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DOMAIN DECOMPOSITION METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL DIFFERENCE EQUATIONS WITH LAYER BEHAVIOR

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Abstract: *In this paper, a domain decomposition method has been presented for solving singularly perturbed differential difference equations with delay as well as advances whose solution exhibits boundary layer behavior. By introducing a terminal point, the original problem is divided into inner and outer region problems. An implicit terminal boundary condition at the terminal point has been determined. The outer region problem with the implicit boundary condition is solved and produces an explicit boundary condition for the inner region problem. Then, the modified inner region problem (using the stretching transformation) is solved as a two-point boundary value problem. Fourth order stable central difference method has been used to solve both the inner and outer region problems. The proposed method is iterative on the terminal point. To demonstrate the applicability of the method, some numerical examples have been solved for different values of the perturbation parameter, delay and advance parameters. The stability and convergence of the scheme has also investigated.*

Key words: Singular perturbation, Differential difference equations, Finite Differences, Terminal Boundary Condition, Boundary layer

1. Introduction

The boundary value problems for singularly perturbed differential difference equations with delay as well as advance are ubiquitous in the mathematical modeling of various practical phenomena in biology and physics, such as in variational problems in control theory, in describing the human pupil-light reflex, in a variety of models for physiological processes or diseases and first exit time problems in the modeling of the determination of expected time for the generation of action potential in nerve cells by random synaptic inputs in dendrites. These biological applications motivate the study of boundary value problems for singularly perturbed differential difference equations with delay as well as advance. For further study of neurophysiological and mathematical aspects of the above class of models, readers can refer to Stein [1, 2], Tuckwell [3-5], Tuckwell and Wan [6], Lange and Miura [7], Derstine and et al [8], Longtin and Milton [9], Wazewska-Czyzewska and Lasota [10], Mackey and Glass [11], etc.

To find the approximate solution of a class of singularly perturbed differential difference equation, one encounters two major difficulties, namely: i) due to presence of the small singular perturbation parameter which is multiplied to the highest order derivative term and ii) due to existence of delay and advance parameters in the argument of reaction terms. To deal with the first difficulty, there are two approaches, namely asymptotic and numerical. Here, we adopt the

numerical approach. When the singular perturbation parameter tends to zero, a breakdown occurs and the solution of the singularly perturbed problem often behave analytically quite differently from a solution of the original equation in the narrow region of the domain. The solution changes rapidly and form boundary or transition layers in these narrow regions. Owing to this, many numerical methods have been developed to solve singularly perturbed ODEs with delay and advance. Lange and Miura gave a series of papers (see [12-17]) investigating different classes of BVPs of singularly perturbed differential difference equations by extending the method of matched asymptotic expansions developed for ODEs. First-order numerical algorithms based on finite difference schemes are found in Sharma [18], fitted methods based on finite difference method Patidar and Sharma [19] and Kadalbajoo et al. [20], parameter uniform numerical method Kumar and Kadalbajoo [21]. To tackle with the second difficulty, we use Taylor series approximation for the terms containing delay as well as advance provided the delay and advance parameter are sufficiently small.
Due to the singular behavior of the solution of the problem in the inner regions, the classical numerical schemes are found to be inadequate to approximate the solution of the singularly perturbed problems. To get rid of this problem, here in the present paper, we introduce a terminal boundary point and decompose the domain into inner and outer regions to treat the original problem as inner region and outer region problems separately. A terminal boundary condition in the implicit form is determined at the terminal point and then, the outer region problem with the implicit boundary condition is solved as a two-point boundary-value problem. From the solution of the outer region problem, an explicit terminal boundary condition is obtained. The inner region problem is modified and solved as a two- point boundary value problem using the obtained explicit terminal boundary condition. Finally, we combine the solutions of both the inner region and outer region problems to get the approximate solution of the original problem. The present method is iterative on the terminal point. We repeat the process (numerical scheme) for various choices of the terminal point, until the solution profiles do not differ materially from iteration to iteration.

2. Description of the Method

Consider singularly perturbed differential equation with small delay as well as advance of the form:

$$\varepsilon y''(x) + a(x)y'(x) + \alpha(x)y(x-\delta) + \omega(x)y(x) + \beta(x)y(x+\eta) = f(x) \tag{1}$$

$\forall x \in (0,1)$ and subject to the interval and boundary conditions

$$y(x) = \phi(x), \quad \text{on} \quad -\delta \le x \le 0 \tag{2}$$

$$y(x) = \gamma(x), \quad on \quad 1 \le x \le 1+\eta \tag{3}$$

Where $a(x), \alpha(x), \beta(x), \omega(x), f(x), \phi(x), and\ \gamma(x)$ are bounded and continuously differentiable functions on (0, 1), $0 < \varepsilon << 1$ is the singular perturbation parameter; and $0 < \delta = o(\varepsilon)$ and $0 < \eta = o(\varepsilon)$ are the delay and the advance parameters respectively. When the shifts are zero (i.e., $\delta = 0$, $\eta = 0$), the solution of the corresponding ordinary differential equation exhibits layer behavior or turning point behavior depending on the coefficient of the convection

term, i.e., if *a(x)* does not change the sign or changes the sign on the underlying interval. The layer behavior of the problem under consideration is maintained only for $\delta \neq 0$ and $\eta \neq 0$, but sufficiently small. In general, the solution of problem (1)-(3) exhibits boundary layer behavior at one end of the interval [0, 1] depending on the sign of $a(x)-\delta\alpha(x)+\eta\beta(x)$.
In this paper, we consider the problems whose solution exhibits the layer behavior on the left side of the interval. We assume that $\alpha(x)+\beta(x)+\omega(x)\leq 0$, $a(x)-\delta\alpha(x)+\eta\beta(x)\geq M>0$ throughout the interval [0, 1], where M is some constant. Under these assumptions, Eq. (1) has a unique solution $y(x)$ which in general, exhibits a boundary layer on the left side of the underlying interval.
By using Taylor series expansion in the neighborhood of the point *x,* we have

$$y(x-\delta)\approx y(x)-\delta y'(x) \tag{4}$$
$$y(x+\eta)\approx y(x)+\eta y'(x) \tag{5}$$

