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Connected Square Network Graphs

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Abstract

In this study, connected square network graphs are introduced and two different definitions are given. Firstly, connected square network graphs are shown to be a Hamilton graph. Further, the labelling algorithm of this graph is obtained by using gray code. Finally, its topological properties are obtained, and conclusion are given.

1. Introduction

Nowadays, many types of interconnection network topologies have been extensively studied by researchers. The most popular of these are trees, cycles, grids, tori, meshes and hypercubes. A new network topology will be introduced in this study. This new network structure is obtained using squares and called the Connected Square Network Graph (CSNG). Two different definitions are given for the connected square network graph. Firstly, it is obtained by combining a finite number of squares in 2D space. Secondly, it is obtained recursively from square and compound cubes in the first way.

In the literature, hypercube, and its variants (Folded Hypercube, Crossed Cube and the Hierarchical Cubic Network) have been studied extensively in the interconnection network [1–9]. Karcı and Selçuk introduced new hypercube variants and investigated it's Hamilton-like features. These; Fractal Cubic Network Graph (FCNG) [10] uses the fractal structures and Connected Cubic Network Graph [11] uses hypercube. They investigated the topological properties of new hypercube variants.

Motivated by the [10] and [11], a new network structure will be defined in this study. The outline of this study is as follows. Section 2 informs basic information about graph theory and explains the definitions of CSNG. Section 3 investigates the analytical properties of CSNG and is obtained Hamiltonian properties of CSNG is obtained. Labelling algorithm for this graph is given in Section 4. In Section 5, topological features of CSNG are obtained and a projection for future work is presented.

2. Preliminaries

Rest of the study, $G=(V,E)$ is a graph where V is a vertex set and E is a edge set. (x,y) is an edge in E where $(x,y)\in G$. The degree of vertex $x\in V(G)$ is denoted by $\deg(x)$ and $d(x,y)$ is a shortest path from x to y in G.

"$||$" indicates the concatenation of two strings. The Hamming distance is $\sum_{i=0}^{n-1}(a_i\oplus b_i)$ since $\oplus$ is bitwise-XOR operation. $S(2)$ is denoted a square in $2D$ space. The $2D$ coordinate system is given below:

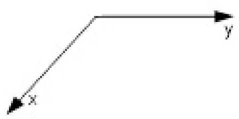

Figure 2.1: Two-dimensional coordinate space

Two different definitions be given to obtain these graphs, in this section.

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Definition 2.1. ($CSNG$): Let $CSNG(0,0) = S(2)$ (Fig 2.(a)). $CSNG(k,m)$ can be defined in two steps.

Case I. Construction in one direction

(i) Suppose $\sum_{i=0}^{m} 2^i$ squares with common two nodes (an edge) are connected along the y-axis. This graph will be called a $CSNG(0,m)$. For example, the mesh structure given in Figure 2(b)-(c) are $CSNG(0,1)$ and $CSNG(0,2)$.

(ii) Suppose $\sum_{i=0}^{k}$ squares with common two nodes (an edge) are connected along the x-axis. This graph will be called a $CSNG(k,0)$.

Case II. Construction in two directions

(i) Suppose $\sum_{i=0}^{m} 2^i$ $CSNG(k,0)$s with common two nodes (an edge) are connected along the y-axis. This graph will be called a CSNG(k,m).

(ii) Suppose $\sum_{i=0}^{k} 2^i$ $CSNG(0,m)$s with common lower and upper surfaces (one surface) are connected along the x-axis. This graph will be called a $CSNG(k,m)$.

$CSNG(0,0)$ is represented by $S(2)$ in the Figure 2.2-(a). In Figure 2.2-(b) (Figure 2.2-(c)), $CSNG(0,1)$ ($CSNG(0,2)$) is obtained by combining 3 (7) squares with one side in common. $CSNG(1,2)$ is obtained by combining 3-$CSNG(0,2)$s which have top and bottom horizontal surfaces to be in common in Figure 3.1-(b).

