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On Quasi-Conformally Flat Generalized Sasakian-Space Forms

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Abstract: In this paper, we classify quasi-conformally flat generalized Sasakian-space forms under the assumption that the characteristic vector field is Killing. Also, we classify quasi-conformally Weyl-symmetric generalized Sasakian-space forms.

Keywords: Generalized Sasakian-space forms, quasi-conformally flat, quasi-conformally Weyl-symmetric.

1. Introduction

In Riemannian geometry, many authors have studied curvature properties and to what extent they determined the manifold itself. Two important curvature properties are quasi-conformal flatness and Weyl-symmetry.

In [1], Alegre, Blair and Carriazo introduced and studied generalized Sasakian-space forms. These spaces are defined as follows: Given an almost contact metric manifold (M,ϕ,ξ,η,g), they say that M is a generalized Sasakian-space form if there exist three functions f_1, f_2 and f_3 on M such that

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1}$$

for any vector fields X,Y,Z on M, where R denotes the curvature tensor of M. In such a case, we will write $M(f_1,\ f_2,\ f_3)$.

Then, Kim studied conformally flat generalized Sasakian space forms [5].

In this paper, we study quasi-conformally flat generalized Sasakian-space forms and quasi-conformally Weyl-symmetric generalized Sasakian-space forms.

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2. Preliminaries

An odd-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η such that

$$\eta(\xi) = 1, \tag{2}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{3}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{4}$$

for any vector fields X, Y on M [2]. Also,

$$\phi\xi = 0 \tag{5}$$

and

$$\eta \circ \phi = 0 \tag{6}$$

are deducible from these conditions. We define the fundamental 2-form Φ on M by $\Phi(X, Y) = g(X, \phi Y)$. An almost contact metric manifold M is said to be a contact metric manifold if $g(X, \phi Y) = d\eta(X, Y)$. If ξ is a Killing vector field, then the contact metric manifold is said to be a K-contact manifold. The almost contact metric structure of M is said to be normal if $[\phi, \phi](X, Y) = -2d\eta(X, Y)\xi$, for any X, Y, where $[\phi, \phi]$ denotes the Nijenhuis torsion tensor of ϕ. A normal contact metric manifold is called a Sasakian manifold. A normal almost contact metric manifold M with closed forms η and Φ is called a cosymplectic manifold. Cosymplectic manifolds are characterized by $\nabla_X \xi = 0$ and $(\nabla_X \phi)Y = 0$ for any vector fields X, Y on M. Given an almost contact metric manifold (M, ϕ, ξ, η, g), a ϕ-section of M at $p \in M$ is a plane section $\pi \subseteq T_p M$ spanned by a unit vector X_p orthogonal to ξ_p and ϕX_p. The ϕ-sectional curvature of π is defined by $g(R(X, \phi X)\phi X, X)$. A cosymplectic space-form, i.e., a cosymplectic manifold with constant ϕ-sectional curvature c, is a generalized Sasakian space-form with $f_1 = f_2 = f_3 = \frac{c}{4}$ [6]. It is known that the ϕ-sectional curvature of a generalized Sasakian-space form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$ [1].

For a $(2n+1)$-dimensional almost contact metric manifold (M, ϕ, ξ, η, g), $n \geq 1$, its Schouten tensor L is defined by

$$L = -\frac{1}{2n-1}Q + \frac{\tau}{4n(2n-1)}I, \tag{7}$$

where Q denotes the Ricci operator and τ is the scalar curvature of M. The Weyl conformal curvature tensor is given by

$$C(X,Y)Z = R(X,Y)Z - [g(LX,Z)Y - g(Y,Z)LX - g(LY,Z)X + g(X,Z)LY]. \quad (8)$$

In dimension > 3, that is $n > 1$, M is conformally flat if and only if $C = 0$, and in this case, L satisfies $(\nabla_X L)Y - (\nabla_Y L)X = 0$ for any vector fields X, Y on M. In dimension 3, that is $n = 1$, $C = 0$ is automatically satisfied and M is conformally flat if and only if L satisfies $(\nabla_X L)Y - (\nabla_Y L)X = 0$ for any vector fields X, Y on M.

A symmetric tensor field T of type $(1,1)$ is a Codazzi tensor if it satisfies

$$(\nabla_X T)Y - (\nabla_Y T)X = 0.$$

For the later use, we give the following lemma which was proved Derdzinski.