Using Eqs. (4) and (5) in Eq.(1) we get an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\varepsilon y''(x)+q(x)y'(x)+r(x)y(x)=f(x) \tag{6}$$

with boundary conditions

$$y(0)=\phi(0)=\phi_0 \tag{7}$$
$$y(1)=\gamma(1)=\gamma_1 \tag{8}$$

where $q(x)=a(x)-\delta\alpha(x)+\eta\beta(x)$ (9)

$$r(x)=\alpha(x)+\beta(x)+\omega(x) \tag{10}$$

The transition from Eq. (1) to Eq. (6) is admitted, because of the condition that $0<\delta<<1$ and $0<\eta<<1$ are sufficiently small. Further details on the validity of this transition can be found in Elsgolt's and Norkin [22]. This replacement is significant from the computational point of view. Thus, the solution of Eq. (6) will provide a good approximation to the solution of Eq. (1).
Now, we divide the problem into two: inner region and outer region problems.
Let x_p ($0<x_p<<1$) be the terminal point or width or thickness of the boundary layer (inner region), then the inner and outer region problems are defined on $0\leq x\leq x_p$ and $x_p\leq x\leq 1$ respectively.
By using Taylor's expansion, we have

$$y(x-x_p)\approx y(x)-x_p y'(x)+\frac{x_p^{\,2}}{2}y''(x) \tag{11}$$

Using Eq. (11) in to Eq. (6) and evaluating at $x=x_p$ we get

$$c_1 y'(x_p)+c_2 y(x_p)=c_3 \tag{12}$$

where $c_1=x_p(2\varepsilon+x_p q(x_p))$ (13)

$$c_2=x_p^{\,2}r(x_p)-2\varepsilon \tag{14}$$

$$c_3 = x_p^2 f(x_p) - 2\varepsilon y(0) \tag{15}$$

which is in implicit form and is taken as the terminal boundary condition at $x = x_p$ (the terminal point). Using the terminal boundary condition in Eq. (12), which is in implicit form, we solve the outer region problem as a two point boundary value problem

$$\varepsilon y''(x) + q(x)y'(x) + r(x)y(x) = f(x) \ , \quad x_p \le x \le 1 \tag{16}$$

with boundary conditions

$$c_1 y'(x_p) + c_2 y(x_p) = c_3 \tag{17}$$

and $y(1) = \gamma(1)$ (18)

Solving this two point boundary value problem, we get the solution $y(x)$ over $[x_p, 1]$. From the solution $y(x)$ of the outer region problem in Eq. (16)-(18) on the interval $x_p \le x \le 1$, we get the value of $y(x_p)$ which is the explicit terminal boundary condition and let us denote it by $y(x_p) = \zeta$.

Since the terminal point x_p is common to both the inner and outer regions, we can formulate the asymptotically equivalent inner region problem as a two-point boundary-value problem

$$\varepsilon y''(x) + q(x)y'(x) + r(x)y(x) = f(x) \ , \quad 0 \le x \le x_p \tag{19}$$

with boundary conditions

$$y(0) = \phi(0) \tag{20}$$

$$y(x_p) = \zeta \tag{21}$$

In order to solve the inner region problem in Eq. (19)-(21), we choose the transformation

$$t = \frac{x}{\varepsilon} \tag{22}$$

Then, using Eq. (22) we can transform the functions in Eq. (19) into

$$y(x) = y(t\varepsilon) = Y(t) \tag{23}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{24}$$

$$y''(x) = \frac{y''(t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon^2} \tag{25}$$

$$q(x) = q(t\varepsilon) = Q(t) \tag{26}$$

$$r(x) = r(t\varepsilon) = R(t) \tag{27}$$

$$f(x) = f(t\varepsilon) = F(t) \tag{28}$$

Substituting Eq. (23)-(28) in Eq. (19) we get the new inner region problem of the form:

$$Y''(t) + Q(t)Y'(t) + \varepsilon R(t)Y(t) = \varepsilon F(t),\ 0 \le t \le t_p \tag{29}$$

with $Y(0) = \phi(0)$ (30)

and $Y(t_p) = y(x_p) = \zeta$ (31)

where $t_p = \dfrac{x_p}{\varepsilon}$. Solving this new inner region problem in Eq. (29)-(31), we obtain the solutions over the interval $0 \le t \le t_p$.

To solve the two-point boundary value problems given in Eq. (16)-(18) (outer region problem) and Eq. (29)-(31) (inner region problem), we make use of fourth order stable central difference method (Choo and Schultz [23]). In fact, any standard analytical or numerical method can be used. Finally, we combine the solutions of both the inner region defined on $0 \le x \le x_p$ and outer region defined on $x_p \le x \le 1$ problems to get the approximate solution of the original problem in Eq. (1)-(3) over the interval $0 \le x \le 1$.

We repeat the process (numerical scheme) for various choices of x_p (the terminal point), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criterion, namely

$$\left| y^{m+1}(x) - y^{m}(x) \right| \le \sigma \quad 0 \le x \le x_p \tag{32}$$

Where $y^m(x)$ = the solution for the m^{th} iterate of x_p and σ = the prescribed tolerance bound.

To set up the difference equation of the outer region problem in Eq. (16)-(18) we divide $[x_p,1]$ into N equal parts, each of length h, $x_p = x_0 < x_1 < x_2 < ... < x_N = 1$. Then, we have $x_i = x_p + ih$; $i = 0,1,2,..., N$. For simplicity let $q(x_i) = q_i, r(x_i) = r_i, f(x_i) = f_i, y(x_p) = y_0$, $y(x_i) = y_i, y(x_i + h) = y_{i+1}, y(x_i - h) = y_{i-1}, y'(x_i) = y_i', y''(x_i) = y_i''$, etc. By Taylor expansion, we obtain the following central difference formulas for y'' and y' at x assuming that y has continuous fourth derivatives on $[0,1]$.