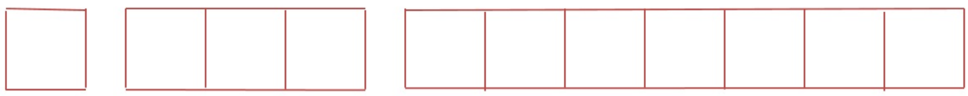

Figure 2.2: a. $CSNG(0,0)$, b. $CSNG(0,1)$, c. $CSNG(0,2)$, respectively

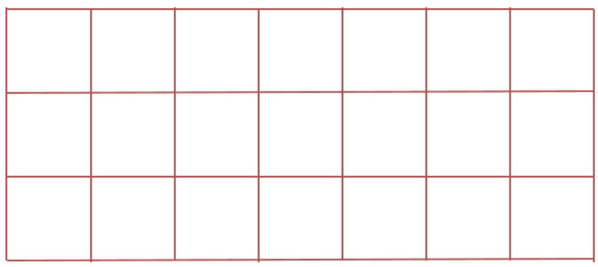

Figure 2.3: $CSNG(1,2)$

Definition 2.2. Two $CSNG(k,m-1)$s (or $CSNG(k-1,m)$s) can be merged to construct a new mesh of size doubling the size of $CSNG(k,m) = G(V,E), k \geq 0, m \geq 0$. There are two situations:

(i) If doubling dimension is x, then the nodes and edges in $0||CSNG(k-1,m)$ and $1||CSNG(k-1,m)$ are also included in $CSNG(k,m) = G(V_x,E_x)$. If $\forall v_i \in V,\ p = 0,...,k+m-1,\ 2^p \leq Label(v_i) \leq 2^p+1,\ |k-m| \leq 1$, then $\forall (0||v_i, 1||v_i) \in E_x$.

(ii) If doubling dimension is y, then the nodes and edges in $0||CSNG(k,m-1)$ and $1||CSNG(k,m-1)$ are also included in $CSNG(k,m) = G(V_y,E_y)$. If $\forall v_i \in V, Label(v_i)$ is even, $Label(v_i) < 2^{k+m}$, $|k-m| \leq 1$, then $\forall (0||v_i, 1||v_i) \in E_y$.

$CSNG(0,1)$ and $CSNG(0,2)$ can be constructed using definition 2.2-(i) in Fig. 2.4-(a) and Fig. 3.1-(a), respectively. Similarly, $CSNG(1,0)$ and $CSNG(1,2)$ can be constructed using definition 2.2-(ii) in Fig. 2.4-(b) and Fig. 3.2-(a), respectively.

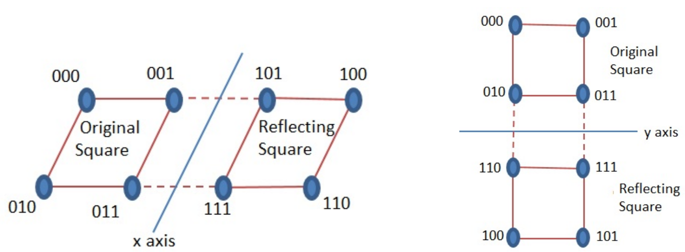

Figure 2.4: a. $CSNG(0,1)$, b. $CSNG(0,1)$, respectively

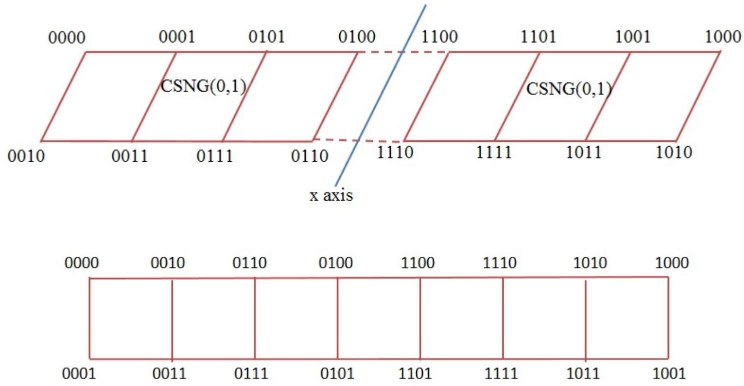

Figure 3.1: a. Construction of $CSNG(0,2)$ using Definiton 2.2, b. Labelling of $CSNG(0,2)$, respectively

3. Topological Features of Connected Square Network Graphs

3.1. Hamilton features of $CSNG(0,m)$ ($CSNG(k,0)$)

In this subsection, we analyzed Hamilton features of $CSNG(0,m)$ ($CSNG(k,0)$). Firstly, we give an example. $CSNG(0,2)$ is a Hamilton graph labelled with a 4-bit gray code in Fig. 3.1-(b).