Lemma 2.1 *[3, 4] Let T be a Codazzi tensor on a Riemannian manifold M. Then, we have the following:*

If T has more than one eigenvalue, then the eigenspaces for each eigenvalue v form an integrable subbundle V_v of constant multiplicity on open sets: If the multiplicity is greater than 1, then the integral submanifolds are umbilical submanifolds and each eigenfunction is constant along the integral submanifolds of its subbundle. Moreover, if v is constant on M, then the integral submanifolds of V_v are totally geodesic.

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space form. Then, the curvature tensor R of M is given by (1). From (1), we can easily see that

$$QX = \{2nf_1 + 3f_2 - f_3\}X - \{3f_2 + (2n-1)f_3\}\eta(X)\xi, \quad (9)$$

$$\tau = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \quad (10)$$

Moreover, we can see that

$$LX = \{-\frac{1}{2}f_1 - \frac{3}{2(2n-1)}f_2\}X + \{\frac{3}{2n-1}f_2 + f_3\}\eta(X)\xi. \quad (11)$$

Therefore, the Weyl conformal curvature tensor C can be written as

$$\begin{aligned} C(X,Y)Z &= \frac{-3}{2n-1}f_2\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + 2g(X,\phi Y)\phi Z\} \quad (12) \\ &- \frac{3}{2n-1}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned}$$

The notion of the quasi-conformal curvature tensor was defined by Yano and Sawaki [8]. According to them a quasi-conformal curvature tensor is defined by

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z \\ &+b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &-\frac{\tau}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{13}$$

where a and b are constants, S is the Ricci tensor, Q is the Ricci operator and τ is the scalar curvature of the manifold M^{2n+1}. A Riemannian manifold (M^{2n+1}, g), $(n > 1)$, is called quasi-conformally flat if the quasi-conformal curvature tensor $\tilde{C} = 0$. If $a = 1$ and $b = \frac{-1}{2n-1}$, then the quasi-conformal curvature tensor is reduced to the Weyl conformal curvature tensor.

A Riemannian manifold is said to be quasi-conformally Weyl-symmetric manifold if

$$R(X,Y) \cdot \tilde{C} = 0,$$

where $\tilde{C}$ is the quasi-conformal curvature tensor.

On the other hand, from (1), we have

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \tag{14}$$

and

$$R(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}. \tag{15}$$

3. Quasi-Conformally Flat Generalized Sasakian-Space Forms

Theorem 3.1 *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space form. Then, we have the following: (i) If $n > 1$, then M is quasi-conformally flat if and only if $f_2 = -\frac{(a+(2n-1)b)}{3(an+b)}f_3$, (ii) If M is quasi-conformally flat and ξ is a Killing vector field, then it is flat, or of constant curvature, or locally the product $N^1 \times N^{2n}$, where N^1 is a 1-dimensional manifold and N^{2n} is a $2n$-dimensional almost Hermitian manifold of constant curvature. In any case, M is locally symmetric and has constant ϕ-sectional curvature.*

Proof Assume that $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian−space form. Using (1), (9), (10) and equation $S(X,Y) = g(QX,Y)$ in (13), we obtain

$$\begin{aligned}\tilde{C}(X,Y)Z &= \frac{1}{2n+1}[(-3a+6b)f_2 + (2a+2(2n-1)b)f_3]\{g(Y,Z)X - g(X,Z)Y\} \\ &+af_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+[(a+(2n-1)b)f_3 + 3bf_2]\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}.\end{aligned} \tag{16}$$

If $a = 1$ and $b = -\frac{1}{2n-1}$, then we obtain (13), that is, the quasi-conformal curvature tensor is reduced to the conformal curvature tensor.

Suppose that $M(f_1, f_2, f_3)$ is quasi-conformally flat and $n > 1$. Then, we have $\tilde{C} = 0$.

