International Electronic Journal of Algebra
Volume 31 (2022) 38-48
DOI: 10.24330/ieja.1058393

COREGULAR SEQUENCES AND TOP LOCAL HOMOLOGY MODULES

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Received: 7 August 2020; Accepted: 8 October 2021
Communicated by A. Çiğdem Özcan

Abstract. In this paper, we show that if M is a non-zero Artinian R-module and $\underline{x} := x_1, \ldots, x_n$ is an M-coregular sequence, then $x_1, \ldots, x_n$ is a $D(H_n^{\underline{x}}(M))$-coregular sequence. Moreover, if R is complete with respect to I-adic topology and $d = \mathrm{Ndim} M$, then $\dim H_d^I(M) \leq d$ and $\mathrm{depth} H_I^d(M) \geq \min\{2, d\}$ whenever $H_d^I(M) \neq 0$.

Mathematics Subject Classification (2020): 13C14, 13C15, 13D07
Keywords: Coregular sequence, local homology, Co-Cohen-Macaulay module

1. Introduction

In this paper, R is a Noetherian commutative ring with identity, I is an ideal of R and M is an R-module. The theory of local homology was initiated by Matlis [10] in 1974. This theory was studied and improved by Simon [15], Greenlees and May [8], Tarrío, López and Lipman [17]. In [4], Cuong and Nam defined the local homology modules $H_i^I(M)$ of M with respect to I by

$$H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(R/I^t, M).$$

This concept in some sense is dual to Grothendiecks definition of local cohomology modules. Also, this definition of local homology modules coincides with the definition of Greenlees and May [8] when M is an Artinian R-module.

The aim of this paper is to provide some properties of coregular sequences which are used in investigation top local homology modules. This work is inspired by some results in [2]. We see in Theorem 3.7 that if M is a non-zero Artinian R-module and $\underline{x} := x_1, \ldots, x_n$ is an M-coregular sequence, then $x_1, \ldots, x_n$ is a $D(H_n^{\underline{x}}(M))$-coregular sequence. Let M be an Artinian R-module of Noetherian dimension n and I an ideal of R, the top local homology module $H_n^I(M)$ is not finitely generated in general. Under condition R is complete with respect to I-adic topology, we shall prove in Theorem 3.12 that $\dim H_n^I(M) \leq n$ and $\mathrm{depth} H_I^n(M) \geq \min\{2, n\}$ whenever $H_n^I(M) \neq 0$.

Throughout, for any ideal I of R, the set $\{m \in M \mid Im = 0\}$ is denoted by $0 :_M I$. For each R-module M, we denote by $\mathrm{Coass}_R M$ the set of all coassociated prime ideals of M (see [18]). Also, we say that R is complete with respect to I-adic topology if $R \cong H_0^I(R)$. If $(R, \mathfrak{m})$ is a local ring and M is an R-module, then the Matlis duality of M is denoted by $D(M)$. The projective dimension and flat dimension of R-module M are denoted by $\mathrm{pd}_R\, M$ and $\mathrm{fd}_R M$, respectively. Many basic properties of local homology modules can be found in [4].

2. Preliminaries

First, we recall the definition of an M-coregular sequence. The concept of coregular sequence was introduced by Matlis [10] as a natural dual of the concept regular sequence.

Definition 2.1. ([10]) *Let M be an R-module. An element $x \in R$ is called coregular for M (or M-coregular) if $M = xM$.*

A sequence $x_1, x_2, \dots, x_n \in R$ is called M-coregular sequence if

(i) *x_i is $0 :_M (x_1, x_2, \dots x_{i-1})$-coregular for all $i = 1, 2, \dots, n$;*

(ii) $0 :_M (x_1, x_2, \dots x_n) \neq 0$.

