Fundamental Journal of Mathematics and Applications, 5 (2) (2022) 89-97
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: *www.dergipark.org.tr/en/pub/fujma*
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1049969

Examples of Almost Paracontact Metric Structures on 5-Dimensions

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Article Info

Keywords: *Almost paracontact metric structure, $\eta-$ Einstein manifold, Nilpotent Lie algebra*
2010 AMS: *53C25, 53D15*
Received: *29 December 2021*
Accepted: *19 March 2022*
Available online: *28 March 2022*

Abstract

In this study, the classes of several almost paracontact metric structures on 5 dimensional nilpotent Lie algebras are determined. It is also shown that there are no $\eta-$ Einstein structures on 5 dimensional nilpotent Lie algebras.

1. Introduction

Differentiable manifolds having almost paracontact structures were introduced by [1] and after the work of [2] many authors have made contribution, see [2]-[6] and references therein. Almost paracontact metric manifolds were classified according to the covariant derivative of the structure tensor. The space of tensors having the same symmetry properties as the structure tensor is decomposed into the direct sum of twelve subspaces. Thus there are 12 basic classes and 2^{12} classes of almost paracontact metric structures. The defining relations and projections onto each subspace are given in [4] and [3].
There are six classes of non-isomorphic non-abelian nilpotent Lie algebras in five dimensions [7]. In this work, we give the explicit classes of some almost paracontact metric structures defined on 5-dimensional nilpotent Lie algebras by calculating projections onto each subspace. In addition, we show that a 5-dimensional nilpotent Lie algebra does not have the structure of an $\eta-$ Einstein manifold. For the existence of some classes of almost paracontact metric structures on 5-dimensional nilpotent Lie algebras, see [8].

2. Preliminaries

Let M^{2n+1} be an odd dimensional differentiable manifold. An ordered triple (φ,ξ,η) of an endomorphism, a vector field and a 1-form, respectively, with the following properties is called an almost paracontact structure on M

$$\varphi^2 = I-\eta\otimes\xi, \qquad \eta(\xi)=1, \varphi(\xi)=0,$$

$$\text{there is a distribution } \mathbb{D}: p\in M \longrightarrow \mathbb{D}_p = Ker\eta.$$

In this case M is called an almost paracontact manifold. If M also admits a semi-Riemannian metric g with the property that

$$g(\varphi(u),\varphi(v)) = -g(u,v)+\eta(u)\eta(v)$$

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for all $u, v \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the set of smooth vector fields on M, then M is called an almost paracontact metric manifold. The 2-form defined by

$$\Phi(u,v) = g(\varphi u, v)$$

for all $u, v \in \mathfrak{X}(M)$, is called the fundamental 2-form. We denote the vector fields and tangent vectors by letters u, v, w. 2^{12} classes of almost paracontact manifolds are obtained by using the covariant derivative of Φ. Consider the tensor F defined by

$$F(u,v,w) = g((\nabla_u \varphi)(v), w),$$

for all $u, v, w \in T_pM$, where T_pM is the tangent space at p and ∇ denotes the covariant derivative of g. Then F satisfies

$$F(u,v,w) = -F(u,w,v), \tag{2.1}$$

$$F(u,\varphi v,\varphi w) = F(u,v,w) + \eta(v)F(u,w,\xi) - \eta(w)F(u,v,\xi). \tag{2.2}$$

The Lee forms associated with F are

$$\theta(u) = g^{ij}F(e_i,e_j,u), \quad \theta^*(u) = g^{ij}F(e_i,\varphi e_j,u), \quad \omega(u) = F(\xi,\xi,u),$$

where $u \in T_pM$, $\{e_i, \xi\}$ is a basis for T_pM and g^{ij} is the inverse of the matrix g_{ij}.
Let $\mathscr{F}$ be the set of $(0,3)$ tensors over T_pM having properties (2.1), (2.2). $\mathscr{F}$ is the direct sum of four subspaces W_i, $i = 1, \ldots, 4$ where projections F^{W_i} onto W_i are

$$F^{W_1}(u,v,w) = F(\varphi^2 u, \varphi^2 v, \varphi^2 w),$$

$$F^{W_2}(u,v,w) = -\eta(v)F(\varphi^2 u, \varphi^2 w, \xi) + \eta(w)F(\varphi^2 u, \varphi^2 v, \xi),$$

$$F^{W_3}(u,v,w) = \eta(u)F(\xi, \varphi v, \varphi w),$$

$$F^{W_4}(u,v,w) = \eta(u)\{\eta(v)F(\xi,\xi,w) - \eta(w)F(\xi,\xi,v)\}.$$

In addition W_1 can be written as a direct sum of subspaces $\mathbb{G}_i$, $i = 1, \ldots, 4$, W_2 is a direct sum of subspaces $\mathbb{G}_i$, $i = 5, \ldots, 10$, and denote W_3 and W_4 as $\mathbb{G}_{11}$ and $\mathbb{G}_{12}$, respectively. Then $\mathscr{F}$ is a direct sum of twelve subspaces $\mathbb{G}_i$, $i = 1, \ldots, 12$. An almost paracontact manifold is said to be in the class $\mathbb{G}_i \oplus \mathbb{G}_j$, etc if the tensor F is in the class $\mathbb{G}_i \oplus \mathbb{G}_j$ over T_pM for all $p \in M$. The defining relations of basic classes $\mathbb{G}_i$ of almost paracontact metric structures and projections F^i onto each $\mathbb{G}_i$ are listed below [3, 4].