If we put $X = \phi Y$ in (16), then we find

$$\begin{aligned}
&\frac{1}{2n+1}[3(2b-a)f_2 + 2(a+(2n-1)b)f_3]\{g(Y,Z)\phi Y - g(\phi Y, Z)Y\} \\
&+af_2\{g(\phi Y, \phi Z)\phi Y - g(Y, \phi Z)\phi^2 Y + 2g(\phi Y, \phi Y)\phi Z\} \\
&+[(a+(2n-1)b)f_3 + 3bf_2]\{\eta(\phi Y)\eta(Z)Y - \eta(Y)\eta(Z)\phi Y \\
&+g(\phi Y, Z)\eta(Y)\xi - g(Y,Z)\eta(\phi Y)\xi\} = 0
\end{aligned} \tag{17}$$

or using (3) and (4) in (17), we obtain

$$\begin{aligned}
&\frac{1}{2n+1}[3(2b-a)f_2 + a(2n+1)f_2 \\
&+2(a+(2n-1)b)f_3]\{g(Y,Z)\phi Y - g(\phi Y, Z)Y\} \\
&+[af_2 + (a+(2n-1)b)f_3 + 3bf_2]\{-\eta(Y)\eta(Z)\phi Y - g(Y, \phi Z)\eta(Y)\xi\} \\
&+af_2\{2g(Y,Y)\phi Z - 2\eta(Y)\eta(Y)\phi Z\} = 0.
\end{aligned} \tag{18}$$

If we choose a unit vector U such that $g(U, \xi) = 0$ and put $Y = U$ in (18), then we have

$$\frac{1}{2n+1}[\{(2(n-1)a+6b)f_2+2(a+(2n-1)b)f_3\}\{g(U,Z)\phi U - g(\phi U, Z)U\} + 2(2n+1)af_2\phi Z] = 0. \tag{19}$$

Putting $Z = U$ in (19), we get

$$\{(2(n-1)a + 6b + 2(2n+1)a)f_2 + 2(a+(2n-1)b)f_3\}\phi U = 0.$$

Thus, we have

$$(2(n-1)a + 6b + 2(2n+1)a)f_2 + 2(a+(2n-1)b)f_3 = 0.$$

From this equation, we get

$$f_2 = -\frac{(a+(2n-1)b)}{3(an+b)}f_3. \tag{20}$$

Conversely, if $f_2 = -\frac{(a+(2n-1)b)}{3(an+b)}f_3$, then from (16), we have $\tilde{C}(X,Y)Z = 0$ and hence, $M(f_1, f_2, f_3)$ is quasi-conformally flat. Therefore, when $n > 1$, $M(f_1, f_2, f_3)$ is conformally flat if and only if $f_2 = -\frac{(a+(2n-1)b)}{3(an+b)}f_3$. Thus, the first part (i) of the Theorem 3.1 is proved.

For the proof of the second part (ii), we assume that $M(f_1, f_2, f_3)$ is quasi-conformally flat and ξ is Killing. Then, the Schouten tensor L of the manifold is a Codazzi tensor, that is,

$$(\nabla_X L)Y - (\nabla_Y L)X = 0 \tag{21}$$

for any vector fields X, Y on M. Also, if $n > 1$, then we have $f_2 = -\frac{(a+(2n-1)b)}{3(an+b)} f_3$ by the first part (i) and hence from (12), we obtain

$$LX = [-\frac{1}{2}f_1 + \frac{1}{2(na+b)}(\frac{a}{2n-1} + b)f_3]X + [\frac{(2n+1)(n-1)}{(2n-1)(na+b)}]af_3\eta(X)\xi. \tag{22}$$

Using (7), from (13), we get

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z - (2n-1)b[g(LY,Z)X - g(LX,Z)Y \\ &+ g(Y,Z)LX - g(X,Z)LY] \\ &- \frac{\tau}{2n(2n+1)}(a + (2n-1)b)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{23}$$

If $n = 1$, then from (23), we get

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z - b[g(LY,Z)X - g(LX,Z)Y \\ &+ g(Y,Z)LX - g(X,Z)LY] \\ &- \frac{\tau}{6}(a+b)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{24}$$

