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Research Article

Estimation on the Spin^c twisted Dirac operators

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Abstract

We generalize the lower bound estimates for eigenvalues of the twisted Dirac operator on compact Riemannian Spin^c−submanifold obtained by Roger Nakad and Julien Roth in (Archiv der Mathematik 104(5), 453-461, 2015).

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1. Introduction

The Dirac operator, is the fundamental part of the Seiberg−Witten equations, has been investigated since more than three decades by both mathematicians and physicists [2, 6, 10, 19, 20]. Since the Schrödinger−Lichnerowicz formula is expressed depending on both the Dirac operator and the scalar curvature of the manifold, it allows to obtain a lower bound for the eigenvalue of the Dirac operator, and depending on the minimum value of the lower bound basic information about the topology and geometry of the manifold can be obtained [3, 4, 12–14, 21]. Accordingly, the first lower bound corresponding to the eigenvalue of the Dirac operator was given by A. Lichnerowicz in 1963 as follows [16]:

$$\lambda^2 \geq \frac{R}{4}. \tag{1.1}$$

Here R denotes scalar curvature of the manifold. The above estimation is obtained by using the classical spinorial Levi−Civita connection defined on Spin−manifolds. T. Friedrich improved (1.1) by summing the classical spinorial Levi−Civita connection with an associated spinor field which is multiplying with a scalar function defined on manifold. T. Friedrich's estimation is given as follows [5]:

$$\lambda^2 \geq \frac{n}{4(n-1)}R, \tag{1.2}$$

where n denotes dimension of the manifold. Equality case is characterized by a non−trivial real−Killing spinor and the manifold is an Einstein. At this stage, O. Hijazi improved (1.2) based on the first eigenvalue of the Yamabe operator as follows [10]:

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1, \tag{1.3}$$

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where $n \geq 3$ and μ_1 is an eigenvalue of the Yamabe operator defined as:

$$Y = 4\frac{n-1}{n-2}\Delta + R. \tag{1.4}$$

Here Δ is a Laplacian acting on functions. Afterwards, these estimates were investigated on the Hypersurface manifolds and submanifolds endowed with Spin$-$structure and new estimates were obtained in [12, 13]. The lower bound estimation for a first eigenvalue of the twisted Dirac operator defined on a submanifold endowed with a Spin$-$structure is given by N. Ginoux and B. Morel [7].

Throughout this paper, we deal with the eigenvalues of the twisted Dirac operator defined on the compact Riemannian Spin$^c-$submanifold. Accordingly, first estimation is given by M. Herzlich and A.Moroianu as follows [9]:

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1, \tag{1.5}$$

where μ_1 denotes the first eigenvalue of the perturbed Yamabe operator Y^Ω given as:

$$Y^\Omega \quad = \quad Y - c_n|\Omega|, \tag{1.6}$$

here $c_n = 2\left[\frac{n}{2}\right]^{\frac{1}{2}}$ and $|\Omega|$ is a norm of Ω curvature form defined on a line bundle of a given manifold. Later, some estimates are obtained by adding some geometric invariant such as Energy$-$Momentum tensor [4, 8, 17].

Just like in the Spin$-$submanifolds, R. Nakad and J. Roth applied N. Ginoux and B. Morel's method to obtain estimates on the compact Riemannian Spin$^c-$submanifold in [18]. The lower bound for the first eigenvalue of the twisted Dirac operator defined on an $r-$dimensional Riemannian Spin$^c-$submanifold M isometrically embedded into an $(r+s)$ dimensional Riemannian Spin^c manifold $\widetilde{M}$ is as follows: If

$$R + R^N_\Phi - c_r|\Omega| > \frac{r-1}{r}||H||^2 > 0 \tag{1.7}$$

and $M_\Phi = \{x \in M | \Phi(x) \neq 0\}$ then

$$\lambda_H^2 \quad \geq \quad \frac{1}{4}\inf_{M_\Phi}\left(\sqrt{\frac{r}{r-1}(R + R^N_\Phi - c_r|\Omega|)} - \|H\|\right)^2. \tag{1.8}$$

Here R^N_Φ is a spinorial curvature of the normal bundle NM and Φ is a spinorfield [18].