$$\begin{aligned}
\mathbb{G}_1: \ F(u,v,w) \quad &= \quad \frac{1}{2(n-1)}\{g(u,\varphi v)\theta_F(\varphi w) - g(u,\varphi w)\theta_F(\varphi v) - g(\varphi u,\varphi v)\theta_F(\varphi^2 w) \\
&\quad + g(\varphi u,\varphi w)\theta_F(\varphi^2 v)
\end{aligned}$$

$$\mathbb{G}_2: \ F(\varphi u,\varphi v,w) = -F(u,v,w), \theta_F = 0$$

$$\mathbb{G}_3: \ F(\xi,v,w) = F(u,\xi,w) = 0, F(u,v,w) = -F(v,u,w)$$

$$\begin{aligned}
\mathbb{G}_4: \ F(\xi,v,w) \quad &= \quad F(u,\xi,w) = 0, \\
\sum_{cyc} F(u,v,w) \quad &= \quad 0 \ \text{ where } \sum_{cyc} \text{ denotes the cyclic sum over } u,v,w
\end{aligned}$$

$$\mathbb{G}_5: \ F(u,v,w) = \frac{\theta_F(\xi)}{2n}\{g(\varphi u,\varphi w)\eta(v) - g(\varphi u,\varphi v)\eta(w)\}$$

$$\mathbb{G}_6: \ F(u,v,w) = -\frac{\theta_F^*(\xi)}{2n}\{g(u,\varphi w)\eta(v) - g(u,\varphi v)\eta(w)\}$$

$$\begin{aligned} \mathbb{G}_7: \ F(u,v,w) &= -\eta(v)F(u,w,\xi) + \eta(w)F(u,v,\xi), \\ F(u,v,\xi) &= -F(v,u,\xi) = -F(\varphi u,\varphi v,\xi), \ \ \theta_F^*(\xi) = 0 \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathbb{G}_8: \ F(u,v,w) &= -\eta(v)F(u,w,\xi) + \eta(w)F(u,v,\xi), \\ F(u,v,\xi) &= F(v,u,\xi) = -F(\varphi u,\varphi v,\xi), \ \ \theta_F(\xi) = 0 \end{aligned}$$

$$\begin{aligned} \mathbb{G}_9: \ F(u,v,w) &= -\eta(v)F(u,w,\xi) + \eta(w)F(u,v,\xi), \\ F(u,v,\xi) &= -F(v,u,\xi) = F(\varphi u,\varphi v,\xi) \end{aligned}$$

$$\begin{aligned} \mathbb{G}_{10}: \ F(u,v,w) &= -\eta(v)F(u,w,\xi) + \eta(w)F(u,v,\xi), \\ F(u,v,\xi) &= F(v,u,\xi) = F(\varphi u,\varphi v,\xi) \end{aligned}$$

$$\mathbb{G}_{11}: \ F(u,v,w) = \eta(u)F(\xi,\varphi v),\varphi w))$$

$$\mathbb{G}_{12}: \ F(u,v,w) = \eta(u)\{\eta(v)F(\xi,\xi,w) - \eta(w)F(\xi,\xi,v)\}$$

Projections F^i onto each subspace $\mathbb{G}_i$ are

$$\begin{aligned} F^1(u,v,w) &= \frac{1}{2(n-1)}\{g(u,\varphi v)\theta_{F^1}(\varphi w) - g(u,\varphi w)\theta_{F^1}(\varphi v) \\ &\quad -g(\varphi u,\varphi v)\theta_{F^1}(\varphi^2 w) + g(\varphi u,\varphi w)\theta_{F^1}(\varphi^2 v)\}, \end{aligned}$$

$$F^2(u,v,w) = \frac{1}{2}\{F(\varphi^2 u,\varphi^2 v,\varphi^2 w) - F(\varphi u,\varphi^2 v,\varphi w)\} - F^1(u,v,w)$$

$$\begin{aligned} F^3(u,v,w) &= \frac{1}{6}\{F(\varphi^2 u,\varphi^2 v,\varphi^2 w) + F(\varphi u,\varphi^2 v,\varphi w) \\ &\quad +F(\varphi^2 v,\varphi^2 w,\varphi^2 u) + F(\varphi v,\varphi^2 w,\varphi u) \\ &\quad +F(\varphi^2 w,\varphi^2 u,\varphi^2 v) + F(\varphi w,\varphi^2 u,\varphi v)\} \end{aligned}$$