Since $M(f_1, f_2, f_3)$ is quasi-conformally flat, we can write $\tilde{C}(X,Y)Z = 0$, then we get

$$\begin{aligned} R(X,Y)Z &= \frac{b}{a}[g(LY,Z)X - g(LX,Z)Y \\ &+ g(Y,Z)LX - g(X,Z)LY] \\ &+ \frac{\tau}{6}\frac{(a+b)}{a}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{25}$$

for any vector fields X, Y, Z. In the 3-dimensional manifold $M(f_1, f_2, f_3)$, the Schouten tensor is given by (11),

$$LX = -\frac{1}{2}(f_1 + 3f_2)X + (3f_2 + f_3)\eta(X)\xi. \tag{26}$$

From (25) and (26), we obtain

$$\begin{aligned} R(X,Y)Z &= [f_1 + (\frac{a-2b}{a})f_2 - \frac{2}{3}(\frac{a+b}{a})f_3]\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{b}{a}(3f_2 + f_3)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\}. \end{aligned} \tag{27}$$

If we take

$$\begin{cases} f_1^* = f_1 + (\frac{a-2b}{a})f_2 - \frac{2}{3}(\frac{a+b}{a})f_3, \\ f_3^* = \frac{b}{a}(3f_2 + f_3), \end{cases} \tag{28}$$

then we can write

$$\begin{aligned} R(X,Y)Z &= f_1^*\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_3^*\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\}. \end{aligned}$$

Equation (26) gives

$$L\xi = (-\frac{1}{2}f_1 + \frac{3}{2}f_2 + f_3)\xi. \tag{29}$$

If X is a vector orthogonal to ξ, then we get

$$LX = -\frac{1}{2}(f_1 + 3f_2)X. \tag{30}$$

For $n > 1$, then from (22), we get

$$L\xi = -\frac{1}{2}[f_1 - \{\frac{1}{na+b}[(\frac{4n^2-2n-1}{2n-1})a + b]\}f_3]\xi. \tag{31}$$

If X is a vector orthogonal to ξ, then we have

$$LX = [-\frac{1}{2}f_1 + \frac{1}{2(na+b)}(\frac{a}{2n-1} + b)f_3]X. \tag{32}$$

Let $\xi, E_1, E_2, ..., E_{2n}$ be local orthonormal vector fields on $M(f_1, f_2, f_3)$. Then from (21), (22) and (32), we get

$$\begin{aligned} (\nabla_{E_i}L)E_j - (\nabla_{E_j}L)E_i &= -\frac{1}{2}(E_i f_1)E_j + \frac{1}{2}(E_j f_1)E_i \\ &+ \frac{1}{2(na+b)}(\frac{a}{2n-1} + b)[(E_i f_3)E_j - (E_j f_3)E_i] \\ &+ \frac{(2n+1)(n-1)}{(2n-1)(na+b)} a f_3 \eta(\nabla_{E_i}E_j - \nabla_{E_j}E_i)\xi = 0. \end{aligned} \tag{33}$$

Taking inner product with E_j in (33), we have

$$(E_j f_1) = \frac{1}{(na+b)}(\frac{a}{2n-1} + b)(E_j f_3). \tag{34}$$

Using (31), we obtain

$$(\nabla_{E_j} L)\xi + L\nabla_{E_j}\xi = -\frac{1}{2}\{f_1 - \frac{1}{(na+b)}[(\frac{4n^2-2n-1}{2n-1})a+b]f_3\}\nabla_{E_j}\xi$$
$$-\frac{1}{2}(E_jf_1)\xi + \frac{1}{2(na+b)}[(\frac{4n^2-2n-1}{2n-1})a+b](E_jf_3)\xi. \quad (35)$$

If we use (34) in (35), then we get

$$(\nabla_{E_j} L)\xi + L\nabla_{E_j}\xi = -\frac{1}{2}\{f_1 - \frac{1}{(na+b)}[(\frac{4n^2-2n-1}{2n-1})a+b]f_3\}\nabla_{E_j}\xi$$
$$+\frac{(2n+1)(n-1)}{(2n-1)(na+b)}a(E_jf_3)\xi. \quad (36)$$

Since $\nabla_{E_j}\xi$ is orthogonal to ξ, using (32), we get

$$L(\nabla_{E_j}\xi) = [-\frac{1}{2}f_1 + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)f_3]\nabla_{E_j}\xi. \quad (37)$$

Thus from (36), we obtain

$$(\nabla_{E_j} L)\xi = [\frac{(2n+1)(n-1)}{(2n-1)(na+b)}a]((E_jf_3)\xi + f_3\nabla_{E_j}\xi). \quad (38)$$