In this paper, by defining modified spinorial Levi$-$Civita connections we give two estimates containing all inequalities obtained by R. Nakad and J. Roth as special cases. In doing so, we will consider the modified scalar curvature in terms of the scalar curvature of the submanifolds, its normal scalar curvature of the submanifold and arbitrary real$-$valued functions which are defined as in the following section.

2. Twisted Dirac operator on the Spin$^c-$submanifolds

Assume that $(\widetilde{M}, \widetilde{g})$ is an $(r+s)-$dimensional compact Riemannian Spin$^c-$manifold and (M, g) is an $r-$dimensional submanifold isometrically immersed into $\widetilde{M}$. Here M is endowed with Spin$^c-$structure. Also, N is an $s-$dimensional normal bundle of M. Recall that M and $\widetilde{M}$ defines a unique Spin$^c-$structure on the normal bundle N for more information see [18]. The curvature 2$-$form of the corresponding auxiliary line bundle are denoted by $i\widetilde{\Omega}$ and $i\Omega$. The spinor bundles over the manifolds M, N, $\widetilde{M}$ defined on $\Gamma(\mathbb{S})$ are denoted by $\mathbb{S}_M$, $\mathbb{S}_N$ and $\mathbb{S}_{\widetilde{M}}$, respectively and $\mathbb{S} := \mathbb{S}_{\widetilde{M}}|_M$ is defined as follows:

$$\mathbb{S} := \begin{cases} \mathbb{S}_M \otimes \mathbb{S}_N, & \text{if } r \text{ or } s \text{ is even,} \\ \mathbb{S}_M \otimes \mathbb{S}_N \oplus \mathbb{S}_M \otimes \mathbb{S}_N, & \text{otherwise.} \end{cases} \tag{2.1}$$

The spinorial Levi−Civita connection on $\mathbb{S}_{\widetilde{M}}$ and the induced spinorial Levi−Civita connection on $\mathbb{S}_M$ are denoted by $\widetilde{\nabla}$ and ∇, respectively. Accordingly, $\widetilde{\nabla}$ is defined as follows

$$\widetilde{\nabla} = \begin{cases} \left(\nabla^{\mathbb{S}_M} \otimes Id + Id \otimes \nabla^{\mathbb{S}_N}\right) \oplus \left(\nabla^{\mathbb{S}_M} \otimes Id + Id \otimes \nabla^{\mathbb{S}_N}\right), & \text{if } r \text{ and } s \text{ are odd,} \\ \left(\nabla^{\mathbb{S}_M} \otimes Id + Id \otimes \nabla^{\mathbb{S}_N}\right), & \text{otherwise.} \end{cases}$$

Consider $\{e_1, ..., e_r, v_1, ..., v_s\}$ positively oriented local orthonormal basis of $T\widetilde{M}\big|_M$ at a fixed point $p \in M$. Here $\{e_1, ..., e_r\}$ and $\{v_1, ..., v_s\}$ are positively oriented local orthonormal basis of TM and NM, respectively.

For any spinor field $\Phi = \Psi\big|_M, \Psi \in \Gamma(\mathbb{S}_{\widetilde{M}})$, the Clifford multiplication between $\Gamma(\mathbb{S}_M)$ and $\Gamma(\mathbb{S}_{\widetilde{M}})\big|_M$ is defined by

$$V \cdot_M \Phi \quad = \quad \big(V \cdot \omega_\perp \cdot \Psi\big)_M, \tag{2.2}$$

where $\omega_\perp$ is given by

$$\omega_\perp = \begin{cases} \omega_s, & \text{for } s \text{ even,} \\ -i\omega_s, & \text{for } s \text{ odd,} \end{cases} \tag{2.3}$$

here ω_s denotes the complex volume form:

$$\omega_s = i^{[\frac{s+1}{2}]} v_1 \cdot ... \cdot v_s, \tag{2.4}$$

and $\omega_\perp^2 = (-1)^s$. On the spinor bundle $\mathbb{S}$ one can define a Hermitian inner product denoted by $(\ ,\ )$ and satisfies the following properties:

$$\begin{aligned} \big(V \cdot \omega_\perp \cdot \Phi, \Phi\big) &= -\big(\omega_\perp \cdot \Phi, V \cdot \Phi\big) \\ &= (-1)^{s+1}\big(\Phi, \omega_\perp \cdot V \cdot \Phi\big) \\ &= -\big(\Phi, V \cdot \omega_\perp \cdot \Phi\big), \end{aligned} \tag{2.5}$$

where $V \in TM$ and $\Phi \in \Gamma(\mathbb{S})$ [15,16]. The spinorial Gauss formula is defined as [1] :

$$\widetilde{\nabla}_V \Phi = \nabla_V \Phi + \frac{1}{2}\sum_{j=1}^{r} e_j \cdot B(V, e_j) \cdot \Phi, \tag{2.6}$$

where $\Phi \in \Gamma(\mathbb{S})$ and B is the component of the second fundamental form at p. Locally, Dirac operators are defined as follows:

$$\widetilde{D}\Phi = \sum_{i=1}^{r} e_i \cdot \widetilde{\nabla}_i \Phi, \ \ D\Phi = \sum_{i=1}^{r} e_i \cdot \nabla_i \Phi, \tag{2.7}$$

and the twisted Dirac operator D_H is defined by

$$D_H\Phi := (-1)^s \omega_\perp \cdot \widetilde{D}\Phi = (-1)^s \omega_\perp \cdot D\Phi + \frac{1}{2} H \cdot \omega_\perp \cdot \Phi, \tag{2.8}$$

where $H = \sum_{i=1}^{r} h(e_i, e_i)$ denotes the mean curvature vector field. On the submanifold (M, g) the well−known formula twisted Schrödinger−Lichnerowicz formula is descibed by

$$\big(D^2\Phi, \Phi\big) \quad = \quad \big(\nabla^*\nabla\Phi, \Phi\big) + \frac{1}{4}\big(R + R^N_\Phi\big)|\Phi|^2 + \frac{i}{2}\big(\Omega \cdot \Phi, \Phi\big), \tag{2.9}$$

where ∇^* denotes adjoint of ∇ and $R^N_\Phi := 2\sum_{i,j=1}^{r}\Big(e_i \cdot e_j \cdot I_d \otimes R^N_{e_i,e_j}\Phi, \frac{\Phi}{|\Phi|^2}\Big)$ on $M_\Phi := \{x \in M : \Phi(x) \neq 0\}$, and $R^N_{e_i,e_j}$ stands for spinorial normal curvature tensor [7]. Integrating over M one gets:

$$\begin{aligned} \int_M |\nabla\Phi|^2 v_g \quad = \quad & \int_M \Big[\lambda_H^2|\Phi|^2 + \frac{1}{4}\|H\|^2|\Phi|^2 + \lambda_H Re\big(\omega_\perp \cdot \Phi, H \cdot \Phi\big) - \frac{1}{4}(R + R^N_\Phi)|\Phi|^2 \\ & -\frac{i}{2}\big(\Omega \cdot \Phi, \Phi\big)\Big] v_g. \end{aligned} \tag{2.10}$$

3. Estimating lower bounds

For any real−valued functions γ, u on $\widetilde{M}$ the modified scalar curvature is defined as [6,12,13]:

$$R_{\gamma,u,\Phi} = R + R^N_\Phi - 4\gamma\nabla u + 4\nabla\gamma\nabla u - 4\big(1-\frac{1}{r}\big)\gamma^2|du|^2. \tag{3.1}$$

In case γ and u are vanished identically (3.1) induced to $R_{\gamma,u,\Phi} = R + R^N_\Phi$. This means that all estimates obtained in this paper coincides with R.Nakad and J.Roth estimates given in [18].

Proposition 3.1. *The following inequality is satisfied for any $\Phi \in \Gamma(\mathbb{S})$,*

$$\big(i\Omega\cdot\Phi, \Phi\big) \geq -\frac{c_r}{2}|\Omega||\Phi|^2, \tag{3.2}$$

where $|\Omega|$ is the norm of Ω with respect to g given on M [18].