$$F^4(u,v,w) = \frac{1}{2}\{F(\varphi^2 u,\varphi^2 v,\varphi^2 w) + F(\varphi u,\varphi^2 v,\varphi w)\} - F^3(u,v,w) \tag{2.4}$$

$$F^5(u,v,w) = \frac{\theta_{F^5}(\xi)}{2n}\{\eta(v)g(\varphi u,\varphi w) - \eta(w)g(\varphi u,\varphi v)\} \tag{2.5}$$

$$F^6(u,v,w) = -\frac{\theta_{F^6}^*(\xi)}{2n}\{\eta(v)g(u,\varphi w) - \eta(w)g(u,\varphi v)\}$$

$$\begin{aligned} F^7(u,v,w) &= -\frac{1}{4}\eta(v)\{F(\varphi^2 u,\varphi^2 w,\xi) - F(\varphi u,\varphi w,\xi) \\ &\quad -F(\varphi^2 w,\varphi^2 u,\xi) + F(\varphi w,\varphi u,\xi)\} + \frac{1}{4}\eta(w)\{F(\varphi^2 u,\varphi^2 v,\xi) \\ &\quad -F(\varphi u,\varphi v,\xi) - F(\varphi^2 v,\varphi^2 u,\xi) + F(\varphi v,\varphi u,\xi)\} - F^6(u,v,w) \end{aligned}$$

$$
\begin{aligned}
F^{8}(u,v,w) &= -\frac{1}{4}\eta(v)\left\{F(\varphi^2u,\varphi^2w,\xi)-F(\varphi u,\varphi w,\xi)\right.\\
&\left.+F(\varphi^2w,\varphi^2u,\xi)-F(\varphi w,\varphi u,\xi)\right\}+\frac{1}{4}\eta(w)\left\{F(\varphi^2u,\varphi^2v,\xi)\right.\\
&\left.-F(\varphi u,\varphi v,\xi)+F(\varphi^2v,\varphi^2u,\xi)-F(\varphi v,\varphi u,\xi)\right\}-F^{5}(u,v,w)
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
F^{9}(u,v,w) &= -\frac{1}{4}\eta(v)\left\{F(\varphi^2u,\varphi^2w,\xi)+F(\varphi u,\varphi w,\xi)\right.\\
&\left.-F(\varphi^2w,\varphi^2u,\xi)-F(\varphi w,\varphi u,\xi)\right\}+\frac{1}{4}\eta(w)\left\{F(\varphi^2u,\varphi^2v,\xi)\right.\\
&\left.+F(\varphi u,\varphi v,\xi)-F(\varphi^2v,\varphi^2u,\xi)-F(\varphi v,\varphi u,\xi)\right\}
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
F^{10}(u,v,w) &= -\frac{1}{4}\eta(v)\left\{F(\varphi^2u,\varphi^2w,\xi)+F(\varphi u,\varphi w,\xi)\right.\\
&\left.+F(\varphi^2w,\varphi^2u,\xi)+F(\varphi w,\varphi u,\xi)\right\}+\frac{1}{4}\eta(w)\left\{F(\varphi^2u,\varphi^2v,\xi)\right.\\
&\left.+F(\varphi u,\varphi v,\xi)+F(\varphi^2v,\varphi^2u,\xi)+F(\varphi v,\varphi u,\xi)\right\}
\end{aligned}
\tag{2.8}
$$

$$F^{11}(u,v,w)=\eta(u)F(\xi,\varphi^2v,\varphi^2w) \tag{2.9}$$

$$F^{12}(u,v,w)=\eta(u)\{\eta(v)F(\xi,\xi,\varphi^2w)-\eta(w)F(\xi,\xi,\varphi^2v)\}. \tag{2.10}$$

It is known that ξ is Killing in $\mathbb{G}_1\oplus\mathbb{G}_2\oplus\mathbb{G}_3\oplus\mathbb{G}_4\oplus\mathbb{G}_5\oplus\mathbb{G}_8\oplus\mathbb{G}_9\oplus\mathbb{G}_{11}$, that is $F^6=F^7=F^{10}=F^{12}=0$ in this case and ξ is parallel in the basic classes $\mathbb{G}_1$, $\mathbb{G}_3$, $\mathbb{G}_4$, $\mathbb{G}_{11}$. Also for five dimensional manifolds, the dimension of $\mathbb{G}_3$ is zero, so $F^3=0$ [3].

A K-paracontact manifold (M,φ,η,ξ,g) is called an η-Einstein manifold if its Ricci tensor is of the form

$$Ric(u,v)=ag(u,v)+b\eta(u)\eta(v),$$

where a, b are constants. Also, the Ricci curvature in the direction of ξ satisfies

$$Ric(\xi,\xi)=-2n \tag{2.11}$$

on a K-paracontact metric manifold of dimension $2n+1$ [2].

