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Research Article

An efficient numerical method for a singularly perturbed Volterra-Fredholm integro-differential equation

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Abstract

The scope of this study is to establish an effective approximation method for linear first order singularly perturbed Volterra-Fredholm integro-differential equations. The finite difference scheme is constructed on Shishkin mesh by using appropriate interpolating quadrature rules and exponential basis function. The recommended method is second order convergent in the discrete maximum norm. Numerical results illustrating the preciseness and computationally attractiveness of the proposed method are presented.

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1. Introduction

Volterra-Fredholm integro-differential equations (VFIDEs) have arisen in different areas of science and engineering. Population dynamics, oceanopraphy, fluid mechanics, financial mathematics, plasma physics, artificial neural networks, electromagnetic theory and biological processes are among these fields (see, e.g., [10, 29]).

In this paper, the following SPVFIDE is being analyzed:

$$Lu := L_1 u + \int_0^x K_1(x,s)u(s)ds + \lambda \int_0^l K_2(x,s)u(s)ds = f(x),\ x \in (0,l]\,, \tag{1.1}$$

$$u(0) = A, \tag{1.2}$$

where $L_1 u = \varepsilon u' + a(x)u$, $\varepsilon \in (0,1]$ is a perturbation parameter and λ is a real parameter. We presume that $f(x)$, $a(x) \geq \alpha > 0$, $(x \in [0,l])$, $K_1(x,s)$ and $K_2(x,s)((x,s) \in [0,l]^2)$ are the sufficiently smooth functions satisfying certain regularity conditions to be specified.

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Many papers have been written about different types of VFIDEs. Existence and uniqueness of the solution were discussed in [16, 22]. Furthermore, numerous analytical and numerical methods have been presented for solving VFIDEs. For instance, Adomian decomposition method, spectral collocation method, Legendre wavelet method, 2D Block-Pulse functions method, finite difference method, Legendre collocation method, Bernstein polynomials method, Homotopy perturbation method [6,7,14,23,30,31]. The above-mentioned studies were only related to the regular situations. (i.e. when the boundary layers are absent).

Singularly perturbed problems (SPPs) are mostly characterized by a small parameter ε that multiplies some or all of the higher-order terms in the equation, because boundary layers are generally found in their solutions. The approximation solutions of SPPs and their applications have been studied in many papers and books, one can refer to [20,26,32]. SPPs are widely used in vast number of applications in the field of population dynamics, fluid dynamics, heat transport problem, nanofluid, neurobiology, mathematical biology, viscoelasticity and simultaneous control systems etc. It is remarkable that, when a small ε parameter is multiplied with the derivative, the vast majority of classic numerical methods on uniform meshes fail to solve problems until the step-size of discretization is considerably reduced. So, as the ε perturbation parameter gets small, the truncation error happens boundless. To solve SPPs, the so-called the fitted finite difference method is used for many approaches (see, e.g., [11, 13, 25, 27, 28]).

In the literature, there have been studies in which different techniques were applied regarding SPVFIDEs. By using Richardson extrapolation, the order of convergence of numerical scheme for singularly perturbed Volterra integro-differential equation (SPVIDE) was improved in [24]. Delay forms of SPVIDEs were discretized in [21, 35]. Amiraliyev et al. recently constructed an exponential-difference scheme with an accuracy of $O\left(N^{-1}\right)$ for the first-order linear singularly perturbed Fredholm integro-differential equation (SPFIDE) on a uniform grid in [1], and finite difference scheme with an accuracy of $O\left(N^{-2}\ln N\right)$ on a Shishkin grid for the second-order linear SPFIDE in [12]. The first and the second order difference schemes were proposed in [4, 34]. In recent years, many authors have applied different methods such as homotopy analysis method, modified variational iteration method, Adomian decomposition method that is named Laplace discrete Adomian decomposition method, modified homotopy perturbation method to obtain approximate analytical solutions for Volterra, Fredholm, Volterra-Fredholm, fuzzy Volterra-Fredholm integro-differential equations in [8, 9, 15, 17–19].

Until now, numerical investigations of SPVFIDEs have not common yet. Solving of such kind of problems is so difficult. Because of existence of the perturbation parameter, traditional numerical methods do not give reliable results. Therefore, we need uniform and robust numerical techniques. The major contribution of this article is to present a robust and effective numerical technique for solving SPFVIDEs.