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi) + T_1 \tag{33}$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y'''(\varsigma) + T_2 \tag{34}$$

Where $T_1 = {-2h^4 y^{(6)}(\xi)}/{6!}$ and $T_2 = {-h^4 y^{(5)}(\varsigma)}/{5!}$ for $\xi, \varsigma \in [x_i - h, x_i + h]$.

Substituting Eq. (33) and (34) into Eq. (16) we can write the central difference approximation of Eq. (16) in the form that includes all the $O(h^2)$ error terms as follows:

$$\left[\frac{\varepsilon}{h^2} - \frac{q_i}{2h}\right] y_{i-1} + \left[r_i - \frac{2\varepsilon}{h^2}\right] y_i + \left[\frac{\varepsilon}{h^2} + \frac{q_i}{2h}\right] y_{i+1} - \frac{h^2}{12}\left[2q_i y_i''' + \varepsilon y_i^{(4)}\right] + T = f_i \tag{35}$$

Where $T = \varepsilon T_1 + q_i T_2$. Now, from (6) we have

$$\varepsilon y_i'' = -q_i y_i' - r_i y_i + f_i \tag{36}$$

Differentiating both sides of Eq. (36) we get

$$\varepsilon y_i''' = -\left(q_i y_i'' + q_i' y_i' + r_i' y_i + r_i y_i'\right) + f_i \tag{37}$$

$$y_i''' = -\frac{1}{\varepsilon}\left[q_i y_i'' + \left(q_i' + r_i\right) y_i' + r_i' y_i\right] + \frac{f_i}{\varepsilon} \tag{38}$$

Differentiating both sides of Eq. (37) again we have

$$\varepsilon y_i^{(4)} = -\left[q_i y_i''' + \left(2q_i' + r_i\right) y_i'' + \left(q_i'' + 2r_i'\right) y_i' + r_i'' y_i\right] + f_i'' \tag{39}$$

By making use of Eq. (38) and (39) into Eq. (35) for y_i''' and $y_i^{(4)}$ we obtain

$$\begin{aligned}&\left(\frac{\varepsilon}{h^2} - \frac{q_i}{2h}\right) y_{i-1} + \left(r_i - \frac{2\varepsilon}{h^2}\right) y_i + \left(\frac{\varepsilon}{h^2} + \frac{q_i}{2h}\right) y_{i+1} + \frac{h^2}{12}\left(\frac{q_i^{\,2}}{\varepsilon} + 2q_i' + r_i\right) y_i'' \\ &+ \frac{h^2}{12}\left(\frac{q_i q_i'}{\varepsilon} + \frac{q_i r_i}{\varepsilon} + q_i'' + 2r_i'\right) y_i' + \frac{h^2}{12}\left(\frac{q_i r_i'}{\varepsilon} + r_i''\right) y_i + T = f_i + \frac{h^2}{12}\left(\frac{q_i f_i'}{\varepsilon} + f_i''\right)\end{aligned} \tag{40}$$

Approximating the converted error term, this has a stabilizing effect, in Eq. (40) by using the central difference formulas given in Eqs. (33) and (34) for y_i'' and y_i' we obtain

$$\begin{aligned}&\left(\frac{\varepsilon}{h^2} - \frac{q_i}{2h}\right) y_{i-1} + \left(r_i - \frac{2\varepsilon}{h^2}\right) y_i + \left(\frac{\varepsilon}{h^2} + \frac{q_i}{2h}\right) y_{i+1} + \frac{1}{12}\left(\frac{q_i^{\,2}}{\varepsilon} + 2q_i' + r_i\right) y_{i+1} \\ &- \frac{1}{6}\left(\frac{q_i^{\,2}}{\varepsilon} + 2q_i' + r_i\right) y_i + \frac{1}{12}\left(\frac{q_i^{\,2}}{\varepsilon} + 2q_i' + r_i\right) y_{i-1} + \frac{h}{24}\left(\frac{q_i q_i'}{\varepsilon} + \frac{q_i r_i}{\varepsilon} + q_i'' + 2r_i'\right) y_{i+1} \\ &- \frac{h}{24}\left(\frac{q_i q_i'}{\varepsilon} + \frac{q_i r_i}{\varepsilon} + q_i'' + 2r_i'\right) y_{i-1} + \frac{h^2}{12}\left(\frac{q_i r_i'}{\varepsilon} + r_i''\right) y_i = f_i + \frac{h^2}{12}\left(\frac{q_i f_i'}{\varepsilon} + f_i''\right) + \widetilde{T}\end{aligned} \tag{41}$$

where $\widetilde{T} = \left(\frac{q_i^{\,2}}{\varepsilon} + 2q_i' + r_i\right)\frac{h^4}{144} y_i^{(4)} + \left(\frac{q_i q_i'}{\varepsilon} + \frac{q_i r_i}{\varepsilon} + q_i'' + 2r_i'\right)\frac{h^4}{72} y_i''' - T$ is the truncation error and $T = \varepsilon T_1 + q_i T_2 = O\left(h^4\right)$. Rearranging Eq. (41) we obtain the three term recurrence relation of the form

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i,\ i = 0,1,\ldots,N-1 \tag{42}$$