Let G be a connected Lie group and (φ,ξ,η,g) a left invariant almost paracontact metric structure on G, that is,

$$\varphi\circ L_a=L_a\circ\varphi,\ \ L_a(\xi)=\xi,$$

where L_a is the left multiplication by $a\in G$ in G and g is left invariant. The almost paracontact metric structure on G induces an almost paracontact metric structure on the Lie algebra $\mathfrak{g}$ of G denoted by (φ,ξ,η,g).

In this study, we determine the classes of some almost paracontact metric structures on 5-dimensional nilpotent Lie algebras. We use the classification of 5 dimensional nilpotent Lie algebras in [7]. There are six non-isomorphic non-abelian algebras $\mathfrak{g}_i$ with basis $\{e_1,\dots,e_5\}$ and non-zero brackets:

$$
\begin{aligned}
\mathfrak{g}_1 &: [e_1,e_2]=e_5,[e_3,e_4]=e_5\\
\mathfrak{g}_2 &: [e_1,e_2]=e_3,[e_1,e_3]=e_5,[e_2,e_4]=e_5\\
\mathfrak{g}_3 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_2,e_3]=e_5\\
\mathfrak{g}_4 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5\\
\mathfrak{g}_5 &: [e_1,e_2]=e_4,[e_1,e_3]=e_5\\
\mathfrak{g}_6 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_2,e_3]=e_5.
\end{aligned}
$$

In addition, we show that a five-dimensional almost paracontact metric manifold (G,φ,ξ,η,g) can not be an η-Einstein manifold, where G is a connected Lie group with 5 dimensional nilpotent Lie algebra.

3. Classes of almost paracontact metric structures on $\mathfrak{g}_i$

Assume that (φ,ξ,η,g) is a left invariant almost paracontact metric structure on a connected Lie group G_i with corresponding Lie algebra $\mathfrak{g}_i$. Denote the corresponding almost paracontact metric structure on $\mathfrak{g}_i$ by the same quadruple.
The algebra $\mathfrak{g}_1$: Consider the basis $\{e_1,\dots,e_5\}$ with non-zero brackets

$$[e_1,e_2]=e_5,[e_3,e_4]=e_5.$$

Let g be the semi-Riemannian metric such that $\{e_1,\dots,e_5\}$ is orthonormal and $\varepsilon_i=g(e_i,e_i)=\pm1$. The nonzero covariant derivatives are evaluated in [8] by Kozsul's formula:

$$\nabla_{e_1}e_2=\frac{1}{2}e_5,\ \ \nabla_{e_1}e_5=-\frac{1}{2}\varepsilon_2\varepsilon_5e_2,$$

$$\nabla_{e_2}e_1=-\frac{1}{2}e_5,\ \ \nabla_{e_2}e_5=\frac{1}{2}\varepsilon_1\varepsilon_5e_1,$$

$$\nabla_{e_3}e_4=\frac{1}{2}e_5,\ \ \nabla_{e_3}e_5=-\frac{1}{2}\varepsilon_4\varepsilon_5e_4,$$

$$\nabla_{e_4}e_3=-\frac{1}{2}e_5,\ \ \nabla_{e_4}e_5=\frac{1}{2}\varepsilon_3\varepsilon_5e_3,$$

$$\nabla_{e_5}e_1=-\frac{1}{2}\varepsilon_2\varepsilon_5e_2,\ \ \nabla_{e_5}e_2=\frac{1}{2}\varepsilon_1\varepsilon_5e_1,\ \ \nabla_{e_5}e_3=-\frac{1}{2}\varepsilon_4\varepsilon_5e_4,\ \ \nabla_{e_5}e_4=\frac{1}{2}\varepsilon_3\varepsilon_5e_3.$$

For each Lie algebra we consider two different almost paracontact metric structures and determine the class of the structure.

- Let (φ,ξ,η,g) be the quadruple such that
  $g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=-g(e_4,e_4)=-g(e_5,e_5)=1$,
  $\xi=e_1$, $\eta=e^1$,
  $\varphi(e_1)=0$, $\varphi(e_2)=e_4$, $\varphi(e_3)=e_5$, $\varphi(e_4)=e_2$, $\varphi(e_5)=e_3$.

$$(\varphi,\xi,\eta,g) \tag{3.1}$$

is an almost paracontact metric structure on $\mathfrak{g}_1$. Note that $\xi=e_1$ is not Killing and $\eta=e^1$ is the metric dual of $\xi=e_1$ such that $\eta(x)=g(x,e_1)$ for all vectors x. We evaluate the projections F^i and determine the class of the structure. The nonzero structure constants $F(e_i,e_j,e_k)=g((\nabla_{e_i}\varphi)(e_j),e_k)$ are given below.