The rest of the paper is arranged as follows: We state asymptotic estimates of the exact solution and construct the finite difference scheme on a Shishkin mesh in Section 2. In Section 3, we present error approximations and convergence analysis. A numerical example is given in Section 4 which validate the theoretical analysis in practice the method is second order convergent. The paper ends with "Conclusion" section.

2. The mesh and difference scheme

First, we have remarked some analytical bounds that will be utilized subsequently during error analysis.

Lemma 2.1. *Assume that* $f, a \in C^2[0,l]$ *and* $\frac{\partial^m K_1}{\partial x^m} \in C[0,l]^2$, $\frac{\partial^m K_2}{\partial x^m} \in C[0,l]^2$, $(m = 0,1,2)$. *Moreover*

$$e^{\alpha^{-1}\overline{K}_1 l}\alpha^{-1}|\lambda| \max_{0\le x\le l}\int_0^l |K_2(x,s)|\, ds < 1.$$

Then the solution $u(x)$ *of the problem (1.1)-(1.2) satisfies the bounds*

$$\|u\|_\infty \le C, \tag{2.1}$$

$$\left|u^{(k)}(x)\right| \le C\left\{1 + \frac{1}{\varepsilon^k}e^{-\frac{\alpha x}{\varepsilon}}\right\}, \quad x \in [0,l], \quad k = 1,2, \tag{2.2}$$

where

$$\overline{K}_1 = \max_{[0,l]^2}|K_1(x,s)|.$$

Proof. The proof is done by similar approach as in [5, 12]. □

Now, we turn to establishment of the difference scheme. Let ω_N be any non-uniform mesh on $[0,l]$:

$$\omega_N = \{0 < x_1 < ... < x_N = l, \ \ h_i = x_i - x_{i-1}\}, \quad \overline{\omega}_N = \omega_N \cup \{x_0 = 0\}.$$

To any mesh function $v(x)$ identified on $\overline{\omega}_N$, we use

$$v_i = v(x_i), \quad v_{\overline{x},i} = \frac{v_i - v_{i-1}}{h_i}, \quad \|v\|_\infty \equiv \|v\|_{\infty,\overline{\omega}_N} := \max_{0\le i\le N}|v_i|.$$

We construct the difference scheme on Shishkin mesh to solve the problem (1.1)-(1.2). For an even number N, we divide each of the subintervals $[0,\sigma]$ and $[\sigma,l]$ into $\frac{N}{2}$ equidistant subintervals. The transition point σ is determined as

$$\sigma = \min\left\{\frac{l}{2}, \alpha^{-1}\varepsilon\ln N\right\}.$$

We use the notation h for the mesh width in $[0,\sigma]$ and the notation H for the width in $[\sigma,l]$. Hence, the mesh stepsizes are

$$h = \frac{2\sigma}{N}, \quad H = \frac{2(l-\sigma)}{N}.$$

x_i node points are specified as

$$\overline{\omega}_N = \begin{cases} x_i = ih, & i = 0,1,...,\frac{N}{2}; \quad x_i \in [0,\sigma]; \\ x_i = \sigma + \left(i - \frac{N}{2}\right)H, & i = \frac{N}{2}+1,...,N; \quad x_i \in [\sigma,l]. \end{cases}$$

We construct the numerical method using the identity

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i} L_1u\left(x\right)\varphi_i(x)dx \;+\; \chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left(\int\limits_{0}^{x} K_1\left(x,s\right)u\left(s\right)ds\right)\varphi_i(x)dx$$

$$+\;\chi_i^{-1}h_i^{-1}\lambda\int\limits_{x_{i-1}}^{x_i}\left(\int\limits_{0}^{l} K_2\left(x,s\right)u\left(s\right)ds\right)\varphi_i(x)dx$$

$$=\;\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx, \qquad 1\le i\le N, \tag{2.3}$$

with the basis functions

$$\varphi_i(x)=e^{-\frac{a_i(x_i-x)}{\varepsilon}}$$

and

$$\chi_i=h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\varphi_i(x)dx=\frac{1-e^{-a_i\rho_i}}{a_i\rho_i}, \qquad \rho_i=\frac{h_i}{\varepsilon}.$$