Where

$$
\begin{aligned}
E_i &= \frac{\varepsilon}{h^2} - \frac{q_i}{2h} + \frac{1}{12}\left(\frac{q_i^2}{\varepsilon} + 2q_i' + r_i\right) - \frac{h}{24}\left(\frac{q_i q_i'}{\varepsilon} + \frac{q_i r_i}{\varepsilon} + q_i'' + 2r_i'\right) \\
F_i &= \frac{2\varepsilon}{h^2} - r_i + \frac{1}{6}\left(\frac{q_i^2}{\varepsilon} + 2q_i' + r_i\right) - \frac{h^2}{12}\left(\frac{q_i r_i'}{\varepsilon} + r_i''\right) \\
G_i &= \frac{\varepsilon}{h^2} + \frac{q_i}{2h} + \frac{1}{12}\left(\frac{q_i^2}{\varepsilon} + 2q_i' + r_i\right) + \frac{h}{24}\left(\frac{q_i q_i'}{\varepsilon} + \frac{q_i r_i}{\varepsilon} + q_i'' + 2r_i'\right) \\
H_i &= f_i + \frac{h^2}{12}\left(\frac{q_i f_i'}{\varepsilon} + f_i''\right)
\end{aligned}
\tag{43}
$$

Equation (42) gives a system of N equations with $N+1$ unknowns y_{-1} to y_{N-1}. To eliminate the unknown y_{-1}, we make use of the implicit boundary condition in Eq. (17) and then by employing the second order central difference approximation in it, we get

$$
y_{-1} = \frac{2hc_2}{c_1} y_0 + y_1 - \frac{2hc_3}{c_1} \tag{44}
$$

Where c_1, c_2 and c_3 are defined in Eq. (13)-(15). Making use of Eq. (44) in the first equation of the recurrence relation in Eq. (42) at $i = 0$, we get

$$
-(F_0 + \frac{2hc_2}{c_1} E_0) y_0 + (E_0 + G_0) y_1 = H_0 + \frac{2hc_3}{c_1} E_0 \tag{45}
$$

Now, Eq. (42) and Eq. (45) give N by N tri-diagonal system which can be easily solved by using Thomas Algorithm.

Similarly, to set up the difference equation for the inner region problem in Eq. (29)-(31) we divide the interval $0 \le t \le t_p$ in to N subintervals of equal mesh length $h = \frac{t_p - 0}{N}$ with mesh points $0 = t_0 < t_1 < t_2, \dots, < t_N = t_p$. Following the same procedures/steps in Eq. (33)-(41), we obtain the three term recurrence relation

$$
\tilde{E}_i y_{i-1} - \tilde{F}_i y_i + \tilde{G}_i y_{i+1} = \tilde{H}_i,\ i = 1, 2, \dots, N-1 \tag{46}
$$

Where

$$
\begin{aligned}
\tilde{E}_i &= \frac{1}{h^2} + \frac{1}{12}\left(Q_i^2 + 2Q_i' + \varepsilon R_i\right) - \frac{Q_i}{2h} - \frac{h}{24}\left(Q_i Q_i' + \varepsilon Q_i R_i + Q_i'' + 2\varepsilon R_i'\right) \\
\tilde{F}_i &= \frac{2}{h^2} + \frac{1}{6}\left(Q_i^2 + 2Q_i' + \varepsilon R_i\right) - \varepsilon R_i - \frac{h^2}{12}\left(\varepsilon Q_i R_i' + \varepsilon R_i''\right) \\
\tilde{G}_i &= \frac{1}{h^2} + \frac{1}{12}\left(Q_i^2 + 2Q_i' + \varepsilon R_i\right) + \frac{Q_i}{2h} + \frac{h}{24}\left(Q_i Q_i' + \varepsilon Q_i R_i + Q_i'' + 2\varepsilon R_i'\right) \\
\tilde{H}_i &= \varepsilon F_i + \frac{\varepsilon h^2}{12}\left(Q_i F_i' + F_i''\right)
\end{aligned}
\tag{47}
$$

To solve the tri diagonal system in Eq. (46), we used Thomas Algorithm

3. Lower Bound for the Terminal Point t_p

To gain further insight to the choice of t_p, terminal point of the boundary layer region which is not unique, consider the problem in Eq. (1)-(3) and choose t_p such that $t_p << 1/\varepsilon$.
By making use of the stretching transformation Eq. (22) into Eq. (6)-(8) and taking the limit as $\varepsilon \to 0$, we get

$$Y''(t) + q(0)Y'(t) = 0 \tag{48}$$

Where $Y(t) = y(t\varepsilon)$ and the boundary conditions

$$Y(0) = \phi(0) \quad \text{and} \quad Y(t_p) = \zeta \tag{49}$$

Now solving the two point boundary value problem in Eq. (48)-(49) analytically, we get the solution

$$Y(t) = \frac{\zeta - \phi(0)}{1 - e^m} + \frac{\phi(0) - \zeta}{1 - e^m} e^{mt} \tag{50}$$

where $m = -q(0) = -(a(0) - \delta\alpha(0) + \eta\beta(0))$
As suggested by Hsiao and Jordan [24] and Lorenz [25], t_p can be determined by taking the inequality

$$e^{mt_p} < \varepsilon \tag{51}$$

Taking ln of both sides of Eq. (51) and rearranging we get

$$t_p \geq -\frac{\ln \varepsilon}{a(0) - \delta\alpha(0) + \eta\beta(0)} \tag{52}$$

For $\varepsilon = 10^{-\mu}$, we can get the crude estimate for the lower bound of t_p as

$$t_p \geq \frac{\mu}{a(0) - \delta\alpha(0) + \eta\beta(0)} \ln(10) \approx \frac{2.3\mu}{a(0) - \delta\alpha(0) + \eta\beta(0)}. \tag{53}$$

4. Stability and Convergence Analysis

Remark: Here we shall use the definition of the stability of the difference operator given in Keller [26].
Definition: The linear difference operator L_h is stable if for sufficiently small h, there exists a constant k, independent of h, such that

$$\left|v_j\right| \le k\left\{\max\left(\left|v_0\right|,\left|v_N\right|\right)+\max_{1\le i\le N-1}\left|L_h v_i\right|\right\}, \quad j=0,\ 1,\ldots,N$$

for any mesh function $\left\{v_j\right\}_{j=0}^{N}$.