Theorem 3.1. Suppose $\sum_{i=0}^{m} 2^i$-squares with common two nodes (an edge) are connected along the y-axis in definition 2.1-(a). This graph, $CSNG(0,m)$, has $3\times 2^{m+1}-2$ edges and 2^{m+2} nodes. Further, $CSNG(0,m)$ is a Hamilton graph and is labeled with a $m+2$-bit gray code.

Proof. The total node number of nodes of $CSNG(0,m)$ can be calculated by using definition 2.1-(a) and mathematical induction.

First Step: Let $m=2$. Suppose $\sum_{i=0}^{2} 2^i = 7$-squares with common two nodes (an edge) are connected along the y-axis. The total number node is along the y-axis $2\left(\sum_{i=0}^{2} 2^i+1\right)=2^{2+2}$.

Hypothesis Step: Let $m=n-1$. Suppose $\sum_{i=0}^{n-1} 2^i$-squares with common two nodes (an edge) are connected along the y-axis. Assume that $CSNG(0,n-1)$ has 2^{n+1} nodes.

Final Step: Let $m=n$. Suppose $\sum_{i=0}^{n} 2^i$-squares with common two nodes (an edge) are connected along the y-axis. The following equation applies for the proof of final step:

$$\sum_{i=0}^{n} 2^i = \sum_{i=0}^{n-1} 2^i + 2^n.$$

$CSNG(0,n)$ is obtained by adding 2^n $S(2)$ to the $CSNG(0,n-1)$ with 2 edges in common. Namely,

$$\left(\sum_{i=0}^{n} 2^i\right) S(2) = \left(\sum_{i=0}^{n-1} 2^i\right) S(2) + 2^n S(2),$$

$$CSNG(0,n) = CSNG(0,n-1) + 2^n S(2).$$

Hence, total node number of $CSNG(0,n)$ is $2^{n+1}+2^n\times 2 = 2^{n+2}$.

Secondly, the total number edge is along the x-axis $2\sum_{i=0}^{m} 2^i = 2\left(2^{m+1}-1\right) = 2.2^{m+1}-2$. The total number edge is along the y-axis $\sum_{i=0}^{m} 2^i + 1 = 2^{m+1}$. Total edge number of $CSNG(0,m)$ is $3.2^{m+1}-2$.

Finally, we showed that $CSNG(0,m)$ is a Hamilton graph. Mathematical induction will be used for proof.

First Step: Let $m=2$. $CSNG(0,2)$ is a Hamilton graph which is labelled with help of 4-bit Gray code seen in Fig. 3.1-(b).

Hypothesis Step: Let $m=n-1$. Suppose $CSNG(0,n-1)$ is a Hamilton graph which is labelled with help of $n+1$-bit Gray code and has 2^{n+1} nodes.

Final Step: Let $m=n$. The following equality is obtained

$$CSNG(0,n) = 0||CSNG(0,n-1) \cup 1||CSNG(0,n-1)$$

since $CSNG(0,n)$ has $2^{n+2}=2.2^{n+1}$ nodes. Suppose x_i and x_j are two nodes in $CSNG(0,n-1)$ and $x_i \oplus x_j = 1$. The edges $(0||Label(x_i), 1||Label(x_i))$ and $(0||Label(x_j), 1||Label(x_j))$ are in $CSNG(0,n-1)$ and they are in Hamilton circuit in $CSNG(0,n)$. Namely, $CSNG(0,n)$ is a Hamilton graph which is labelled with help of $n+2$-bit Gray code and has 2^{n+2} nodes.

Similar results can be obtained in $CSNG(k,0)$.

3.2. Hamilton features of $CSNG(k,m)$

In this subsection, we analyzed Hamilton features of $CSNG(k,m)$. Firstly, we give an example. $CSNG(1,2)$ is a Hamilton graph labelled with a 5-bit gray code in Fig. 3.2-(b).