We note that the function $\varphi_i(x)$ is the solution of the problem

$$-\varepsilon\varphi'(x)+a_i\varphi(x)=0, \quad x_{i-1}<x<x_i, \qquad \varphi(x_i)=1.$$

Using the method of exact difference schemes [2, 34] (see also [33], pp. 207–214), for the first term in the left side of (2.3) we obtain

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[\varepsilon u'(x)+a(x)u(x)\right]\varphi_i(x)dx=\varepsilon\theta_i u_{\bar{x},i}+a_iu_i$$

$$+\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[a(x)-a(x_i)\right]u(x)\varphi_i(x)dx \tag{2.4}$$

with

$$\theta_i=\frac{a_i\rho_i}{1-e^{-a_i\rho_i}}e^{-a_i\rho_i}. \tag{2.5}$$

By Newton interpolation formula in respect to mesh points x_{i-1}, x_i we have

$$a(x)-a(x_i)=(x-x_i)a_{\bar{x},i}+\frac{a''(\xi_i(x))}{2}(x-x_{i-1})(x-x_i).$$

Therefore we get

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[a(x)-a(x_i)\right]u(x)\varphi_i(x)dx=a_{\bar{x},i}\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}(x-x_i)u(x)\varphi_i(x)dx$$

$$+\frac{1}{2}\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}a''(\xi_i(x))(x-x_{i-1})(x-x_i)u(x)\varphi_i(x)dx. \tag{2.6}$$

Also using

$$u(x)=u(x_i)-\int\limits_{x}^{x_i}u'(s)ds,$$

in the first term at the right side of (2.6), we have

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[a(x)-a(x_i)\right]u(x)\varphi_i(x)dx=\left(a_{\overline{x},i}\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}(x-x_i)\varphi_i(x)dx\right)u_i+R_i^{(1)},$$

where

$$R_i^{(1)}=\frac{1}{2}\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}a''(\xi_i(x))(x-x_{i-1})(x-x_i)u(x)\varphi_i(x)dx$$
$$-a_{\overline{x},i}\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}(x-x_i)\varphi_i(x)\Big(\int\limits_{x}^{x_i}u'(s)ds\Big)dx. \tag{2.7}$$

Simple calculation gives

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}(x-x_i)\varphi_i(x)dx=h_i\delta_i$$

with

$$\delta_i=\frac{e^{-a_i\rho_i}}{1-e^{-a_i\rho_i}}-\frac{1}{a_i\rho_i}. \tag{2.8}$$

It is easy to see that $-1\leq\delta_i\leq 0$.
After that, the identity (2.4) reduces to

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left[\varepsilon u'(x)+a(x)u(x)\right]\varphi_i(x)dx=\varepsilon\theta_i u_{\overline{x},i}+\bar{a}_i u_i+R_i^{(1)}, \tag{2.9}$$

where

$$\bar{a}_i=a_i+a_{\overline{x},i}h_i\delta_i \tag{2.10}$$

and δ_i is given by (2.8).
Analogously we derive

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}f(x)\varphi_i(x)dx=\bar{f}_i+R_i^{(2)}, \tag{2.11}$$

where

$$\bar{f}_i=f_i+f_{\overline{x},i}h_i\delta_i, \tag{2.12}$$

$$R_i^{(2)}=\frac{1}{2}\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}f''(\eta_i(x))(x-x_{i-1})(x-x_i)\varphi_i(x)dx. \tag{2.13}$$

For second term in the left side of (2.3), using the Taylor expansion

$$K_2(x,s)=K_2(x_i,s)+(x-x_i)\frac{\partial}{\partial x}K_2(x_i,s)+\frac{(x-x_i)^2}{2}\frac{\partial^2}{\partial x^2}K_2(\xi_i(x),s),$$

we get

$$\chi_i^{-1}h_i^{-1}\lambda\int\limits_{x_{i-1}}^{x_i}\varphi_i(x)\left(\int\limits_0^l K_2(x,s)u(s)ds\right)dx$$
$$=\lambda\int\limits_0^l K_2(x_i,s)u(s)ds+R_i^{(3)}+h_i\delta_i\lambda\int\limits_0^l\frac{\partial}{\partial x}K_2(x_i,s)u(s)ds$$
$$\equiv\lambda\int\limits_0^l \mathcal{K}_2(x_i,s)u(s)ds+R_i^{(3)}, \tag{2.14}$$