Theorem 1: Under the assumptions $r(x)\equiv -\theta<0$ for positive constant θ, $\frac{q_i^2}{\varepsilon}+2q_i'+r_i\ge 0$ and

$$h<\min\frac{2\left(\frac{q_i^2}{\varepsilon}+2q_i'+r_i\right)}{\left|\frac{q_i}{\varepsilon}\left(q_i'+r_i\right)+q_i''\right|},$$

$i=1,\ 2,\ldots,\ N-1$, the difference operator defined on Eq. (42) is stable with $k=\max\left\{1,\frac{1}{\theta}\right\}$.

Proof: Let $L_h(.)$ denote the difference operator on left hand side of Eq. (42) and w_i be any mesh function satisfying

$$L_h(w_i)=H_i \tag{54}$$

If $\max\left|w_i\right|$ occurs for $i=0$ or $i=N$, then the definition holds trivially, since $k\ge 1$. So assume that $\max\left|w_i\right|$ occurs for $i=1,\ 2,\ \ldots,N-1$.

Under the given assumptions, $E_i>0$, $G_i>0$, $F_i\ge E_i+G_i$ and $\left|E_i\right|\le\left|G_i\right|$. This implies that the tri-diagonal system in Eq. (42) is diagonally dominant and its solution exists and is unique (Greenspan and Casulli [27]).

Then by rearranging the difference Eq. (42) and using the non negativity of the coefficients, we have

$$\begin{aligned}&F_i\left|w_i\right|\le E_i\left|w_{i-1}\right|+G_i\left|w_{i+1}\right|+\left|H_i\right|\\ \Rightarrow\ &F_i\left|w_i\right|\le E_i\left|w_{i-1}\right|+G_i\left|w_{i+1}\right|+\left|L_h w_i\right|\end{aligned} \tag{55}$$

Since $r(x)\equiv-\theta$, a constant, by assumption, $r'(x)=0$. Thus, from (43) we have

$$F_i=\frac{2\varepsilon}{h^2}-r_i+\frac{1}{6}\left(\frac{q_i^2}{\varepsilon}+2q_i'+r_i\right)=\frac{2\varepsilon}{h^2}+\frac{1}{6}\left(\frac{q_i^2}{\varepsilon}+2q_i'+r_i\right)+\theta$$

Now using the fact,

$$E_i+G_i=\frac{2\varepsilon}{h^2}+\frac{1}{6}\left(\frac{q_i^2}{\varepsilon}+2q_i'+r_i\right)$$ and equation Eq. (55) we get

$$\begin{aligned}\left(\frac{2\varepsilon}{h^2}+\frac{1}{6}\left(\frac{q_i^2}{\varepsilon}+2q_i'+r_i\right)+\theta\right)\left|w_i\right|&\le E_i\left|w_{i-1}\right|+G_i\left|w_{i+1}\right|+\left|L_h w_i\right|\\ &\le\left(E_i+G_i\right)\max_{1\le k\le N-1}\left|w_k\right|+\max_{1\le k\le N-1}\left|L_h w_k\right|\end{aligned} \tag{56}$$

Since the inequality in Eq. (56) holds for every i, it follows that

$$\left(\frac{2\varepsilon}{h^2}+\frac{1}{6}\left(\frac{q_i^{\ 2}}{\varepsilon}+2q_i'+r_i\right)+\theta\right)\max_{1\le i\le N-1}|w_i| \le \left(\frac{2\varepsilon}{h^2}+\frac{1}{6}\left(\frac{q_i^{\ 2}}{\varepsilon}+2q_i'+r_i\right)\right)\max_{1\le k\le N-1}|w_k|+\max_{1\le k\le N-1}|L_h w_k|$$

This implies

$$\theta\max_{1\le i\le N-1}|w_i| \le \max_{1\le k\le N-1}|L_h w_k|$$

Hence,

$$\max_{1\le i\le N-1}|w_i| \le \frac{1}{\theta}\max_{1\le k\le N-1}|L_h w_k| \le \frac{1}{\theta}\left\{\max(|w_0|,|w_N|)+\max_{1\le k\le N-1}|L_h w_k|\right\}$$

Therefore,

$$|w_i| \le k\left\{\max(|w_0|,|w_N|)+\max_{1\le k\le N-1}|L_h w_k|\right\}$$

Hence, L_h is stable and this implies that the solutions to the system of the difference equation Eq. (42) are uniformly bounded, independent of mesh size h and the parameter ε. Hence the scheme is stable for all step sizes.

Corollary: Under the conditions for theorem 1, the error $e_i = y(x_i) - y_i$ between the solution $y(x)$ of the continuous problem and the solution y_i of the discrete problem, with the boundary conditions satisfies the estimate

$$|e_i| \le k\max_{1\le i\le N-1}|\tau_i| \tag{57}$$

where

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}}\left\{\left(\frac{q_i q_i'}{\varepsilon}+\frac{q_i r_i}{\varepsilon}+q_i''+2r_i'\right)\frac{h^4}{72}|y_i'''|\right\}+\max_{x_{i-1}\le x\le x_{i+1}}\left\{\left(\frac{q_i^{\ 2}}{\varepsilon}+2q_i'+r_i\right)\frac{h^4}{144}|y^{(4)}{}_i|\right\}$$
$$+\max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{h^4 q_i}{120}|y^{(5)}{}_i|\right\}+\max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{\varepsilon h^4}{360}|y^{(6)}{}_i|\right\}$$

is the truncation error.

Proof: Under the given conditions one can easily show that the error e_i satisfies

$$L_h(e(x_i)) = L_h(y(x_i)-y_i) = \tau_i,\quad i=1,\ 2,\ \ldots,N-1$$

and $e_0 = e_N = 0$.

Then theorem 1, the stability of L_h, implies that

$$|y(x_i)-y_i| = |e_i| \le k\max_{1\le i\le N-1}|\tau_i| \tag{58}$$

Hence the estimate in Eq. (57) establishes the convergence of the scheme for the fixed values of the perturbation parameter ε.