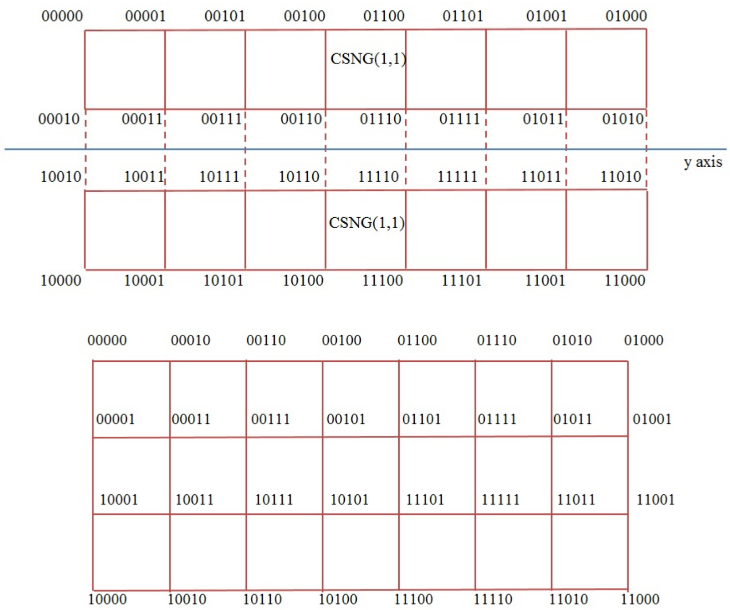

Figure 3.2: a. Construction of $CSNG(1,2)$ using Definiton 2.2, b. Labeling of $CSNG(1,2)$

Theorem 3.2. Suppose $\sum_{i=0}^{k}$-$CSNG(0,m)$s with common lower and upper surfaces (one surface) are connected along the x-axis. This graph will be called a $CSNG(k,m)$. $CSNG(k,m)$ has 2^{k+m+2} nodes and $2^{k+m+3}-2^{m+1}-2^{k+1}$ edges. Further, $CSNG(k,m)$ is a Hamilton graph and is labeled with a $k+m+2$-bit gray code.

Proof. Firstly, $CSNG(k,m)$ is consist of $\sum_{i=0}^{k}$-$CSNG(0,m)$s. Besides, $CSNG(0,m)$ is consist of $\sum_{j=0}^{m}$-$CSNG(0,0)$s where $CSNG(0,0)$ is a $S(2)$ square. Hence,

$$CSNG(k,m)=\sum_{i=0}^{k}CSNG(0,m)=\sum_{i=0}^{k}\sum_{j=0}^{m}CSNG(0,0)S(2).$$

Node numbers of $CSNG(k,m)$ is

$$\left(\sum_{i=0}^{k}2^{i}+1\right)\left(\sum_{j=0}^{m}2^{j}+1\right)=2^{k+1}2^{m+1}=2^{k+m+2}.$$

Because there are $\sum_{i=0}^{k}2^{i}+1$ nodes along the x-axis and $\sum_{j=0}^{m}2^{j}+1$ nodes along the y-axis.

Secondly, total number of edges along the x-axis is $\left(\sum_{i=0}^{m}2^{i}\right)\left(\sum_{j=0}^{k}2^{j}+1\right)$ and, total number of edges along the y-axis is $\left(\sum_{i=0}^{k}2^{i}\right)\left(\sum_{j=0}^{m}2^{j}+1\right)$.

Total number of edges of $CSNG(k,m)$ is

$$\left(\sum_{i=0}^{m}2^{i}\right)\left(\sum_{j=0}^{k}2^{j}+1\right)+\left(\sum_{i=0}^{k}2^{i}\right)\left(\sum_{j=0}^{m}2^{j}+1\right)=(2^{m+1}-1)2^{k+1}+(2^{k+1}-1)2^{m+1}=2^{k+m+3}-2^{m+1}-2^{k+1}.$$

A similar proof of Theorem 3.1 can be done to show that $CSNG(k,m)$ is a Hamilton graph.

4. Labelling Algorithm

In this section, an algorithm will be designed to label $CSNG(k,m)$ with the help of the reference [12].

Example 4.1. Let $k=1,m=2$ and $S=\{00\ 01\ 11\ 10\}$, $inv\_S=\{10\ 11\ 01\ 00\}$. Assume that $CSNG(0,0)=S$, $inv\_CSNG(0,0)=inv\_S$ where inv_CSNG is reverse sorting of $CSNG$. It can be calculation for $k+m=3$ iteration.