Theorem 3.2. *Assume that (M,g) is a compact $(r\geq 2)-$dimensional Riemannian Spin$^c-$submanifold isometrically immersed into $(r+s)-$dimensional Riemannian Spin$^c-$manifold $(\widetilde{M},\widetilde{g})$ and λ_H is the first nonzero eigenvalue of D_H associated with the eigenspinorfield $\Phi\in\Gamma(\mathbb{S})$. If*

$$\Omega_{\gamma,u,\Phi} = \{(\gamma,u,\Phi)|r\big(R_{\gamma,u,\Phi} - c_m|\Omega|\big) > (r-1)\|H\|^2 > 0\}, \tag{3.3}$$

where γ, u are real−valued functions, then one has

$$\lambda_H^2 \quad \geq \quad \frac{1}{4}\sup_{\Omega_{\gamma,u,\Phi}}\inf_{M_\Phi}\Big(\sqrt{\frac{r}{r-1}\big(R_{\gamma,u,\Phi} - c_r|\Omega|\big)} - \|H\|\Big)^2. \tag{3.4}$$

Proof. For any real−valued functions μ,γ and $\Phi\in\Gamma(\mathbb{S})$, consider the following modified spinorial Levi−Civita connection defined on $\Gamma(\mathbb{S})$ by

$$\nabla^u_i\Phi = \nabla_i\Phi + \frac{\mu-1}{2(r\mu-1)}e_i\cdot H\cdot\Phi + \mu\lambda_H e_i\cdot\omega_\perp\cdot\Phi + \gamma\nabla_i u\Phi + \frac{\gamma}{r}\nabla_j u e_i\cdot e_j\cdot\Phi. \tag{3.5}$$

Here μ in nowhere equals to $\frac{1}{m}$. Accordingly, the norm of (3.5) is obtained as follows:

$$\begin{aligned}
|\nabla^u\Phi|^2 \quad = \quad & |\nabla\Phi|^2 - \frac{\mu-1}{r\mu-1}Re\big(D\Phi, H\cdot\Phi\big) - 2\lambda_H\mu Re\big(D\Phi,\omega_\perp\cdot\Phi\big)\\
&+2\gamma\sum_{i=1}^{r}\nabla_i u Re\big(\nabla_i\Phi,\Phi\big) + r\Big[\frac{\mu-1}{2(r\mu-1)}\Big]^2\|H\|^2|\Phi|^2\\
&+r\Big[\frac{\mu-1}{(r\mu-1)}\Big]\mu\lambda_H Re\big(H\cdot\Phi,\omega_\perp\cdot\Phi\big) - \frac{\mu-1}{(r\mu-1)}\gamma Re\big(H\cdot\Phi, du\cdot\Phi\big)\\
&+r\Big[\frac{\mu-1}{r\mu-1}\Big]\frac{\gamma}{r}Re\big(H\cdot\Phi, du\cdot\Phi\big) + r\mu^2\lambda_H^2|\Phi|^2\\
&-2\mu\gamma\lambda_H Re\big(\omega_\perp\cdot\Phi, du\cdot\Phi\big) + 2\mu\gamma\lambda_H Re\big(\omega_\perp\cdot\Phi, du\cdot\Phi\big) + \gamma^2|du|^2|\Phi|^2\\
&-2\frac{\gamma^2}{r}|du|^2|\Phi|^2 + \frac{\gamma^2}{r}|du|^2|\Phi|^2.
\end{aligned} \tag{3.6}$$

Using the equality obtained in (2.10) and (3.2), we get

$$\begin{aligned}
\int_M \big(1+r\mu^2-2\mu\big)\lambda_H^2|\Phi|^2 v_g \quad \geq \quad & \int_M \Big[\frac{(R+R^N_\Phi)}{4} - \frac{c_r}{2}|\Omega| + \big(1-\frac{1}{r}\big)\gamma^2|du|^2 - \gamma\Delta u\\
&+\nabla\gamma\nabla u - \big(\frac{r^2\mu^2 - 2r\mu - r\mu^2 + 2\mu - 1)}{4(r\mu-1)^2}\big)\|H\|^2\Big]|\Phi|^2 v_g.
\end{aligned} \tag{3.7}$$

Making use of (3.1) in (3.7), one has

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\gamma,u,\mu}\inf_M\Big(\frac{R_{\gamma,u,\Phi} - c_r|\Omega|}{1+r\mu^2-2\mu} - \frac{(r-1)}{(r\mu-1)^2}\|H\|^2\Big). \tag{3.8}$$