$$F(e_2,e_1,e_3)=F(e_1,e_3,e_2)=-F(e_1,e_2,e_3)=-F(e_2,e_3,e_1)=1/2,$$

$$F(e_1,e_5,e_4)=F(e_5,e_1,e_4)=-F(e_1,e_4,e_5)=-F(e_5,e_4,e_1)=1/2,$$

$$F(e_3,e_5,e_2)=F(e_5,e_3,e_2)=-F(e_5,e_2,e_3)=-F(e_3,e_2,e_5)=1/2,$$

$$F(e_3,e_3,e_4)=-F(e_3,e_4,e_3)=F(e_5,e_5,e_4)=-F(e_5,e_4,e_5)=1/2.$$

By Theorem 3.4 in [3] the dimension of $\mathbb{G}_3$ is zero in 5-dimensions. Thus for each almost paracontact metric structure in 5 dimensions $F^3=0$. Since $(\nabla_{e_1}\varphi)(e_1)=0$, we have
$F(\xi,\xi,\varphi^2z)=F(e_1,e_1,\varphi^2z)=g((\nabla_{e_1}\varphi)(e_1),\varphi^2z)=0$ and $F^{12}=0$ from (2.10).
For any vector $u=\sum u_ie_i$, $\varphi^2(u)=u_2e_2+u_3e_3+u_4e_4+u_5e_5$ and from (2.9),

$$\begin{aligned}F^{11}(u,v,w) &= u_1F(e_1,v_2e_2+v_3e_3+v_4e_4+v_5e_5,w_2e_2+w_3e_3+w_4e_4+w_5e_5)\\ &= \frac{1}{2}u_1\{-v_2w_3+v_3w_2-v_4w_5+v_5w_4\}\\ &\neq 0.\end{aligned}$$

Now since

$$F(\varphi^2u,\varphi^2w,\xi)=-\frac{1}{2}\{u_2w_3+u_5w_4\}$$

and

$$F(\varphi u,\varphi w,\xi)=-\frac{1}{2}\{u_4w_5+u_3w_2\},$$

we have

$$F(\varphi^2u,\varphi^2w,\xi)+F(\varphi u,\varphi w,\xi)=F(\varphi^2w,\varphi^2u,\xi)+F(\varphi w,\varphi u,\xi).$$

Thus from (2.8), (2.7), (2.5), (2.6) respectively, we get

$$\begin{aligned} F^{10}(u,v,w) &= \frac{1}{4}y_1\{u_2w_3+u_3w_2+u_4w_5+u_5w_4\} \\ &\quad -\frac{1}{4}w_1\{u_2v_3+u_5v_4+u_4v_5+u_3v_2\} \\ &\neq 0, \end{aligned}$$

$F^9=0$, $F^5+F^8=0$ and thus $F^5=F^8=0$. Also since $F^{W_2}=F^5+F^6+F^7+F^8+F^9+F^{10}$ and $F^5=F^8=F^9=0$, we get

$$\begin{aligned} (F^6+F^7)(u,v,w) &= F^{W_2}(u,v,w)-F^{10}(u,v,w) \\ &= \frac{1}{4}u_2v_1w_3+\frac{1}{4}u_5v_1w_4-\frac{1}{4}u_4v_1w_5 \\ &\quad -\frac{1}{4}u_3v_1w_2-\frac{1}{4}u_2v_3w_1-\frac{1}{4}u_5v_4w_1 \\ &\quad +\frac{1}{4}u_4v_5w_1+\frac{1}{4}u_3v_2w_1. \end{aligned}$$

Let $T=F^6+F^7$. The nonzero structure constants of the tensor T are

$$T(e_2,e_1,e_3)=-T(e_2,e_3,e_1)=-T(e_3,e_1,e_2)=T(e_3,e_2,e_1)=1/4,$$

$$T(e_5,e_1,e_4)=-T(e_4,e_1,e_5)=-T(e_5,e_4,e_1)=T(e_4,e_5,e_1)=1/4.$$

We show that T satisfies the defining relation (2.3) of $\mathbb{G}_7$.

$$\begin{aligned} & -\eta(v)T(u,w,\xi)+\eta(w)T(u,v,\xi) \\ = & -v_1\{-\frac{1}{4}u_2w_3-\frac{1}{4}u_5w_4+\frac{1}{4}u_4w_5+\frac{1}{4}u_3w_2\} \\ & +w_1\{-\frac{1}{4}u_2v_3-\frac{1}{4}u_5v_4+\frac{1}{4}u_4v_5+\frac{1}{4}u_3v_2\} \\ = & \; T(u,v,w), \end{aligned}$$

$$-T(v,u,\xi)=-T(v,u,e_1)=\frac{1}{4}v_2u_3+\frac{1}{4}v_5u_4-\frac{1}{4}v_4u_5-\frac{1}{4}v_3u_2=T(u,v,\xi),$$

$$\begin{aligned} T(\varphi u,\varphi v,\xi) &= T(u_4e_2+u_5e_3+u_2e_4+u_3e_5,v_4e_2+v_5e_3+v_2e_4+v_3e_5,e_1) \\ &= -\frac{1}{4}u_4v_5-\frac{1}{4}u_3v_2+\frac{1}{4}u_2v_3+\frac{1}{4}u_5v_4=-T(u,v,\xi). \end{aligned}$$