5. Numerical Examples

To demonstrate the applicability of the method we have considered three boundary value problems of the type given by Eq. (1)-(3) with left-end boundary layer. The approximate solution is compared with exact solution and the inner layer solutions also plotted by using graphs for different values of ε and the terminal points. The exact solution of such boundary value problems having constant coefficients (i.e. $a(x)=a$, $\alpha(x)=\alpha$, $\beta(x)=\beta$, $\omega(x)=\omega$, $f(x)=f$, $\phi(x)=\phi$ and $\gamma(x)=\gamma$ are constants) is given by:

$$y(x)=c_1\exp(m_1x)+c_2\exp(m_2x)+\frac{f}{c}, \tag{59}$$

where

$$\begin{aligned}
c_1 &= \frac{-f+\gamma c+\exp(m_2)(f-\phi c)}{(\exp(m_1)-\exp(m_2))c}\\
c_2 &= \frac{f-\gamma c+\exp(m_1)(-f+\phi c)}{(\exp(m_1)-\exp(m_2))c}\\
m_1 &= \frac{-(a-\alpha\delta+\beta\eta)+\sqrt{(a-\alpha\delta+\beta\eta)^2-4\varepsilon c}}{2\varepsilon}\\
m_2 &= \frac{-(a-\alpha\delta+\beta\eta)-\sqrt{(a-\alpha\delta+\beta\eta)^2-4\varepsilon c}}{2\varepsilon}\\
c &= \alpha+\beta+\omega
\end{aligned} \tag{60}$$

Example 1: Consider the model boundary value problem given by Eq. (1)-(3) with $a(x)=1,\ \alpha(x)=2,\ \beta(x)=0,\ \omega(x)=-3,\ f(x)=0,\ \phi(x)=1,\ \gamma(x)=1$.
The exact solution of the problem is given by Eq. (59)-(60). The numerical results are given in tables 1, 2 for ε=0.001 and 0.0001 respectively.

Table 1 Numerical Results for Example 1, $\varepsilon=10^{-3}$ and $\delta=0.1\varepsilon=\eta$

x	$t_p=10$	$t_p=20$	$t_p=30$	Exact Sol.
	y(x)	y(x)	y(x)	
0.00000	1.0000000	1.0000000	1.0000000	1.0000000
0.00020	0.8891662	0.8863916	0.8856926	0.8854601
0.00040	0.7984523	0.7934058	0.7921346	0.7917112
0.00060	0.7242065	0.7173000	0.7155601	0.7149817
0.00080	0.6634415	0.6550117	0.6528881	0.6521843
0.01000	<u>0.3925705</u>	0.3771362	0.3732482	0.3718990
0.02000		<u>0.3819092</u>	0.3779721	0.3756049
0.03000			<u>0.3828013</u>	0.3793768
0.10000	0.4072314	0.4072314	0.4072314	0.4068620
0.20000	0.4500288	0.4500288	0.4500288	0.4496161
0.40000	0.5495898	0.5495898	0.5495898	0.5490746
0.60000	0.6711769	0.6711769	0.6711769	0.6705341
0.80000	0.8196632	0.8196632	0.8196632	0.8188615
0.90000	0.9058052	0.9058052	0.9058052	0.9049096
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

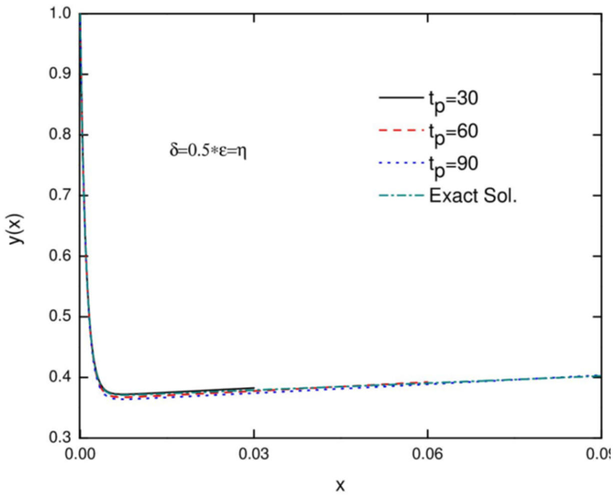

Fig. 1: Graph of inner layer solutions of example 1 for $\varepsilon = 0.001$ and different terminal points

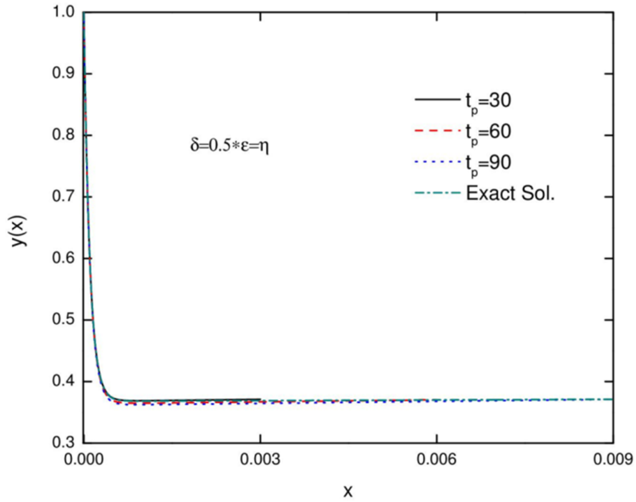

Fig. 2: Graph of inner layer solutions of example 1 for $\varepsilon = 0.0001$ and different terminal points