1. Iteration ($k=0,m=1$):

$$\begin{aligned}CSNG(0,1)&=0||CSNG(0,0)\cup 1||inv\_CSNG(0,0)\\&=0||\{00\ 01\ 11\ 10\}\cup 1||\{10\ 11\ 01\ 00\}\\&=\{000\ 001\ 011\ 010\ 110\ 111\ 101\ 100\}\end{aligned}$$

and

$$\begin{aligned}inv\_CSNG(0,1)&=1||CSNG(0,0)\cup 0||inv\_CSNG(0,0)\\&=1||\{00011110\}\cup 0||\{10110100\}\\&=\{100\ 101\ 111\ 110\ 010\ 011\ 001\ 000\}\end{aligned}$$

2. Iteration ($k=0, m=2$):

$$\begin{aligned} CSNG(0,2) &= 0||CSNG(0,1)\cup 1||inv\_CSNG(0,1) \\ &= 0||\{000\ 001\ 011\ 010\ 110\ 111\ 101\ 100\}\cup 1||\{100\ 101\ 111\ 110\ 010\ 011\ 001\ 000\} \\ &= \{0000\ 0001\ 0011\ 0010\ 0110\ 0111\ 0101\ 0100\ 1100\ 1101\ 1111\ 1110\ 1010\ 1011\ 1001\ 1000\} \end{aligned}$$

and

$$\begin{aligned} inv\_CSNG(0,2) &= 1||CSNG(0,1)\cup 0||inv\_CSNG(0,1) \\ &= 1||\{000\ 001\ 011\ 010\ 110\ 111\ 101\ 100\}\cup 0||\{100\ 101\ 111\ 110\ 010\ 011\ 001\ 000\} \\ &= \{1000\ 1001\ 1011\ 1010\ 1110\ 1111\ 1101\ 1100\ 0100\ 0101\ 0111\ 0110\ 0010\ 0011\ 0001\ 0000\} \end{aligned}$$

3. Iteration ($k=1, m=2$).

$$\begin{aligned} CSNG(1,2) &= 0||CSNG(0,2)\cup 1||inv\_CSNG(0,2) \\ &= 0||\{0000\ 0001\ 0011\ 0010\ 0110\ 0111\ 0101\ 0100\ 1100\ 1101\ 1111\ 1110\ 1010\ 1011\ 1001\ 1000\}\cup \\ &\quad 1||\{1000\ 1001\ 1011\ 1010\ 1110\ 1111\ 1101\ 1100\ 0100\ 0101\ 0111\ 0110\ 0010\ 0011\ 0001\ 0000\} \end{aligned}$$

That is, labelling of nodes of $CSNG(1,2)$ is

00000 00001 00011 00010 00110 00111 00101 00100
01100 01101 01111 01110 01010 01011 01001 01000
11000 11001 11011 11010 11110 11111 11101 11100
10100 10101 10111 10110 10010 10011 10001 10000.

Remark 4.2. The Algorithm 1 finds the labeling of $CSNG(k,m)$ using recursive process. The running time of the Algorithm 1 is $O(p)$ where $p=\max(k,m)$. (Algorithm 1 in Appendix)

5. Comparison Results

Connected square network graphs are scalable. It has been shown that $CSNG(k,m)$ is an Hamiltonian graph and is not an Euler graph. $CSNG(0,m)$ (or $CSNG(k,0)$) has nodes with 2 and 3 are degree nodes and total node number of is 2^{m+2} in Table 1. $CSNG(k,m)$ has nodes with $2,3$ and 4 are degree nodes and total node number of is 2^{k+m+2} in Table 2. The edge-node relationship for $CSNG(0,m)$ and $CSNG(k,m)$ is given in Table 3 and Table 4, respectively. (Tables in Appendix)