According to the basis $\{f_1,f_2,f_3,f_4,f_5\}=\{e_2,e_3,e_4,e_5,\xi=e_1\}$, since $g_{ij}=diag(1,1,-1,-1,1)$ and $g^{ij}=diag(1,1,-1,-1,1)$, we have

$$\begin{aligned} \theta_T^*(\xi) &= \theta_T^*(e_1)=g^{ij}T(f_i,\varphi f_j,\xi) \\ &= T(f_1,\varphi f_1,e_1)+T(f_2,\varphi f_2,e_1)-T(f_3,\varphi f_3,e_1)-T(f_4,\varphi f_4,e_1) \\ &= T(e_2,\varphi e_2,e_1)+T(e_3,\varphi e_3,e_1)-T(e_4,\varphi e_4,e_1)-T(e_5,\varphi e_5,e_1) \\ &= T(e_2,e_4,e_1)+T(e_3,e_5,e_1)-T(e_4,e_2,e_1)-T(e_5,e_3,e_1) \\ &= 0. \end{aligned}$$

As a result $T=F^6+F^7\in\mathbb{G}_7$, in particular $F^6=0$ and $F^7\neq 0$. In addition,

$$\begin{aligned} F(\varphi^2u,\varphi^2v,\varphi^2w) &= -\frac{1}{2}u_3v_2w_5+\frac{1}{2}u_3v_3w_4-\frac{1}{2}u_3v_4w_3+\frac{1}{2}u_3v_5w_2 \\ &\quad -\frac{1}{2}u_5v_2w_3+\frac{1}{2}u_5v_3w_2-\frac{1}{2}u_5v_4w_5+\frac{1}{2}u_5v_5w_4 \\ &= F(\varphi u,\varphi^2v,\varphi w) \end{aligned}$$

together with (2.4) implies

$$\begin{aligned}
F^4(u,v,w) &= \frac{1}{2}\{F(\varphi^2 u,\varphi^2 u,\varphi^2 w)+F(\varphi u,\varphi^2 v,\varphi w)\}-F^3(u,v,w)\\
&= F(\varphi^2 u,\varphi^2 v,\varphi^2 w)\\
&= F^{W_1}(u,v,w)\\
&= -\frac{1}{2}u_3v_2w_5+\frac{1}{2}u_3v_3w_4-\frac{1}{2}u_3v_4w_3+\frac{1}{2}u_3v_5w_2\\
&\quad -\frac{1}{2}u_5v_2w_3+\frac{1}{2}u_5v_3w_2-\frac{1}{2}u_5v_4w_5+\frac{1}{2}u_5v_5w_4\\
&\neq 0.
\end{aligned}$$

Since $F^{W_1}=F^1+F^2+F^3+F^4=F^4$, we obtain $F^1=F^2=0$.
To sum up, since the only nonzero projections are F^4, F^7, F^{10} and F^{11}, the almost paracontact structure (3.1) belongs to the class $\mathbb{G}_4\oplus\mathbb{G}_7\oplus\mathbb{G}_{10}\oplus\mathbb{G}_{11}$.

- Consider now the almost paracontact metric structure

$$(\varphi,\xi,\eta,g) \tag{3.2}$$

defined by $g(e_1,e_1)=g(e_2,e_2)=-g(e_3,e_3)=-g(e_4,e_4)=g(e_5,e_5)$,
$\xi=e_5$, $\eta=e^5$,
$\varphi(e_1)=e_3$, $\varphi(e_2)=e_4$, $\varphi(e_3)=e_1$, $\varphi(e_4)=e_2$, $\varphi(e_5)=0$.
Note that $\xi=e_5$ is Killing [8], and thus, $F^6=F^7=F^{10}=F^{12}=0$ by Proposition 4.7 in [3]. The 1-form $\eta=e^5$ is the metric dual of $\xi=e_5$. Nonzero structure constants of F are

$$F(e_1,e_4,e_5)=-F(e_1,e_5,e_4)=-F(e_2,e_3,e_5)=F(e_2,e_5,e_3)=1/2,$$

$$-F(e_3,e_5,e_2)=F(e_3,e_2,e_5)=-F(e_4,e_1,e_5)=F(e_4,e_5,e_1)=1/2,$$

$$-F(e_5,e_1,e_4)=F(e_5,e_4,e_1)=F(e_5,e_2,e_3)=-F(e_5,e_3,e_2)=1.$$

Then by (2.9),

$$F^{11}(u,v,w)=u_5\{-v_1w_4+v_2w_3-v_3w_2+v_4w_1\}\neq 0.$$

Since

$$\begin{aligned}
F(\varphi^2u,\varphi^2w,\xi) &= \frac{1}{2}\{u_1w_4-u_2w_3+u_3w_2-u_4w_1\}\\
&= F(\varphi u,\varphi w,\xi),
\end{aligned}$$