Since ξ is Killing, then we get

$$(\nabla_\xi L)E_j + L(\nabla_\xi E_j) = [-\frac{1}{2}\xi(f_1) + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)\xi(f_3)]E_j$$
$$+[-\frac{1}{2}f_1 + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)f_3]\nabla_\xi E_j, \quad (39)$$

where

$$L(\nabla_\xi E_j) = -\frac{1}{2}f_1\nabla_\xi E_j + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)f_3\nabla_\xi E_j. \quad (40)$$

Thus from (36), we have

$$(\nabla_\xi L)E_j = [-\frac{1}{2}\xi(f_1) + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)\xi(f_3)]E_j. \quad (41)$$

Since $(\nabla_{E_j} L)\xi = (\nabla_\xi L)E_j$, from (38) and (41), we get

$$[\frac{(2n+1)(n-1)}{(2n-1)(na+b)}a]((E_jf_3)\xi + f_3\nabla_{E_j}\xi) \quad (42)$$
$$= [-\frac{1}{2}\xi(f_1) + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)\xi(f_3)]E_j.$$

Taking inner product with E_j in (42), we obtain

$$\xi(f_1) = \frac{1}{(na+b)}(\frac{a}{2n-1}+b)\xi(f_3). \tag{43}$$

Taking inner product with ξ, from (42), we get

$$[\frac{(2n+1)(n-1)}{(2n-1)(na+b)}a]((E_j f_3)\xi + f_3 \nabla_{E_j} \xi) = 0, \tag{44}$$

this gives $E_j f_3 = 0$ and $f_3 \nabla_{E_j} \xi = 0$ $(j = 1, 2, ..., 2n)$. Combining this with $\nabla_\xi \xi = 0$ gives

$$f_3(\nabla_X \xi) = 0 \tag{45}$$

for any vector field X. From (45), we get

$$(Y f_3)(\nabla_X \xi) + f_3 \nabla_Y \nabla_X \xi = 0.$$

This equation and (45) give

$$(X f_3) \nabla_Y \xi - (Y f_3) \nabla_X \xi + f_3[\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi] = 0.$$

Multiplying this equation with f_3 and using (45), we get

$$f_3^2 R(X,Y)\xi = 0.$$

This equation and (14) give

$$f_3^2(f_1 - f_3)[\eta(Y)X - \eta(X)Y] = 0$$

from which we obtain $f_3(f_1 - f_3) = 0$.

Consider the case $f_1 = 0$. In this case, we have $f_3 = 0$ on M and hence, $f_2 = 0$. Thus, M is flat.

Next consider the case $f_1 \neq 0$. Differentiating $f_3(f_1 - f_3) = 0$ with ξ gives $\{f_1 + [\frac{1}{(na+b)}(\frac{a}{2n-1} + b) - 2]f_3\}\xi(f_3) = 0$. If $f_3(p) = 0$ at a point $p \in M$, then $f_1(p)\xi(f_3)(p) = 0$, where since $f_1 \neq 0$, we get $\xi(f_3) = 0$ at p. If $f_3(p) \neq 0$, then $f_3 = f_1$ in an open neighborhood U of p. Thus, $\{\frac{a(1+n-2n^2)}{(na+b)(2n-1)} f_3\}\xi(f_3) = 0$. For $n > 1$, since $1 + n - 2n^2 \neq 0$, we get $\xi(f_3) = 0$ on U. Thus, we have $\xi(f_3) = 0$ on M. Since $E_j f_3 = 0$ $(j = 1, 2, ..., 2n)$, f_3 is constant on M. Hence, we have:

(a) If $f_3 = 0$, then M is of constant curvature f_1.

(b) If $f_3 \neq 0$, then we have $f_1 = f_3$ and $\nabla_X \xi = 0$ for any vector X on M. Hence, the Schouten tensor L has two distinct constant eigenvalues $\frac{1}{2}f_1$ with multiplicity 1 and $-\frac{1}{2}f_1$ with multiplicity $2n$. Therefore, we have the decomposition $\mathcal{D} \oplus [\xi]$, where $\mathcal{D}$ is the distribution defined by $\eta = 0$ and $[\xi]$ is the distribution spanned by the vector ξ. By Lemma 2.1, $\mathcal{D}$ is integrable. Hence, M is locally product of an integral submanifold N^1 of $[\xi]$ and an integral submanifold N^{2n} of $\mathcal{D}$. Since the eigenvalue is constant on M, N^{2n} is a totally geodesic submanifold of M by Lemma 2.1. If we denote the restriction of ϕ in $\mathcal{D}$ by J, then

$$J^2X = \phi^2 X = -X + \eta(X)\xi = -X$$

for any $X \in \mathcal{D}$. Hence, J defines an almost complex structure on N^{2n}.