where

$$\mathcal{K}_2(x_i,s)=K_2(x_i,s)+h_i\delta_i\frac{\partial}{\partial x}K_2(x_i,s), \tag{2.15}$$

$$R_i^{(3)}=\frac{1}{2}\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}(x-x_i)^2\varphi_i(x)\left(\int\limits_0^l\frac{\partial^2}{\partial x^2}K_2(\xi_i(x),s)u(s)ds\right)dx. \tag{2.16}$$

Next, using the composite trapezoidal integration on $[0,l]$, for $\mathcal{K}_2(x_i,s)u(s)$, we have

$$\int\limits_0^l\mathcal{K}_2(x_i,s)u(s)ds=\sum_{j=0}^N\hbar_j\mathcal{K}_{2ij}u_j+R_i^{(4)}, \tag{2.17}$$

where

$$R_i^{(4)}=\frac{1}{2}\sum_{j=1}^N\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\frac{d^2}{d\xi^2}(\mathcal{K}_2(x_i,\xi)u(\xi))d\xi. \tag{2.18}$$

Eventually, for the fourth term in left side of (2.3), applying the interpolating quadrature rules in [3], it is found

$$\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}\left(\int\limits_0^x K_1\left(x,s\right)u\left(s\right)ds\right)\varphi_i(x)dx=\int\limits_0^{x_i}\mathcal{K}_1(x_i,s)u(s)ds+R_i^{(5)}, \tag{2.19}$$

where

$$\mathcal{K}_1(x_i,s)=K_1(x_i,s)+h_i\delta_i\frac{\partial}{\partial x}K_1(x_i,s), \tag{2.20}$$

$$R_i^{(5)}=\chi_i^{-1}h_i^{-1}\int\limits_{x_{i-1}}^{x_i}dx\varphi_i(x)\int\limits_{x_{i-1}}^{x_i}\frac{d^2}{d\xi^2}\left(\int\limits_0^{\xi}K_1(\xi,s)u(s)ds\right)T_1\left(\xi-s\right)d\xi, \tag{2.21}$$

$$T_s\left(\lambda\right)=\begin{cases}\frac{\lambda^s}{s!}, & \lambda\geq 0;\\ 0, & \lambda<0.\end{cases}$$

After, applying the composite trapezoidal rule on $[0,x_i]$, for $\mathcal{K}_1(x_i,s)u(s)$, we have

$$\int\limits_0^{x_i}\mathcal{K}_1(x_i,s)u(s)ds=\sum_{j=0}^i\hbar_j\mathcal{K}_{1ij}u_j+R_i^{(6)}, \tag{2.22}$$

where

$$R_i^{(6)}=\frac{1}{2}\sum_{j=1}^i\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\frac{d^2}{d\xi^2}(\mathcal{K}_1(x_i,\xi)u(\xi))d\xi. \tag{2.23}$$

Combining (2.9), (2.11), (2.14), (2.17), (2.19) and (2.22), we obtain following difference relation:

$$L_N u_i := \varepsilon\theta_i u_{\bar{x},i} + \bar{a}_i u_i + \sum_{j=0}^{i} \hbar_j \mathcal{K}_{1ij} u_j + \lambda \sum_{j=0}^{N} \hbar_j \mathcal{K}_{2ij} u_j = \bar{f}_i - R_i \tag{2.24}$$

with remainder term

$$R_i = \sum_{k=1}^{6} R_i^{(k)}, \tag{2.25}$$

where $R_i^{(k)}, (k = 1, 2, 3, 4, 5, 6)$ are defined by (2.7), (2.13), (2.16), (2.18), (2.21) and (2.23) respectively.