from (2.7),

$$\begin{aligned}
F^9(u,v,w) &= -\frac{1}{2}v_5\{u_1w_4-u_2w_3+u_3w_2-u_4w_1\}\\
&\quad +\frac{1}{2}w_5\{u_1v_4-u_2v_3+u_3v_2-u_4v_1\}\\
&\neq 0.
\end{aligned}$$

Also since $F(\varphi^2u,\varphi^2w,\xi)=F(\varphi u,\varphi w,\xi)$, by (2.5) and (2.6) we have $F^5+F^8=0$ implying $F^5=F^8=0$. In addition, $F^{W_1}=F^1+F^2+F^3+F^4=0$ and thus $F^1=F^2=F^3=F^4=0$.
As a result the structure (3.2) is in $\mathbb{G}_9\oplus\mathbb{G}_{11}$.

Note that the almost paracontact structures (3.1) and (3.2) can also be considered as almost paracontact structures on other Lie algebras $\mathfrak{g}_i$, $i=1,2,\dots,6$. By calculating projections F^i for each structure, we determine the class of two different structures (3.1) and (3.2) on each Lie algebra. We omit calculations for other Lie algebras since they are similar to those for $\mathfrak{g}_1$. We only write the class of the structures.

The algebra $\mathfrak{g}_2$:

- Let (φ,ξ,η,g) be the almost paracontact structure (3.1) on $\mathfrak{g}_2$. The class of this structure is $\mathbb{G}_1\oplus\mathbb{G}_7\oplus\mathbb{G}_{10}\oplus\mathbb{G}_{11}$.
- (3.2) considered as an almost paracontact structure on $\mathfrak{g}_2$ is in $\mathbb{G}_4\oplus\mathbb{G}_5$.

The algebra $\mathfrak{g}_3$:

- The structure (3.1) on $\mathfrak{g}_3$ belongs to $\mathbb{G}_4\oplus\mathbb{G}_5\oplus\mathbb{G}_6\oplus\mathbb{G}_7\oplus\mathbb{G}_8\oplus\mathbb{G}_{10}\oplus\mathbb{G}_{11}$.
- The structure (3.2) on $\mathfrak{g}_3$ is of type $\mathbb{G}_1\oplus\mathbb{G}_2\oplus\mathbb{G}_4\oplus\mathbb{G}_8$.

The algebra $\mathfrak{g}_4$:

- (3.1) on $\mathfrak{g}_4$ is in $\mathbb{G}_5 \oplus \mathbb{G}_6 \oplus \mathbb{G}_7 \oplus \mathbb{G}_8 \oplus \mathbb{G}_{10} \oplus \mathbb{G}_{11}$.
- (3.2) on $\mathfrak{g}_4$ is in $\mathbb{G}_2 \oplus \mathbb{G}_4 \oplus \mathbb{G}_8 \oplus \mathbb{G}_9 \oplus \mathbb{G}_{11}$.

The algebra $\mathfrak{g}_5$:

- (3.1) on $\mathfrak{g}_5$ lies in $\mathbb{G}_{10}$.
- (3.2) on $\mathfrak{g}_5$ is in the class $\mathbb{G}_4 \oplus \mathbb{G}_5 \oplus \mathbb{G}_8$.

The algebra $\mathfrak{g}_6$:

- (3.1) on $\mathfrak{g}_6$ belongs to $\mathbb{G}_4 \oplus \mathbb{G}_7 \oplus \mathbb{G}_{10} \oplus \mathbb{G}_{11}$.
- (3.2) on $\mathfrak{g}_6$ is in $\mathbb{G}_1 \oplus \mathbb{G}_4 \oplus \mathbb{G}_8 \oplus \mathbb{G}_9 \oplus \mathbb{G}_{11}$.

Note that almost paracontact structures obtained here belong to the given direct sum properly, that is, they contain summand from each subclass, since corresponding projections are nonzero. Thus we give examples of almost paracontact metric structures which contain summands from several classes.