Also, $g'(JX, JY) = g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) = g'(X,Y)$ for any $X, Y \in \mathcal{D}$, where g' is the induced metric on N^{2n} from g. Hence, (N^{2n}, J, g') is an almost Hermitian manifold. Since N^{2n} is a totally geodesic hypersurface of M, the equation of Gauss is given by

$$R(X,Y)Z = R'(X,Y)Z$$

for any vector fields X, Y and Z tangent to N^{2n}, where R' is the curvature tensor of N^{2n}. Thus, we get

$$R'(X,Y)Z = f_1[g'(Y,Z)X - g'(X,Z)Y]$$

and hence, N^{2n} is a space of constant curvature f_1. In any case, from the above arguments, we can easily see that $M(f_1, f_2, f_3)$ is locally symmetric. Since f_1 and f_3 are constants, we can see that M is of constant ϕ-sectional curvature. This completes the proof of the Theorem 3.1. □

The above theorem was proved in another ways by Kim [5] and Sarkar and De [7].

Remark 3.2 *In the Theorem* 1, *the condition "ξ is Killing vector field" cannot be removed. For example, given (N, J, g) with constant curvature c, say, a* 6*-dimensional sphere with nearly Kaehler structure [6], the warped product $M = \mathbb{R} \times_f N$, where $f > 0$ is a nonconstant function on $\mathbb{R}$, can be endowed with an almost contact metric structure (ϕ, ξ, η, g_f).*

4. Quasi-Conformally Weyl-Symmetric Generalized Sasakian-Space Forms

Let us consider a quasi-conformally Weyl-symmetric generalized Sasakian-space form $M(f_1, f_2, f_3)$. Then, the condition

$$R(X,Y) \cdot \tilde{C} = 0$$

holds on $M(f_1, f_2, f_3)$ for every vector fields X, Y. Hence, we have

$$\begin{aligned}(R(X,Y)\cdot\tilde{C})(U,V)W &= R(X,Y)\tilde{C}(U,V)W - \tilde{C}(R(X,Y)U,V)W \\ &\quad -\tilde{C}(U,R(X,Y)V)W - \tilde{C}(U,V)R(X,Y)W = 0. \qquad (46)\end{aligned}$$

So, for $X = \xi$ in (46), we have

$$R(\xi, Y)\tilde{C}(U, V)W - \tilde{C}(R(\xi, Y)U, V)W$$
$$-\tilde{C}(U, R(\xi, Y)V)W - \tilde{C}(U, V)R(\xi, Y)W = 0. \tag{47}$$

From (15), we get

$$(f_1 - f_3)\{g(Y, \tilde{C}(U, V)W)\xi - \eta(\tilde{C}(U, V)W)Y - g(Y, U)\tilde{C}(\xi, V)W$$
$$+\eta(U)\tilde{C}(Y, V)W - g(Y, V)\tilde{C}(U, \xi)W + \eta(V)\tilde{C}(U, Y)W$$
$$-g(Y, W)\tilde{C}(U, V)\xi + \eta(W)\tilde{C}(U, V)Y\} = 0. \tag{48}$$

Taking the inner product of (48) with ξ, we obtain

$$(f_1 - f_3)\{g(Y, \tilde{C}(U, V)W) - \eta(\tilde{C}(U, V)W)\eta(Y) - g(Y, U)\eta(\tilde{C}(\xi, V)W)$$
$$+\eta(U)\eta(\tilde{C}(Y, V)W) - g(Y, V)\eta(\tilde{C}(U, \xi)W) + \eta(V)\eta(\tilde{C}(U, Y)W)$$
$$+\eta(W)\eta(\tilde{C}(U, V)Y)\} = 0. \tag{49}$$