By neglecting the error term in (2.24) the following difference scheme is presented for the approximate solution:

$$L_N y_i := \varepsilon\theta_i y_{\bar{x},i} + \bar{a}_i y_i + \sum_{j=0}^{i} \hbar_j \mathcal{K}_{1ij} y_j + \lambda \sum_{j=0}^{N} \hbar_j \mathcal{K}_{2ij} y_j = \bar{f}_i,\ 1 \le i \le N, \tag{2.26}$$

$$y_0 = A, \tag{2.27}$$

where $\theta_i, \bar{a}_i, \bar{f}_i$, $\mathcal{K}_{1ij}$ and $\mathcal{K}_{2ij}$ are given by (2.5), (2.10), (2.12), (2.20) and (2.15) respectively.

3. Error estimates

Lemma 3.1. *Presume that* $f, a \in C^2[0,l]$ *and* $\dfrac{\partial^m K_1}{\partial x^m}, \dfrac{\partial^m K_2}{\partial x^m}, \dfrac{\partial^{m+1} K_1}{\partial x \partial s^m}, \dfrac{\partial^{m+1} K_2}{\partial x \partial s^m} \in C^2[0,l]^2$, $(m = 0, 1, 2)$. *Then the truncation error function* R_i *satisfies the estimate*

$$\|R\|_{\infty,\bar{\omega}_N} \le CN^{-2} \ln N. \tag{3.1}$$

Proof. Firstly estimate $R_i^{(1)}$. Since $a \in C^2[0,l]$, $|x - x_{i-1}| \le h_i$ and $|x - x_i| \le h_i$, then by using Lemma 2.1, it follows that

$$\left|R_i^{(1)}\right| \le Ch_i^2 + |a_{\bar{x},i}\delta_i|\, h_i \int_{x_{i-1}}^{x_i} |u'(x)|\, dx \le Ch_i \left(h_i + \int_{x_{i-1}}^{x_i} |u'(x)|\, dx \right). \tag{3.2}$$

For $R_i^{(2)}$, since $f \in C^2[0,l]$, analogously we have

$$\left|R_i^{(2)}\right| \le Ch_i^2. \tag{3.3}$$

Next for $R_i^{(3)}$, taking into account the boundedness of $\dfrac{\partial^2 K_1}{\partial x^2}$, from (2.16) it follows that

$$\left|R_i^{(3)}\right| \le Ch_i^2. \tag{3.4}$$

For $R_i^{(5)}$, taking into account the boundedness of $\dfrac{\partial^2 K_2}{\partial x^2}$, analogously we have

$$\left|R_i^{(5)}\right| \le Ch_i^2. \tag{3.5}$$

It remains to estimates $R_i^{(4)}$ and $R_i^{(6)}$. From (2.18), under the condition of Lemma 2.1, we have for this case

$$\left|R_i^{(4)}\right| \le C \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (x_j - \xi)(\xi - x_{j-1})(1 + |u'(\xi)| + |u''(\xi)|) d\xi. \tag{3.6}$$

The same evaluation is similarly obtained for $R_i^{(6)}$. From

$$|R_i| \leq \sum_{k=1}^{4} \left| R_i^{(k)} \right|,$$

after taking into consideration (3.2)-(3.6), therefore we get

$$|R_i| \leq C \left(h_i^2 + h_i \int\limits_{x_{i-1}}^{x_i} |u'(x)|\, dx + \sum_{j=1}^{N} \int\limits_{x_{j-1}}^{x_j} (x_j - \xi)(\xi - x_{j-1})(1 + |u'(\xi)| + |u''(\xi)|) d\xi \right).$$

This inequality by estimates (2.1) and (2.2) reduces to

$$|R_i| \leq C \Bigg(h_i^2 + h_i \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx + \sum_{j=1}^{N} h_j^3 + \sum_{j=1}^{N} \int\limits_{x_{j-1}}^{x_j} (x_j - \xi)(\xi - x_{j-1}) \frac{1}{\varepsilon} e^{\frac{-\alpha \xi}{\varepsilon}} d\xi$$

$$+ \sum_{j=1}^{N} \int\limits_{x_{j-1}}^{x_j} (x_j - \xi)(\xi - x_{j-1}) \frac{1}{\varepsilon^2} e^{\frac{-\alpha \xi}{\varepsilon}} d\xi \Bigg)$$

$$\leq C \left(h_i^2 + \sum_{j=1}^{N} h_j^3 + h_i \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx + \sum_{j=1}^{N} \int\limits_{x_{j-1}}^{x_j} (x_j - \xi)(\xi - x_{j-1}) \frac{1}{\varepsilon^2} e^{\frac{-\alpha \xi}{\varepsilon}} d\xi \right) (\varepsilon \leq 1) \quad (3.7)$$