Suppose that $r\big(R_{\gamma,u,\Phi}-c_r|\Omega|\big)>(r-1)\|H\|^2>0$ on M_Φ. Then by taking the following equality on M_Φ

$$(1-r\mu)^2=\frac{(r-1)\|H\|}{\sqrt{\frac{r}{r-1}\big(R_{\gamma,u,\Phi}-c_r|\Omega|\big)}-\|H\|}, \tag{3.9}$$

and inserting (3.9) into (3.8), we get the desired result given in (3.4). □

Let η_1 be the lowest eigenvalue of the self−adjoint operator $\mathcal{R}^N$ defined by

$$\begin{aligned}\mathcal{R}^N:\Gamma(\mathbb{S}) &\longrightarrow \Gamma(\mathbb{S})\\ \Phi &\longmapsto 2\sum_{i,j=1}^{m} e_i\cdot e_j\cdot Id\otimes R^N_{e_i,e_j}\Phi.\end{aligned} \tag{3.10}$$

Then $R_{\gamma,u,\Phi}$ can be rewritten as:

$$\widetilde{R}=R+\eta_1-4\gamma\nabla u+4\nabla\gamma\nabla u-4\big(1-\frac{1}{r}\big)\gamma^2|du|^2, \tag{3.11}$$

This gives us the following corollary.

Corollary 3.3. *Under the same conditions given in Theorem* (3.2)*, if* $r\geq 2$ *and*

$$\widetilde{\Omega}_{\gamma,\eta_1,u,\Phi}=\{(\gamma,\eta_1,u,\Phi)|m\big(R_{\gamma,\eta_1,u,\Phi}-c_r|\Omega|\big)>(r-1)\|H\|^2>0\}$$

on M*, then*

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\widetilde{\Omega}_{\gamma,\eta_1,u,\Phi}}\inf_{M_\Phi}\Big(\sqrt{\frac{r}{r-1}\big(\widetilde{R}-c_r|\Omega|\big)}-\|H\|\Big)^2. \tag{3.12}$$

Under the conformal change of the Riemannian metric and using the classic arguments given in [11–13] we can improve our estimates as in the following. Before doing this, let's recall some of the identities and definitions that will be used in our proofs.

Let $\mathbb{S}$ and $\overline{\mathbb{S}}$ be the spinor bundles constructed on M with respect to the metric g and $\overline{g}=e^{2u}g$, respectively. Here $\overline{g}$ is a conformal change of the metric defined on $\widetilde{M}$ with a real−valued function u. Accordingly, the following relations are satisfied between the defined Hermitian metrics $(\ ,\ )=(\ ,\ )_{\overline{g}}$ on $\mathbb{S}$ and $\overline{\mathbb{S}}$, respectively:

$$\big(\Psi,\Phi\big)=\big(\overline{\Psi},\overline{\Phi}\big)_{\overline{g}} \text{ and } \overline{V}\bar{\cdot}\overline{\Psi}=\overline{V\cdot\Psi},\ \overline{D}\big(e^{-\frac{(r-1)}{2}u}\overline{\Psi}\big)=e^{-\frac{(r+1)}{2}u}\overline{D\Psi}, \tag{3.13}$$

where $\overline{V}=e^{-u}V$, $\Psi,\Phi\in\Gamma(\mathbb{S})$, $V\in\Gamma(TM)$ and $\overline{D}$ denotes the Dirac operator with respect to $\overline{g}$. The corresponding mean curvature vector field with respect to $\overline{g}=e^{2u}g$ is

$$\widehat{H}=e^{-2u}\big(H-r\,grad^N u\big), \tag{3.14}$$

and if we consider $grad^N u=0$, twisted Dirac operator $\overline{D}_H$ is described as:

$$\overline{D}_H\big(e^{-\frac{(r-1)}{2}u}\overline{\Psi}\big)=e^{-\frac{(r+1)}{2}u}\overline{D_H\Psi}. \tag{3.15}$$

Finally, the curvature operator $\widetilde{R}$ is transformed to

$$\begin{aligned}\overline{\widetilde{R}} &= R+\eta_1+4\big(\frac{r-1}{2}-\gamma\big)\Delta u+4\nabla\gamma\nabla u\\ &\quad -\Big((r-1)(r-2)+4(2-r)\gamma+4\big(1-\frac{1}{r}\big)\gamma^2\Big)|du|^2.\end{aligned} \tag{3.16}$$

In the rest of this paper, by considering the regular conformal change of the metric $\overline{g}$ with $grad^N u=0$, on M we get the following estimates.