Table 2 Numerical Results for Example 1, $\varepsilon=10^{-4}$ and $\delta = 0.1\varepsilon = \eta$

x	t_p =10	t_p =20	t_p =30	Exact Sol.
	y(x)	y(x)	y(x)	
0.00000	1.0000000	1.0000000	1.0000000	1.0000000
0.00002	0.8890539	0.8863192	0.8856640	0.8854204
0.00004	0.7982205	0.7932469	0.7920552	0.7916133
0.00006	0.7238525	0.7170458	0.7154148	0.7148132
0.00008	0.6629651	0.6546576	0.6526669	0.6519369
0.00100	<u>0.3881176</u>	0.3730271	0.3694100	0.3683056
0.00200		<u>0.3740894</u>	0.3704678	0.3686454
0.00300			<u>0.3715705</u>	0.3690142
0.10000	0.4066579	0.4066579	0.4066579	0.4065989
0.20000	0.4494155	0.4494155	0.4494155	0.4493577
0.40000	0.5488909	0.5488909	0.5488909	0.5488380
0.60000	0.6703845	0.6703845	0.6703845	0.6703415
0.70000	0.7408718	0.7408718	0.7408718	0.7408360
0.80000	0.8187701	0.8187701	0.8187701	0.8187438
0.90000	0.9048591	0.9048591	0.9048591	0.9048446
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

Example 2: Consider the model boundary value problem given by Eq. (1)-(3) with $a(x)=1,\ \alpha(x)=0,\ \beta(x)=2,\ \omega(x)=-3,\ f(x)=0,\ \phi(x)=1,\ \gamma(x)=1.$
The exact solution of the problem is given by Eq. (59)-(60). The numerical results are given in tables 3, 4 for ε=0.001 and 0.0001 respectively.

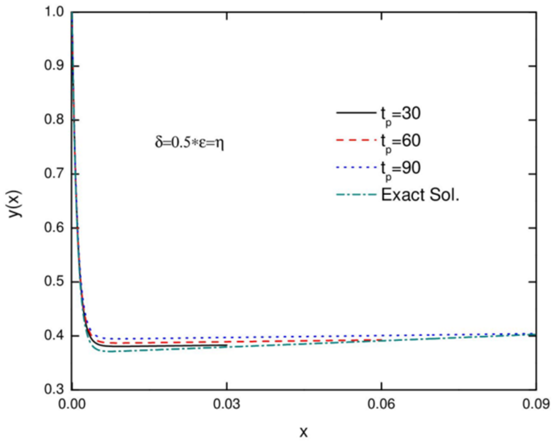

Fig. 3: Graph of inner layer solutions of example 2 for $\varepsilon = 0.001$ and different terminal points

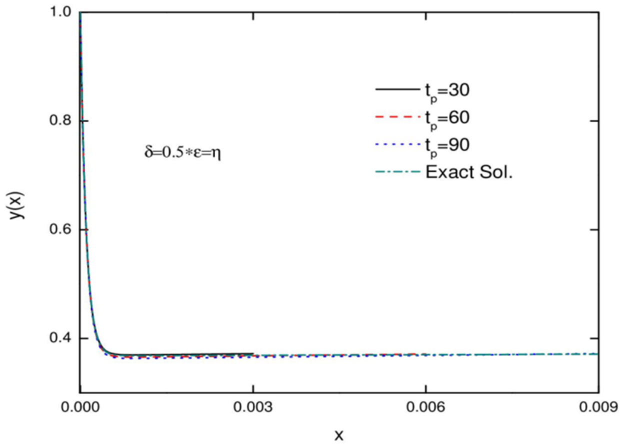

Fig. 4: Graph of inner layer solutions of example 2 for $\varepsilon = 0.0001$ and different terminal points

Table 3 Numerical Results for Example 2, $\varepsilon=10^{-3}$ and $\delta = 0.1\varepsilon = \eta$

	t_p =10	t_p =20	t_p =30	
x	y(x)	y(x)	y(x)	Exact Sol.
0.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.0002000	0.8898741	0.8877456	0.8877023	0.8854528
0.0004000	0.7997113	0.7958402	0.7957615	0.7917016
0.0006000	0.7258931	0.7205952	0.7204876	0.7149733
0.0008000	0.6654564	0.6589903	0.6588590	0.6521794
0.0100000	0.3926501	0.3809045	0.3806655	0.3719724
0.0200000		0.3819941	0.3817548	0.3756783
0.0300000			0.3828873	0.3794502
0.1000000	0.4073182	0.4073182	0.4073182	0.4069350
0.2000000	0.4501143	0.4501143	0.4501143	0.4496878
0.4000000	0.5496684	0.5496684	0.5496684	0.5491403
0.6000000	0.6712409	0.6712409	0.6712409	0.6705877
0.7000000	0.7417666	0.7417666	0.7417666	0.7410401
0.8000000	0.8197021	0.8197021	0.8197021	0.8188941
0.9000000	0.9058264	0.9058264	0.9058264	0.9049277
1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 4 Numerical Results for Example 2, ε=10^{-4} and $\delta = 0.1\varepsilon = \eta$

x	t_p =10	t_p =20	t_p =30	Exact Sol.
	y(x)	y(x)	y(x)	
0.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.0000200	0.8891366	0.8864011	0.8857456	0.8854197
0.0000400	0.7983701	0.7933951	0.7922027	0.7916124
0.0000600	0.7240578	0.7172490	0.7156171	0.7148124
0.0000800	0.6632162	0.6549061	0.6529145	0.6519365
0.0010000	<u>0.3885801</u>	0.3734850	0.3698667	0.3683130
0.0020000		<u>0.3745477</u>	0.3709251	0.3686527
0.0030000			<u>0.3720278</u>	0.3690215
0.1000000	0.4071122	0.4071122	0.4071122	0.4066063
0.2000000	0.4498618	0.4498618	0.4498618	0.4493649
0.4000000	0.5492994	0.5492994	0.5492994	0.5488446
0.6000000	0.6707174	0.6707174	0.6707174	0.6703469
0.7000000	0.7411475	0.7411475	0.7411475	0.7408404
0.8000000	0.8189732	0.8189732	0.8189732	0.8187471
0.9000000	0.9049717	0.9049717	0.9049717	0.9048465
1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Example 3: Consider the model boundary value problem given by Eq. (1)-(3) with $a(x)=1,\ \alpha(x)=-2,\ \beta(x)=1,\ \omega(x)=-5,\ f(x)=0,\ \phi(x)=1,\ \gamma(x)=1.$
The exact solution of the problem is given by Eq. (59)-(60). The numerical results are given in tables 5, 6 for ε=0.001 and 0.0001 respectively.