Now we find a convergence error estimate for the right-side of (3.7) in our special piecewise-uniform mesh. First note that the following estimates are valid for each values of σ :

$$h_i^2 = \begin{cases} h^2 = \left(\frac{\sigma}{N/2} \right)^2 \leq CN^{-2}, & 1 \leq i \leq \frac{N}{2} \\ \\ H^2 = \left(\frac{l-\sigma}{N/2} \right)^2 \leq CN^{-2}, & \frac{N}{2} + 1 \leq i \leq N \end{cases} \quad (3.8)$$

and

$$\sum_{j=1}^{N} h_j^3 = \sum_{j=1}^{N/2} h^3 + \sum_{j=\frac{N}{2}+1}^{N} H^3 = \frac{N}{2} h^3 + \frac{N}{2} H^3 = 4\sigma^3 N^{-2} + 4(l - \sigma)^3 N^{-2} \leq CN^{-2}. \quad (3.9)$$

The case $\sigma = \frac{l}{2}$ can be analysed in the classical way. For this reason, we will consider only the case $\sigma = \alpha^{-1}\varepsilon \ln N < \frac{l}{2}$ and estimate the expression in the right-side in (3.7) on ω_N. The inequalities

$$h_i \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx \leq \frac{h^2}{\varepsilon} = \left(\frac{2\alpha^{-1}\varepsilon \ln N}{N} \right)^2 \frac{1}{\varepsilon} = 4\alpha^{-2} \varepsilon N^{-2} \ln^2 N$$

$$\leq \frac{l}{2} 4\alpha^{-1} N^{-2} \ln N \leq CN^{-2} \ln N, \qquad 1 \leq i \leq \frac{N}{2},$$

$$h_i \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx \leq H\alpha^{-1} \left(e^{\frac{-\alpha x_{i-1}}{\varepsilon}} - e^{\frac{-\alpha x_i}{\varepsilon}} \right) = H\alpha^{-1} e^{\frac{-\alpha x_{i-1}}{\varepsilon}} \left(1 - e^{\frac{-\alpha H}{\varepsilon}} \right)$$

$$\leq H\alpha^{-1} e^{\frac{-\alpha x_{i-1}}{\varepsilon}} \leq H\alpha^{-1} N^{-1} \leq CN^{-2}, \quad \frac{N}{2} \leq i \leq N,$$

imply that

$$h_i \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx \leq CN^{-2} \ln N, \qquad 1 \leq i \leq N. \quad (3.10)$$

Further, consider the splitting

$$\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi=\sum_{j=1}^{N/2}\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi$$
$$+\sum_{j=\frac{N}{2}+1}^{N}\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi,$$

for the first sum on the right side of the above equality, we have

$$\sum_{j=1}^{N/2}\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi=h^2\int\limits_0^{\sigma}\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi$$
$$\leq\frac{h^2}{\varepsilon}\alpha^{-1}\leq 2l\alpha^{-2}N^{-2}\ln N. \tag{3.11}$$

If a partial integration formula is applied for the integral term of second sum, then we have

$$\sum_{j=\frac{N}{2}+1}^{N}\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi=2\alpha^{-1}\sum_{j=\frac{N}{2}+1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-x-\frac{H}{2}\right)\frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}dx$$
$$\leq 2\alpha^{-1}H\int\limits_{\sigma}^{l}\frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}dx=2\alpha^{-2}H\left(e^{\frac{-\alpha\sigma}{\varepsilon}}-e^{\frac{-\alpha l}{\varepsilon}}\right)$$
$$\leq 2\alpha^{-2}HN^{-1}\leq CN^{-2}. \tag{3.12}$$

Thereby from (3.11) and (3.12), we get

$$\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi\leq CN^{-2}\ln N. \tag{3.13}$$

Thus for $\sigma=\alpha^{-1}\varepsilon\ln N$, by (3.10) and (3.13) it follows that

$$h_i\int\limits_{x_{i-1}}^{x_i}\frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}dx+\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi\leq CN^{-2}\ln N,\quad 1\leq i\leq N. \tag{3.14}$$

The estimates (3.8), (3.9), (3.14) along with (3.7) yield (3.1). □

We proceed to estimate the error of the approximate solution $z_i=y_i-u_i$, $(0\leq i\leq N)$. From (2.24) and (2.26) we have

$$L_N z_i=R_i, \tag{3.15}$$
$$z_i=0, \tag{3.16}$$

where the truncation error function R_i is given by (2.25).

