,b_2)=(L_2,a_2,b_2)$. The image of the intersection intuitionistic fuzzy point $(L_1\cap L_2,a_2,b_2)$ under $\bar{f}$ is

$$(f(L_1\cap L_2),a_2,b_2)=(f(L_1)\cap f(L_2),a_2,b_2)=(L_1\cap L_2,a_2,b_2).$$

It is implies that the intuitionistic fuzzy point $(L_1\cap L_2,a_2,b_2)$ remains invariant under the intuitionistic fuzzy collineation $\bar{f}$.

(iii) Let different base lines L_1 and L_2 of (L_1,α_1,β_1) and (L_2,α_2,β_2) be invariant under the collineation f in $\mathscr{P}$. Since $L_1\neq L_2\neq L$, $\alpha_i=a_2$, and $\beta_i=b_2$ $i=1,2$. The intersection point of (L_1,a_2,b_2) and (L_2,a_2,b_2) is the base point (q,a_0,b_0) or any intuitionistic fuzzy point (p,a_1,b_1) on the base line (L,a_1,b_1) of $[\mathscr{P},\lambda,\mu]$. If the intersection point is (q,a_0,b_0), then $a_0=a_1=a_2$ and $b_0=b_1=b_2$. If the intersection point is (p,a_1,b_1), then $a_1=a_2$ and $b_1=b_2$ are obtained.

Theorem 2.7. *Suppose that $\bar{f}$ is any intuitionistic fuzzy collineation of $[\mathscr{P},\lambda,\mu]$ defined by the collineation f of the base plane $\mathscr{P}$. In this case,*

(i) *If M is a pointwise invariant line under the collineation f in the base projective plane $\mathscr{P}$, then the corresponding intuitionistic fuzzy line (M,γ,σ) is also pointwise invariant under the intuitionistic fuzzy collineation $\bar{f}$ in $[\mathscr{P},\lambda,\mu]$.*

(ii) *If two distinct lines L_1 and L_2 are pointwise invariant under the collineation f of the base plane $\mathscr{P}$, then the intersection point of the intuitionistic fuzzy lines (L_1,γ_1,σ_1) and (L_2,γ_2,σ_2) is invariant under the intuitionistic fuzzy collineation $\bar{f}$.*

(iii) *If the base line L and L_1, $L_1\neq L$ are pointwise invariant lines under the collineation f of the base plane $\mathscr{P}$, then the intuitionistic fuzzy collineation $\bar{f}$ is identity collineation in $[\mathscr{P},\lambda,\mu]$.*

Proof **(i)** Let the base line M of the intuitionistic fuzzy line (M,γ,σ) in $[\mathscr{P},\lambda,\mu]$ be pointwise invariant under the collineation f of $\mathscr{P}$. From the definition of $\bar{f}$ and being pointwise invariant of M under collineation f,$\bar{f}(p,\alpha,\beta)=(f(p),\alpha,\beta)=(p,\alpha,\beta)$ for every fuzzy point (p,α,β) on (M,γ,σ). Hence the fuzzy line (M,γ,σ) is pointwise invariant in $[\mathscr{P},\lambda,\mu]$.

(ii) Let the base lines L_1 and L_2 of the intuitionistic fuzzy lines (L_1,γ_1,σ_1) and (L_2,γ_2,σ_2) in $[\mathscr{P},\lambda,\mu]$ be pointwise invariant under the collineation f of the base plane $\mathscr{P}$, respectively. From (i), the fuzzy lines (L_1,γ_1,σ_1) and (L_2,γ_2,σ_2) are pointwise invariant under the intuitionistic fuzzy collineation $\bar{f}$ of $[\mathscr{P},\lambda,\mu]$. Since (L_i,γ_i,σ_i) are pointwise invariant and (p,α,β) is on (L_i,γ_i,σ_i), $i=1,2$, hence the intersection point of (L_1,γ_1,σ_1) and (L_2,γ_2,σ_2) is invariant.

(iii) Let the base line L and L_1, $L_1\neq L$ be pointwise invariant under the collineation f of $\mathscr{P}$. So $\gamma_1=a_2$ and $\sigma_1=b_2$ are obtained. It is well-known that if there are two distinct pointwise lines under a collineation of projective plane $\mathscr{P}$, then the collineation f is identity collineation. From *i)* (L,a_1,b_1) and (L_1,a_2,b_2), are pointwise invariant. The image of (p,a_2,b_2) such that $p\,\phi L$ and $p\,\phi L_1$ is $\bar{f}(p,a_2,b_2)=(f(p),a_2,b_2)=(p,a_2,b_2)$ under $\bar{f}$. Hence every intuitionistic fuzzy point in $[\mathscr{P},\lambda,\mu]$ are invariant, and this means that $\bar{f}$ is the identity collineation of $[\mathscr{P},\lambda,\mu]$.

Corollary If f is identity collineation of $\mathscr{P}$, $\bar{f}$ is identity collineation of $[\mathscr{P},\lambda,\mu]$.

3. Conclusion

The concepts of intuitionistic fuzzy isomorphism and intuitionistic fuzzy collineations between two intuitionistic fuzzy projective planes are introduced and then some important results are obtained. It is seen that the intuitionistic fuzzy collineations of intuitionistic fuzzy projective planes cannot hold the intuitionistic fuzzy versions of some classical properties related to collineations of projective plane. The properties of intuitionistic fuzzy projective plane left invariant under the intuitionistic fuzzy collineations are characterized depending on the base point, base line and the membership degrees of intuitionistic fuzzy projective plane. Consequently, these obtained results on intuitionistic fuzzy isomorphism and intuitionistic fuzzy collineation have an important effect on enriching the theory of intuitionistic fuzzy geometries.

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