Theorem 3.4. *Assume that* (M,g) *is an* $(r\geq 3)-$*dimensional compact Riemannian Spin*$^c-$*submanifold which is isometrically immersed into* $(r+s)-$*dimensional Riemannian Spin*$^c-$*manifold* $(\widetilde{M},\widetilde{g})$ *and* λ_H *is the first nonzero eigenvalue of* D_H *associated with the eigenspinor field* $\Phi\in\Gamma(\mathbb{S})$*. Under the conformal change of the metric* $\overline{g}=e^{2u}g$ *on* $\widetilde{M}$*, assume that*

$$\overline{\widetilde{\Omega}}=\{(\gamma,\eta_1,u,\Phi)|r\big(\overline{\widetilde{R}}-c_r|\Omega|\big)>(r-1)\|H\|^2>0\}$$

on M_Φ. Then we have

$$\lambda_H^2 \quad \geq \quad \frac{1}{4}\sup_{\widetilde{\overline{\Omega}}} \inf_{M_\Psi}\Big(\sqrt{\frac{r}{r-1}\big(\widetilde{\overline{R}} - c_r|\Omega|\big)} - \|H\|\Big)^2. \tag{3.17}$$

Proof. Consider a non−trivial eigenvalue λ_H of D_H associated with the eigenspinor field $\overline{\Phi}$. Here $\overline{\Psi} := e^{-\frac{s-1}{2}u}\overline{\Phi}$. Taking into account $\overline{D}_H\overline{\Psi} = \lambda_H e^{-u}\overline{\Psi}$, $\widehat{H} = e^{-u}\overline{H}$, $\overline{R}^N_{\overline{\Psi}} = e^{-2u}R^N_\Phi$ by applying $\overline{\Phi}$ to (3.7), we get

$$\begin{aligned}\int_M \big(1 + r\mu^2 - 2\mu\big)e^{-2u}\lambda_H^2|\overline{\Psi}|^2 v_{\overline{g}} \quad \geq \quad & \int_M \frac{1}{4}\Big(\widetilde{\overline{R}} - c_r|\Omega| \\ & -\big(\frac{r^2\mu^2 - 2r\mu - r\mu^2 + 2\mu - 1)}{(r\mu - 1)^2}\big)\|H\|^2\Big)e^{-2u}|\overline{\Psi}|^2 v_{\overline{g}}.\end{aligned} \tag{3.18}$$

Applying

$$(1 - r\mu)^2 = \frac{(r-1)\|H\|}{\sqrt{\frac{r}{r-1}\big(\widetilde{\overline{R}} - c_r|\Omega|\big)} - \|H\|}, \text{ defined on } M_\Phi, \tag{3.19}$$

into (3.18) one can obtained the estimates given in (3.17). □

In the following we improve our estimation in terms of the Energy−Momentum tensor.

Theorem 3.5. *Assume that (M, g) is a compact $r-$dimensional Riemannian Spin$^c-$submanifold isometrically immersed into $(r+s)-$dimensional Riemannian Spin$^c-$manifold $(\widetilde{M}, \widetilde{g})$ and λ_H is the first nonzero eigenvalue of D_H associated with the eigenspinor field $\Phi \in \Gamma(\mathbb{S})$. Under the conformal change of the metric $\overline{g} = e^{2u}g$ on $\widetilde{M}$, assume that $r \geq 3$ and*

$$\Omega^{Q_\Phi} = \{(\gamma, \eta_1, u, \Phi)|\big(\widetilde{R} - c_r|\Omega| + 4|Q_\Phi|^2\big) > \|H\|^2 > 0\}, \tag{3.20}$$

on M_Φ and where Q^Φ defined on M_Φ as follows:

$$Q^\Phi_{ij} \quad = \quad \frac{1}{2}\left(e_i \cdot \omega_\perp \cdot \nabla_j\Phi + e_j \cdot \omega_\perp \cdot \nabla_i\Phi, \frac{\Phi}{|\Phi|^2}\right). \tag{3.21}$$