By passing we note that since $a\in C^2[0,l]$ and $|\delta_i|\leq 1$, then exist a number $\bar{\alpha}$ such that for sufficiently large values of N will be $\bar{a}_i\geq\bar{\alpha}>0$.

Theorem 3.2. *Let a,f,K_1 and K_2 satisfy the assumptions from Lemma 3.1. Moreover*

$$(\bar{\alpha})^{-1}e^{2(\bar{\alpha})^{-1}\widetilde{\mathcal{K}}_1 l}|\lambda|\max_{1\leq i\leq N}\sum_{j=1}^{N}\hbar_j\left|\mathcal{K}_{2ij}\right|<1. \tag{3.17}$$

Then for the solution y of the difference problem (2.26)-(2.27) holds the error estimate

$$\|y-u\|_{\infty,\overline{\omega}_N} \leq CN^{-2}\ln N.$$

Proof. Applying Lemma 2.1 for the solution of (3.15)-(3.16) and Lemma 4.1 from [21], we get

$$\begin{aligned} |y_i| &\leq (\bar{\alpha})^{-1}\|R\|_{\infty,\omega_N} + (\bar{\alpha})^{-1}\widetilde{\mathcal{K}}_1\sum_{j=1}^{N}\hbar_j|y_j| + (\bar{\alpha})^{-1}|\lambda|\sum_{j=1}^{N}\hbar_j|\mathcal{K}_{2ij}||y_j| \\ &\leq \eta_N + (\bar{\alpha})^{-1}\widetilde{\mathcal{K}}_1\sum_{j=1}^{i}\hbar_j|y_j|, \end{aligned} \tag{3.18}$$

where

$$\eta_N = (\bar{\alpha})^{-1}\|R\|_{\infty,\omega_N} + |\lambda|(\bar{\alpha})^{-1}\max_{1\leq i\leq N}\sum_{j=1}^{N}\hbar_j|\mathcal{K}_{2ij}|\|z\|_{\infty,\bar{\omega}_N}$$

and

$$\widetilde{\mathcal{K}}_1 = \max_{[0,l]^2}|\mathcal{K}_1(x,s)|.$$

By the difference analogue of Gronwall's inequality to the relation (3.18), we obtain

$$|z_i| \leq \eta_N \exp\left((\bar{\alpha})^{-1}\widetilde{\mathcal{K}}_1\sum_{j=1}^{N}\frac{\hbar_j}{1-(\bar{\alpha})^{-1}\widetilde{\mathcal{K}}_1\hbar_j}\right), \qquad 1\leq i\leq N.$$

Thereby

$$\|z\|_{\infty,\omega_N} \leq \eta_N \exp\left(2(\bar{\alpha})^{-1}\widetilde{\mathcal{K}}_1 l\right),$$

for sufficiently large values of N and together with (3.17), we get

$$\|z\|_{\infty,\overline{\omega}_N} \leq C\|R\|_{\infty,\omega_N}.$$

This inequality together with (3.1) to get desired result. □

4. Numerical results

In this section, theoretical results are tested on two samples.

Example 4.1. We consider the following problem:

$$\varepsilon u'(x) + u(x) + \int_0^x \sin(x-s)\,u(s)\,ds + \int_0^1 su(s)ds = \sin(x), \quad 0 < x \leq 1,$$
$$u(0) = 1.$$

The exact solution to this problem is unknown. Hereby, we use the double mesh principle. We introduce the maximum point-wise errors and the computed ε-uniform maximum point-wise errors as

$$e_\varepsilon^N = \max_i |y_i^{\varepsilon,N} - \widetilde{y}_{2i}^{\varepsilon,2N}|_{\infty,\overline{\omega}_N},$$

$$e^N = \max_\varepsilon e_\varepsilon^N,$$

where $\widetilde{y}_i^{\varepsilon,2N}$ is the approximate solution of the respective method on the mesh

$$\widetilde{\omega}_{2N} = \{x_{i/2} : i = 0, 1, ..., 2N\}$$

with

$$x_{i+1/2} = \frac{x_i + x_{i+1}}{2} \quad \text{for} \quad i = 0, 1, ..., N-1.$$

We also describe the rate of convergence of the form

$$p^N = \frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

Table 1. Computed errors and convergence rates for the Example 4.1.