We denote by $\mathrm{Width}_I(M)$ the supremum of the lengths of all maximal M-coregular sequences in I. If M is an Artinian R-module such that $0 :_M I \neq 0$, then we see in [13] that

$$\mathrm{Width}_I(M) = \inf\{i \mid \mathrm{Tor}_i^R(R/I, M) \neq 0\} < \infty.$$

Moreover, it follows from [6] that

$$\mathrm{Width}_I(M) = \inf\{i \mid H_i^I(M) \neq 0\}.$$

We now recall the concept of Noetherian dimension $\mathrm{Ndim} M$ of an R-module which was introduced by Roberts [14] with the terminology Krull dimension. Kirby [9] changed the terminology of Roberts and referred to Noetherian dimension to avoid confusion with the well-known Krull dimension for finitely generated modules.

Definition 2.2. ([14]) *The Noetherian dimension of M, denoted by* $\mathrm{Ndim} M$, *is defined inductively as follows.*

When $M = 0$ we define $\mathrm{Ndim} M = -1$. *Then by induction, for any integer $r \geq 0$, we define* $\mathrm{Ndim} M = r$ *when*

(i) $\mathrm{Ndim} M < r$ *is false, and*

(ii) *for every ascending chain $M_1 \subseteq M_2 \subseteq \dots$ of submodules of M there exists an integer n_0 such that* $\mathrm{Ndim}(M_{n+1}/M_n) < r$ *for all $n \geq n_0$.*

Thus M is non-zero and finitely generated if and only if $\mathrm{Ndim}M = 0$. If M is Artinian module, then $\mathrm{Ndim}M < \infty$ (see [14]). Also, if M is an Artinian R-module with $\mathrm{Ndim}M = d > 0$ and $x \in R$ is an M-coregular element, then $\mathrm{Ndim}(0 :_M x) = d - 1$ (see [7]). Cuong and Nam [4] showed that if M is an Artinian R-module, then $H_i^I(M) = 0$ for all $i > \mathrm{Ndim}M$. Moreover, if $(R, \mathfrak{m})$ is a local ring and M is non-zero, then $H^{\mathfrak{m}}_{\mathrm{Ndim}M}(M) \neq 0$.

In [16], the authors introduced the concept of Co-Cohen-Macaulay modules by comparing the Noetherian dimension and width for Artinian modules over a local ring.

Definition 2.3. *Let* $(R, \mathfrak{m})$ *be a local ring. An Artinian module* M *is called Co-Cohen-Macaulay if* $\mathrm{Ndim}M = \mathrm{Width}_{\mathfrak{m}}(M)$.

It is clear that if M is a non-zero Co-Cohen-Macaulay R-module, then $H_i^{\mathfrak{m}}(M) = 0$ for all $i \neq \mathrm{Ndim}M$.

3. Results

First, we will study the coregular sequences in connection with local homology modules and homological functor.

Proposition 3.1. *Let* M *be a non-zero Artinian* R*-module and* $\underline{x} := x_1, \ldots, x_n$ *a maximal* M*-coregular sequence. Then*

$$\mathrm{Tor}_i^R(R/\underline{x}, H_n^{\underline{x}}(M)) \cong \mathrm{Tor}_{n+i}^R(R/\underline{x}, M)$$

for all $i \geq 0$.

Proof. It follows from [12, Lemma 3.1(i)] that there is a Grothendieck spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^R(R/\underline{x}, H_q^{\underline{x}}(M)) \underset{p}{\Rightarrow} \mathrm{Tor}_{p+q}^R(R/\underline{x}, M).$$

Let $i \geq 0$ an integer, there exists a filtration Φ of submodules of $H_{n+i} = \mathrm{Tor}_{n+i}^R(R/\underline{x}, M)$

$$0 = \Phi^{-1}H_{n+i} \subseteq \ldots \subseteq \Phi^{n+i-1}H_n \subseteq \Phi^{n+i}H_{n+i} = H_{n+i}$$

such that

$$E^{\infty}_{j,n+i-j} \cong \Phi^j H_{n+i}/\Phi^{j-1}H_{n+i}$$

for all $0 \leq j \leq n + i$. It should be aware that $H_i^{\underline{x}}(M) = 0$ for all $i \neq n$. This indicates $E^{\infty}_{p,q} = 0$ for all $p \geq 0, q \neq n$ and $E^2_{i,n} = E^{\infty}_{i,n}$ for all $i \geq 0$. Consequently, one gets that

$$\Phi^{i-1}H_{n+i} = \ldots = \Phi^0 H_{n+i} = \Phi^{-1}H_{n+i} = 0$$

and

$$\Phi^i H_{n+i} = \ldots = \Phi^{i+n} H_{n+i} = H_{n+i}.$$

Thus, there is an isomorphism $\mathrm{Tor}_i^R(R/\underline{x}, H_n^{\underline{x}}(M)) \cong \mathrm{Tor}_{n+i}^R(R/\underline{x}, M)$, as required. □