Then we have

$$\lambda_H^2 \quad \geq \quad \frac{1}{4}\sup_{\Omega^{Q_\Phi}} \inf_{M_\Phi}\Big(\sqrt{\widetilde{R} - c_r|\Omega| + 4|Q_\Phi|^2} - \|H\|\Big)^2. \tag{3.22}$$

Proof. For any real−valued functions μ, γ and $\Phi \in \Gamma(\mathbb{S})$, consider the following modified spinorial Levi−Civita connection on $\Gamma(\mathbb{S})$:

$$\begin{aligned}\nabla_i^{Q_\Phi}\Phi \quad = \quad & \nabla_i\Phi - \frac{1}{2r\mu}e^i \cdot H \cdot \Phi + \mu\lambda_H e_i \cdot \omega_\perp \cdot \Phi + \gamma\nabla_i u\Phi + \frac{\gamma}{r}\nabla_j u e_i \cdot e_j \cdot \Phi \\ & +(Q_\Phi)_{ij}e_j \cdot \omega_\perp \cdot \Phi,\end{aligned} \tag{3.23}$$

where γ, μ are real−valued functions and μ nonzero for any point on M.

By using the fact that $trQ_\Phi = \lambda_H + \frac{1}{2}Re\Big(H\cdot\Phi, \frac{\omega_\perp\cdot\Phi}{|\Phi|^2}\Big)$, one can easily obtained the norm of above modified spinorial Levi−Civita connection as follows:

$$\begin{aligned}
|\nabla^{Q^\Phi}\Phi|^2 &= |\nabla\Phi|^2 + \frac{1}{r\mu}\lambda_H Re\big(\omega_\perp\cdot\Phi, H\cdot\Phi\big) + \frac{1}{r\mu}||H\|^2|\Phi|^2 - 2\lambda_H^2\mu|\Phi|^2 \\
&-\lambda_H\mu Re\big(H\cdot\Phi,\omega_\perp\cdot\Phi\big) + 2\gamma\sum_{i=1}^{r} Re\big(\nabla_i\Phi,\nabla_i u\Phi\big) - 2|Q_\Phi|^2|\Phi|^2 \\
&+\frac{1}{4r\mu^2}\|H\|^2|\Phi|^2 - \lambda_H Re\big(H\cdot\Phi,\omega_\perp\cdot\Phi\big) + \frac{1}{m\mu}\gamma Re\big(H\cdot\Phi, du\cdot\Phi\big) \\
&-\frac{\gamma}{r\mu}Re\big(H\cdot\Phi, du\cdot\Phi\big) - \frac{1}{r\mu}\lambda_H Re\big(H\cdot\Phi,\omega_\perp\cdot\Phi\big) \\
&-\frac{1}{2r\mu}\frac{Re\big(H\cdot\Phi,\omega_\perp\cdot\Phi\big)^2}{|\Phi|^4}|\Phi|^2 + r\lambda_H^2\mu^2|\Phi|^2 - 2\lambda_H\mu\gamma Re\big(\omega_\perp\cdot\Phi, du\cdot\Phi\big) \\
&+2\gamma\mu\lambda_H Re\big(\omega_\perp\cdot\Phi, du\cdot\Phi\big) + 2\lambda_H^2\mu|\Phi|^2 + \lambda_H\mu Re\big(H\cdot\Phi,\omega_\perp\cdot\Phi\big) \\
&+\gamma^2|du|^2|\Phi|^2 - 2\frac{\gamma^2}{r}|du|^2|\Phi|^2 + \frac{\gamma^2}{r}|du|^2|\Phi|^2 + |Q_\Phi|^2|\Phi|^2.
\end{aligned}$$