ε	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
2^0	0.067501	0.020348	0.005763	0.001502	0.000373
	1.73	1.82	1.94	2.01	
2^{-4}	0.073471	0.022457	0.006449	0.001716	0.000435
	1.71	1.80	1.91	1.98	
2^{-8}	0.073741	0.022854	0.006701	0.001808	0.000468
	1.69	1.77	1.89	1.95	
2^{-12}	0.072343	0.022734	0.006759	0.001849	0.000482
	1.67	1.75	1.87	1.94	
2^{-16}	0.074378	0.023536	0.007046	0.001941	0.000509
	1.66	1.74	1.86	1.93	
e^N	0.074378	0.023536	0.007046	0.001941	0.000509
p^N	1.66	1.74	1.86	1.93	

The values of ε and N for which we resolve Example 4.1 are $\varepsilon = 2^0, 2^{-4}, 2^{-8}, 2^{-12}, 2^{-16}$ and $N = 2^6, 2^7, 2^8, 2^9, 2^{10}$.

Example 4.2. Consider the another problem

$$\varepsilon u'(x) + u(x) + \int_0^x xu(s)\,ds + \frac{1}{10}\int_0^1 u(s)ds = -\frac{\varepsilon}{(1+x)^2} + \frac{1}{1+x} + x\varepsilon\left(1 - e^{-\frac{x}{\varepsilon}}\right)$$
$$+x\ln(1+x) + \frac{1}{10}\left(1 - e^{-\frac{1}{\varepsilon}} + \ln 2\right), \quad 0 < x \le 1,$$
$$u(0) = 2.$$

The exact solution of this problem is given by

$$u(x) = e^{-\frac{x}{\varepsilon}} + \frac{1}{1+x}.$$

We define the exact error e_ε^N as follows:

$$e_\varepsilon^N = \|y - u\|_{\infty,\overline{\omega}_N}$$

where y is the numerical approximation to u for various values of N, ε. The values of ε and N for which we solve Example 4.2 are $\varepsilon = 2^0, 2^{-4}, 2^{-8}, 2^{-12}, 2^{-16}$ and $N = 2^6, 2^7, 2^8, 2^9, 2^{10}$. The resulting values of e^N and p^N are listed in Table 2.

Table 2. Computed errors and convergence rates for the Example 4.2.

ε	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
2^0	0.050413	0.014278	0.003906	0.001011	0.000251
	1.82	1.87	1.95	2.01	
2^{-4}	0.060440	0.017118	0.004683	0.001212	0.000305
	1.82	1.87	1.95	1.99	
2^{-8}	0.067053	0.019123	0.005268	0.001373	0.000348
	1.81	1.86	1.94	1.98	
2^{-12}	0.070560	0.020404	0.005660	0.001475	0.000374
	1.79	1.85	1.94	1.98	
2^{-16}	0.073873	0.021362	0.005967	0.001566	0.000397
	1.79	1.84	1.93	1.98	
e^N	0.073873	0.021362	0.005967	0.001566	0.000397
p^N	1.79	1.84	1.93	1.98	

In Table 1 and Table 2, we observe that the ε-uniform rate of convergence p^N is monotonically increasing towards two, therefore in agreement with the theoretical rate given by Theorem 3.2.

5. Conclusion

A novel second order numerical approach for solving the first order VFIDE with boundary layer has been proposed. It has been done some stability estimates for the exact solution and its derivatives before giving the numerical method. To solve the problem numerically, exponential fitted finite difference approach on Shishkin mesh has been used. The obtained outcomes are shown in Table 1 and Table 2. It has been proven that the order of uniform convergence is almost $O\left(N^{-2}\ln N\right)$. The presented method can also be applied to partial and fractional types of integro-differential equations for future investigations.

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