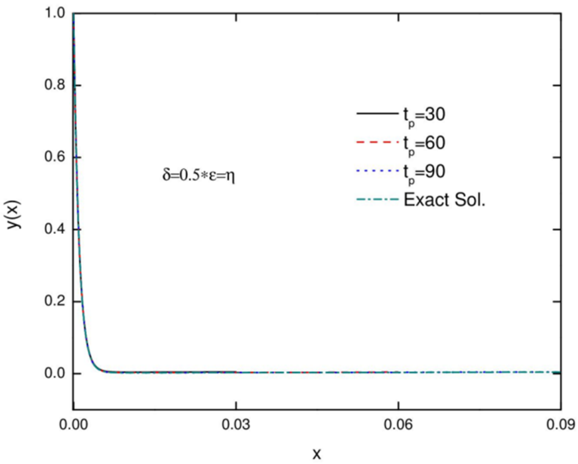

Fig. 3: Graph of inner layer solutions of example 3 for $\varepsilon = 0.001$ and different terminal points

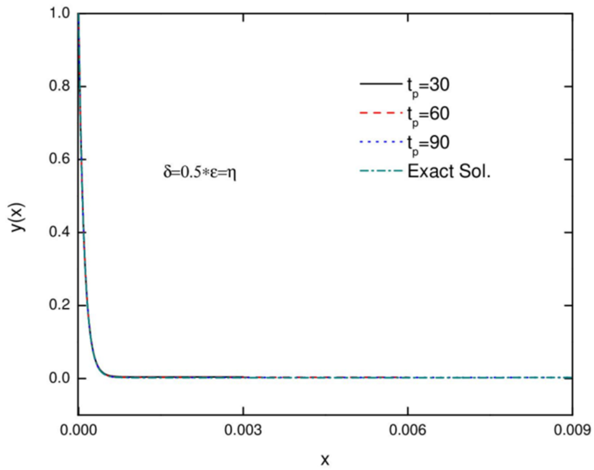

Fig. 4: Graph of inner layer solutions of example 3 for $\varepsilon = 0.0001$ and different terminal points

Table 5 Numerical Results for Example 3, ε=10^{-3} and $\delta = 0.1\varepsilon = \eta$

x	t_p =10	t_p =20	t_p =30	Exact Sol.
	y(x)	y(x)	y(x)	
0.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.0002000	0.8208835	0.8190290	0.8186693	0.8181940
0.0004000	0.6743861	0.6710132	0.6703587	0.6695278
0.0006000	0.5545692	0.5499527	0.5490565	0.5479606
0.0008000	0.4565743	0.4509386	0.4498444	0.4485531
0.0100000	0.0171680	0.0065311	0.0044664	0.0027721
0.0200000		0.0068309	0.0046571	0.0028973
0.0300000			0.0049032	0.0030753
0.1000000	0.0046968	0.0046968	0.0046968	0.0046685
0.2000000	0.0085266	0.0085266	0.0085266	0.0084753
0.4000000	0.0281015	0.0281015	0.0281015	0.0279330
0.6000000	0.0926154	0.0926154	0.0926154	0.0920615
0.7000000	0.1681357	0.1681357	0.1681357	0.1671315
0.8000000	0.3052365	0.3052365	0.3052365	0.3034163
0.9000000	0.5541317	0.5541317	0.5541317	0.5508323
1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 6 Numerical Results for Example 3, ε=10^{-4} and $\delta = 0.1\varepsilon = \eta$

x	t_p=10	t_p =20	t_p =30	Exact Sol.
	y(x)	y(x)	y(x)	
0.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.0000200	0.8217893	0.8198857	0.8195039	0.8190809
0.0000400	0.6758847	0.6724228	0.6717284	0.6709752
0.0000600	0.5564294	0.5516915	0.5507411	0.5497316
0.0000800	0.4586279	0.4528452	0.4516852	0.4504781
0.0010000	0.0169474	0.0064431	0.0043361	0.0025479
0.0020000		0.0064169	0.0043036	0.0025180
0.0030000			0.0043166	0.0025331
0.1000000	0.0045280	0.0045280	0.0045280	0.0045317
0.2000000	0.0082482	0.0082482	0.0082482	0.0082542
0.4000000	0.0273696	0.0273696	0.0273696	0.0273847
0.6000000	0.0908196	0.0908196	0.0908196	0.0908529
0.7000000	0.1654379	0.1654379	0.1654379	0.1654832
0.8000000	0.3013628	0.3013628	0.3013628	0.3014180
0.9000000	0.5489652	0.5489652	0.5489652	0.5490155
1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

6. Discussion and Conclusion

A domain decomposition method has been presented to find the numerical solution of singularly perturbed differential difference equations with delay as well as advance whose solutions exhibits boundary layer behavior. The method is iterative on the terminal point x_p and the process is repeated for different values of x_p (the terminal point which is not unique), until the solution profile stabilizes in both the inner and outer regions. The present method has been implemented on three model examples with left-end boundary layer, by taking $\delta = 0.1\varepsilon = \eta$, $\delta = 0.5\varepsilon = \eta$ and different values of ε. The numerical results have been tabulated and compared with the exact solutions (Tables 1-6). Further, the inner layer solutions for different values of the terminal points and ε have been presented by using graphs (Fig. 1-6). We have given here only a few values although the solutions are computed at all the points with mesh size h. It can be observed from the tables and the graphs that the present method approximates the exact solution very well. The present method is simple, easy and efficient technique for solving singularly perturbed differential difference equations. In fact, our method helps us not only to get good results but also to know the behavior of the solution in the boundary layer/inner region with $h \geq \varepsilon$ as shown by the graphs where most of the existing numerical methods fail to give good results. Thus, the present method provides an alternative technique for solving singularly perturbed boundary value problems involving delay as well as advance parameters.

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