Using the equality obtained in (2.10), we get

$$\begin{aligned}
\int_M |\nabla^{Q_\Phi}\Phi|^2 v_g &= \int_M \Big((1+r\mu^2)\lambda_H^2|\Phi|^2 - \frac{1}{4}\big(R+R^N\big)|\Phi|^2 + \big(1-\frac{1}{r}\big)p^2|du|^2|\Phi|^2 \\
&+\big(\gamma\Delta_u - \nabla\gamma\nabla u\big)|\Phi|^2 - |Q_\Phi|^2|\Phi|^2 \\
&+\big(\frac{r\mu^2+2\mu+1}{4r\mu^2}\big)\|H\|^2|\Phi|^2 - \frac{Re\big(H\cdot\Phi,\omega_\perp\cdot\Phi\big)^2}{2r\mu|\Phi|^4}|\Phi|^2 \\
&-\frac{i}{2}\big(\omega\cdot\Phi,\Phi\big)\Big)v_g.
\end{aligned} \tag{3.24}$$

By the definition given in (3.11) and $\|H\|^2 - \frac{Re\left(H\cdot\Phi,\omega_\perp\cdot\Phi\right)^2}{|\Psi|^4}|\Phi|^2 \geq 0$, one has

$$\begin{aligned}
\int_M (1+r\mu^2)\lambda_H^2|\Phi|^2 v_g \geq \int_M \Bigg[&\frac{\big(\widetilde{R} - c_r|\Omega|\big)}{4}|\Phi|^2 - \big(\frac{1+r\mu^2}{4r\mu^2}\big)\|H\|^2|\Phi|^2 \\
&-\frac{1}{2r\mu}\Big(\|H\|^2 - \frac{Re\big(H\cdot\Phi,\omega_\perp\cdot\Phi\big)^2}{|\Phi|^4}|\Phi|^2\Big) + |Q_\Phi|^2|\Phi|^2\Bigg]v_g.
\end{aligned}$$

This give us

$$\lambda_H^2 \geq \frac{1}{4}\inf_M\Big(\frac{\big(\widetilde{R} - c_r|\Omega| + 4|Q_\Phi|^2\big)}{1+r\mu^2} - \frac{\|H\|^2}{r\mu^2}\Big). \tag{3.25}$$

In case $\big(\widetilde{R} - c_r|\Omega| + 4|Q_\Phi|^2\big) > \|H\|^2 > 0$ on M_Φ, μ can be defined as:

$$\mu = \sqrt{\frac{\|H\|}{\big(r\sqrt{\big(\widetilde{R} - c_r|\Omega| + 4|Q_\Phi|^2\big)} - \|H\|\big)}} \tag{3.26}$$

on M_Φ. □

Under the conformal change of the metric $\overline{g} = e^{2u}g$ and $grad^N u = 0$ on M one can obtained the following theorem:

Theorem 3.6. *Under the same conditions given in Theorem* 3.4*, If* $r \geq 3$ *and*

$$\overline{\Omega}^{Q_\Phi} = \{(\gamma,\eta_1,u,\Phi)|\big(\overline{\widetilde{R}} - c_r|\Omega| + 4|Q_\Phi|^2\big) > \|H\|^2 > 0\}$$

on M_Φ*, then we have*

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\overline{\Omega}^{Q_\Phi}}\inf_{M_\Phi}\Big(\sqrt{\big(\overline{\widetilde{R}} - c_r|\Omega| + 4|Q_\Phi|^2\big)} - \|H\|\Big)^2. \tag{3.27}$$

for some γ, u real−valued functions.

Proof. Applying $\overline{\Psi}$ to (3.25),

$$\int_M (1+r\mu^2)e^{-2u}\lambda_H^2|\overline{\Psi}|^2 v_{\overline{g}} \geq \int_M \frac{1}{4}\Big[\widetilde{\overline{R}} - c_r|\Omega| + 4|Q^\Phi|^2 - \big(\frac{1+r\mu^2)}{r\mu^2}\big)\|H\|^2\Big]e^{-2u}|\overline{\Psi}|^2 v_{\overline{g}}, \tag{3.28}$$

and making use the following defined real−valued function

$$\mu = \sqrt{\frac{\|H\|}{r\left(\sqrt{\widetilde{\overline{R}} - c_r|\Omega| + 4|Q_\Phi|^2} - \|H\|\right)}} \tag{3.29}$$

one can easily obtained the desired result given in (3.27). □

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