In [18], a prime ideal $\mathfrak{p}$ is called coassociated to a non-zero R-module M if there is an Artinian homomorphic image T of M with $\mathfrak{p} = \mathrm{Ann}\, T$. The set of coassociated primes of M is denoted by $\mathrm{Coass}_R M$.

Corollary 3.2. *Let M be a non-zero Artinian R-module and $\underline{x} := x_1, \ldots, x_n$ a maximal M-coregular sequence. Then $\mathrm{Coass}_R H_n^{\underline{x}}(M)$ is a finite set.*

Proof. Proposition 3.1 shows that

$$\mathrm{Tor}_0^R(R/\underline{x}, H_n^{\underline{x}}(M)) \cong \mathrm{Tor}_n^R(R/\underline{x}, M).$$

Combining [5, Theorem 4.2(i)] with [5, Corollary 3.4], we realize that $\mathrm{Coass}_R H_n^{\underline{x}}(M) \subseteq V(\underline{x})$. By [18, Theorem 1.21],

$$\begin{aligned}\mathrm{Coass}_R(\mathrm{Tor}_0^R(R/\underline{x}, H_n^{\underline{x}}(M))) &= V(\underline{x}) \cap \mathrm{Coass}_R H_n^{\underline{x}}(M) \\ &= \mathrm{Coass}_R H_n^{\underline{x}}(M).\end{aligned}$$

Since M is an Artinian R-module, the set $\mathrm{Coass}_R \mathrm{Tor}_n^R(R/\underline{x}, M)$ is finite. Thus, the assertion follows from the above isomorphism. □

If M is a finitely generated module over a local ring $(R, \mathfrak{m})$, then we know that

$$\mathrm{pd}_R M = \sup\{i \mid \mathrm{Tor}_i^R(R/\mathfrak{m}, M) \neq 0\}.$$

In the case where M is an Artinian R-module, the flat dimension of M is defined as (see [1])

$$\mathrm{fd}_R M = \sup\{i \mid \mathrm{Tor}_i^R(R/\mathfrak{m}, M) \neq 0\}.$$

By using two above results, we have the following consequence.

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a complete local ring with respect to $\mathfrak{m}$-adic topology, M a non-zero Artinian R-module. Assume that $x_1, \ldots, x_n$ is a maximal M-coregular sequence such that $\mathfrak{m} = (x_1, \ldots, x_n)$. Then the following statements hold:*

(i) *$\mathrm{pd}_R H_n^{\mathfrak{m}}(M)$ is finite if and only if $\mathrm{fd}_R M$ is finite.*

(ii) *If $\mathrm{fd}_R M < \infty$, then $\mathrm{pd}_R H_n^{\mathfrak{m}}(M) + n = \mathrm{fd}_R M$.*

Proof. It follows from [4, Proposition 4.6] that $H_n^{\mathfrak{m}}(M)$ is a finitely generated R-module. Combining the hypothesis with the isomorphism in Proposition 3.1, one obtains that

$$\mathrm{Tor}_i^R(R/\mathfrak{m}, H_n^{\mathfrak{m}}(M)) \cong \mathrm{Tor}_{n+i}^R(R/\mathfrak{m}, M).$$

Consequently, the assertions follow from the above facts. □

Before, stating and proving the first main result of this paper, we need some lemmas which provide us many properties of coregular sequences.