Remark 5.1. (see [13]) Sum connectivity-index of $CSNG(0,m)$ is calculated as follows

$$\begin{aligned} \chi_\alpha(G) &= \sum_{(x,y)\in E}(\deg x+\deg y)^\alpha \\ &= 2.4^\alpha+4.5^\alpha+(2^{k+m+3}-2^{m+1}-2^{k+1}-6)6^\alpha \end{aligned}$$

and sum connectivity-index of $CSNG(k,m)$ is calculated as follows

$$\begin{aligned} \chi_\alpha(G) &= \sum_{(x,y)\in E}(\deg x+\deg y)^\alpha \\ &= 8.5^\alpha+(2^{m+2}+2^{k+2}-12)6^\alpha+(2^{m+2}+2^{k+2}-8)7^\alpha+(2^{k+m+3}-2^{m+1}-2^{k+1}-2^{m+3}-2^{k+3}+12)8^\alpha \end{aligned}$$

where $\alpha\in R$.
The general Randic index $R_\alpha(G)$ of $CSNG(0,m)$ is calculated as follows

$$\begin{aligned} R_\alpha(G) &= \sum_{(x,y)\in E}(\deg x\deg y)^\alpha \\ &= 2.4^\alpha+4.6^\alpha+(2^{k+m+3}-2^{m+1}-2^{k+1}-6)9^\alpha \end{aligned}$$

and, the general Randic index $R_\alpha(G)$ of $CSNG(k,m)$ is calculated as follows

$$\begin{aligned} R_\alpha(G) &= \sum_{(x,y)\in E}(\deg x\deg y)^\alpha \\ &= 8.6^\alpha+(2^{m+2}+2^{k+2}-12)9^\alpha+(2^{m+2}+2^{k+2}-8)12^\alpha+(2^{k+m+3}-2^{m+1}-2^{k+1}-2^{m+3}-2^{k+3}+12)16^\alpha \end{aligned}$$

where $\alpha=-1,-1/2,1/2,1$.

6. Conclusion

In this paper, connected square network graphs are introduced. Two different definitions are given to obtain connected square network graphs. The topological properties of these graphs have been investigated and it has been proven to be a Hamilton graph. These graphs can be thought of as a hypercube variant. A labeling algorithm is given that reinforces this idea.

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Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

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7. Appendix

Algorithm 1: This algorithm calculate labelled of $CSNG(k,m)$.

Data: k, m, $S = \{00\ 01\ 11\ 10\}$, $inv\_S = \{10\ 11\ 01\ 00\}$
Result: labelled of $CSNG(k,m)$

begin
- $CSNG(0,0) = S$
- $inv\_CSNG(0,0) = inv\_S$
- **for** $j = 1$ *to* m **do**
  - $CSNG(0,j) = 0||CSNG(0,j-1) \cup 1||inv\_CSNG(0,j-1)$
  - $inv\_CSNG(0,j) = 1||CSNG(0,j-1) \cup 0||inv\_CSNG(0,j-1)$
- **for** $i = 1$ *to* k **do**
  - $CSNG(i,j) = 0||CSNG(i-1,j) \cup 1||inv\_CSNG(i-1,j)$
  - $inv\_CSNG(i,j) = 1||CSNG(i-1,j) \cup 0||inv\_CSNG(i-1,j)$
- return $CSNG(i,j)$

Table 1: The number of degree of nodes of $CSNG(0,m)$

$deg(2)$	$deg(3))$	Total Node
4	$2^{m+2}-4$	2^{m+2}

Table 2: The number of degree of nodes of $CSNG(k,m)$

$deg(2)$	$deg(3))$	$deg(4))$	Total Node
4	$2^{m+2}+2^{k+2}-8$	$2^{k+m+2}-2^{m+2}-2^{k+2}+4$	2^{k+m+2}

Table 3: The number of the edges of $CSNG(0,m)$

$(deg(2),deg(2))$	$(deg(2),deg(3))$	$(deg(3),deg(3))$	Total Edge
2	4	$2^{k+m+3}-2^{m+1}-2^{k+1}-6$	$2^{k+m+3}-2^{m+1}-2^{k+1}$

Table 4: The number of the edges of $CSNG(k,m)$

$(deg(2),deg(3))$	$(deg(3),deg(3))$	$(deg(3),deg(4))$	$(deg(4),deg(4))$	Total Edge
8	$2^{m+2}+2^{k+2}12$	$2^{m+2}+2^{k+2}-8$	$2^{k+m+3}-2^{m+1}-2^{k+1}-2^{m+3}-2^{k+3}+12$	$2^{k+m+3}-2^{m+1}-2^{k+1}$