4. η-Einstein manifolds of 5-dimensions

It is known that paracontact structures exist only on $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$ for five dimensional nilpotent Lie algebras. In addition a vector field is Killing iff $\xi \in < e_5 >$, see [8].
We state

Proposition 4.1. *Let G be a connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}_i$, $i = 1, \ldots, 6$. Then a K-paracontact metric structure $(G, \varphi, \xi, \eta, g)$ is not η-Einstein.*

Proof. A five-dimensional almost paracontact metric manifold $(G, \varphi, \xi, \eta, g)$ is not an η-Einstein manifold, if the Lie algebra of the connected Lie group G is isomorphic to $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$ since there are no paracontact structures on $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$, paracontact structures exist only on $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$, see [8]. Thus it is enough to check the existence of η-Einstein manifolds only on $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$. Assume that $(G, \varphi, \xi, \eta, g)$ is η-Einstein, where G is a connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}_1$. Since ξ is Killing, $\xi = \xi_5 e_5$.
$\eta(\xi) = 1 = g(\xi, \xi) = g(\xi_5 e_5, \xi_5 e_5) = \xi_5^2 \varepsilon_5$ implies $\xi_5^2 = 1$ and $\varepsilon_5 = +1$.
From the equation (2.11), we have

$$Ric(\xi, \xi) = \xi_5^2 Ric(e_5, e_5) = Ric(e_5, e_5) = -4.$$

On the other hand, by direct calculation

$$R_{e_5 e_m} e_5 = \nabla_{[e_5, e_m]} e_5 - \nabla_{e_5}(\nabla_{e_m} e_5) + \nabla_{e_m}(\nabla_{e_5} e_5) = -\nabla_{e_5}(\nabla_{e_m} e_5)$$

and

$$R_{e_5 e_1} e_5 = -\nabla_{e_5}(\nabla_{e_1} e_5) = -\nabla_{e_5}(-\frac{1}{2}\varepsilon_2 \varepsilon_5 e_2) = \frac{1}{2}\varepsilon_2 \varepsilon_5 (\frac{1}{2}\varepsilon_1 \varepsilon_5 e_1) = \frac{1}{4}\varepsilon_1 \varepsilon_2 e_1,$$
$$R_{e_5 e_2} e_5 = \frac{1}{4}\varepsilon_1 \varepsilon_2 e_2, \quad R_{e_5 e_3} e_5 = \frac{1}{4}\varepsilon_3 \varepsilon_4 e_3, \quad R_{e_5 e_4} e_5 = \frac{1}{4}\varepsilon_3 \varepsilon_4 e_4$$

yields

$$\begin{aligned} Ric(e_5, e_5) &= \sum_{m=1}^{5} \varepsilon_m g(R_{e_5 e_m} e_5, e_m) \\ &= \varepsilon_1 g(\frac{1}{4}\varepsilon_1 \varepsilon_2 e_1, e_1) + \varepsilon_2 g(\frac{1}{4}\varepsilon_1 \varepsilon_2 e_2, e_2) + \varepsilon_3 g(\frac{1}{4}\varepsilon_3 \varepsilon_4 e_3, e_3) + \varepsilon_4 g(\frac{1}{4}\varepsilon_3 \varepsilon_4 e_4, e_4) \\ &= \frac{1}{2}\varepsilon_1 \varepsilon_2 + \frac{1}{2}\varepsilon_3 \varepsilon_4. \end{aligned}$$

Since $\varepsilon_i = \pm 1$, $Ric(e_5, e_5) = \frac{1}{2}\varepsilon_1 \varepsilon_2 + \frac{1}{2}\varepsilon_3 \varepsilon_4 \neq 4$. Thus $(G, \varphi, \xi, \eta, g)$ can not be η-Einstein. The proof is similar for $\mathfrak{g}_2$ and $\mathfrak{g}_3$. In $\mathfrak{g}_2$, $\xi = \xi_5 e_5$ and by (2.11), $Ric(\xi, \xi) = \xi_5^2 Ric(e_5, e_5) = Ric(e_5, e_5) = -4$. By direct calculation,

$$\begin{aligned} Ric(e_5, e_5) &= \sum_{m=1}^{5} \varepsilon_m g(R_{e_5 e_m} e_5, e_m) \\ &= \frac{1}{2}\varepsilon_1 \varepsilon_3 + \frac{1}{2}\varepsilon_2 \varepsilon_4 \\ &\neq 4. \end{aligned}$$

In $\mathfrak{g}_3$, $\xi = \xi_5 e_5$ and

$$\begin{aligned} Ric(e_5, e_5) &= \sum_{m=1}^{5} \varepsilon_m g(R_{e_5 e_m} e_5, e_m) \\ &= \frac{1}{2}\varepsilon_1 \varepsilon_4 + \frac{1}{2}\varepsilon_2 \varepsilon_3, \end{aligned}$$

which contradicts with (2.11). □

5. Conclusion

In this manuscript new examples of almost paracontact metric structures on some five dimensional nilpotent Lie algebras are given. These examples contain summands from several classes of almost paracontact metric structures. In addition, we show that a K-paracontact metric structure $(G, \varphi, \xi, \eta, g)$ on a connected Lie group G is not η-Einstein.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

This study was supported by Eskişehir Technical University Scientific Research Projects Commission under the Grant No: 22 ADP 011.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author’s contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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