Putting $Y = U$ in (49), we have

$$(f_1 - f_3)\{g(U, \tilde{C}(U, V)W) - \eta(\tilde{C}(U, V)W)\eta(U) - g(U, U)\eta(\tilde{C}(\xi, V)W)$$
$$+\eta(U)\eta(\tilde{C}(U, V)W) - g(U, V)\eta(\tilde{C}(U, \xi)W) + \eta(V)\eta(\tilde{C}(U, U)W)$$
$$+\eta(W)\eta(\tilde{C}(U, V)U\} = 0. \tag{50}$$

From (16), we get

$$\eta(\tilde{C}(X, Y)Z) = (\frac{a + (2n-1)b}{2n+1})[-3f_2 + (1-2n)f_3]\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}. \tag{51}$$

Putting $Z = \xi$, the equation (51) turns into the form

$$\eta(\tilde{C}(X, Y)\xi) = 0. \tag{52}$$

Thus, using (52) in (50), we obtain

$$(f_1 - f_3)\{g(U, \tilde{C}(U, V)W) - g(U, U)\eta(\tilde{C}(\xi, V)W)$$
$$-g(U, V)\eta(\tilde{C}(U, \xi)W) + \eta(W)\eta(\tilde{C}(U, V)U)\} = 0. \tag{53}$$

Let $\{e_i\}$, $1 \leq i \leq 2n+1$, $(e_{2n+1} = \xi)$ be an orthonormal basis of the tangent space at any point. Then, the sum for $U = e_i$, $1 \leq i \leq 2n+1$, of the relation (53) give us

$$(f_1 - f_3)\{g(e_i, \tilde{C}(e_i, V)W) - g(e_i, e_i)\eta(\tilde{C}(\xi, V)W)$$
$$-g(e_i, V)\eta(\tilde{C}(e_i, \xi)W) + \eta(W)\eta(\tilde{C}(e_i, V)e_i)\} = 0. \tag{54}$$

On the other hand, from (51), we have

$$\eta(\tilde{C}(\xi,V)W) = (\frac{a+(2n-1)b}{2n+1})[-3f_2+(1-2n)f_3]\{g(W,V)-\eta(W)\eta(V)\}. \tag{55}$$

Using (55) in (54), we get

$$(f_1-f_3)\{g(e_i,\tilde{C}(e_i,V)W)+2n(\frac{a+(2n-1)b}{2n+1})[3f_2+(1-2n)f_3]g(W,V)\}=0. \tag{56}$$

Also, from (16), we have

$$\begin{aligned}\tilde{C}(e_i,V)W &= \frac{1}{2n+1}[(-3a+6b)f_2+(2a+2(2n-1)b)f_3][g(W,V)e_i-g(W,e_i)V]\\ &+af_2[g(e_i,\phi W)\phi V-g(V,\phi W)\phi e_i+2g(e_i,\phi V)\phi W]\\ &+[(a+(2n-1)b)f_3+3bf_2][\eta(e_i)\eta(W)V-\eta(V)\eta(W)e_i\\ &+g(e_i,W)\eta(V)\xi-g(V,W)\eta(e_i)\xi].\end{aligned} \tag{57}$$

Taking the inner product of (57) with e_i, we get

$$g(\tilde{C}(e_i,V)W,e_i) = (\frac{a+(2n-1)b}{2n+1})(3f_2+(2n-1)f_3)[g(W,V)-(2n+1)\eta(W)\eta(V)]. \tag{58}$$

If we use (58) in (56), we get

$$(f_1-f_3)(a+(2n-1)b)(3f_2+(2n-1)f_3)[g(W,V)-\eta(W)\eta(V)]=0. \tag{59}$$

If $f_1 \neq f_3$ and $a \neq (2n-1)b$, then $3f_2+(2n-1)f_3=0$, that is,

$$f_2 = -\frac{(2n-1)}{3}f_3. \tag{60}$$

Hence, using (60) in (10), we obtain

$$\tau = 2n(2n+1)(f_1-f_3) \tag{61}$$

and using (60) in (9), we get

$$QX = 2n(f_1-f_3)X. \tag{62}$$

So, we have the following result:

Theorem 4.1 *Let $M(f_1,f_2,f_3)$ be a generalized Sasakian-space form. Then, M^{2n+1} $(n>1)$ is quasi-conformally Weyl-symmetric if and only if either $f_1=f_3$ or $f_2=-\frac{(2n-1)}{3}f_3$ (when $f_1 \neq f_3$), where $a \neq (2n-1)b$.*

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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