Lemma 3.4. *Let $\underline{x} := x_1, \ldots, x_n \in R$ be an A-coregular sequence and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ an exact sequence. Then there is an exact sequence*

$$0 \to 0 :_A \underline{x} \to 0 :_B \underline{x} \to 0 :_C \underline{x} \to 0 :_D \underline{x}.$$

Proof. We need only consider the case where x is an A-coregular element. Consider the chain of homomorphisms

$$0 \to 0 :_A x \xrightarrow{\overline{f}} 0 :_B x \xrightarrow{\overline{g}} 0 :_C x \xrightarrow{\overline{h}} 0 :_D \underline{x},$$

where $\overline{f}, \overline{g}, \overline{h}$ are restrictions of f, g, h respectively. It remains to prove that the chain of homomorphisms is exact at $0 :_C x$. Let $c \in \mathrm{Ker}\,\overline{h}$, then $0 = \overline{h}(c) = h(c)$. There exists an element $b \in B$ such that $g(b) = c$ and $xg(b) = xc = 0$. Therefore, we have $a \in A$ satisfying $f(a) = xb$. Since x is an A-coregular element, one gets $a' \in A$ such that $a = xa'$. This implies that $b - f(a') \in 0 :_B x$ and $\overline{g}(b - f(a')) = c$. Consequently, we have $c \in \mathrm{Im}\,\overline{g}$, and the proof is complete. □

It should be pointed out that $0 :_A \underline{x} \cong \mathrm{Hom}(R/\underline{x}, M)$. Therefore, we immediately have a consequence.

Corollary 3.5. *Let $\underline{x} := x_1, \ldots, x_n \in R$ be an M-coregular sequence. Then*

$$\mathrm{Ext}_R^1(R/\underline{x}, M) = 0.$$

Combining [11, Proposition 1.1.7] with [5, Proposition 4.1], there is a Mayer-Vietoris sequence.

Lemma 3.6. *Let I, J be two ideals of R and M an Artinian R-module. Then there is an exact sequence*

$$\ldots \to H_i^{I\cap J}(M) \to H_i^I(M) \oplus H_i^J(M) \to H_i^{I+J}(M) \to H_{i-1}^{I\cap J}(M) \to \cdots.$$

Now, the first main result relates to the Matlis duality of top local homology modules and coregular sequences.

Theorem 3.7. *Let M be a non-zero Artinian R-module. Assume that $\underline{x} := x_1, \ldots, x_n$ is an M-coregular sequence. Then $x_1, \ldots, x_n$ is a $D(H_n^{\underline{x}}(M))$-coregular sequence.*

Proof. The proof is by induction on n. Let $n = 1$, the short exact sequence

$$0 \to 0 :_M x_1 \to M \xrightarrow{x_1} M \to 0$$

induces the following exact sequence

$$0 \to H_1^{(x_1)}(0 :_M x_1) \to H_1^{(x_1)}(M) \xrightarrow{x_1} H_1^{(x_1)}(M) \to H_0^{(x_1)}(0 :_M x_1) \to 0.$$

It is clear that $H_0^{(x_1)}(0 :_M x_1) \cong 0 :_M x_1$ and $H_1^{(x_1)}(0 :_M x_1) = 0$. Hence, there is a short exact sequence

$$0 \to H_1^{(x_1)}(M) \xrightarrow{x_1} H_1^{(x_1)}(M) \to 0 :_M x_1 \to 0$$

which yields the following exact sequence

$$0 \to D(0 :_M x_1) \to D(H_1^{(x_1)}(M)) \xrightarrow{x_1} D(H_1^{(x_1)}(M)) \to 0.$$

Since x_1 is an M-coregular sequence, if follows that $D(0 :_M x_1) \neq 0$. Thus, x_1 is a $D(H_1^{(x_1)}(M))$-coregular sequence.

Now, assume that $n > 1$ and the claim is true for the cases are less than n. The short exact sequence

$$0 \to 0 :_M x_1 \to M \xrightarrow{x_1} M \to 0$$

leads the long exact sequence

$$0 \to H_n^{\underline{x}}(0 :_M x_1) \to H_n^{\underline{x}}(M) \xrightarrow{x_1} H_n^{\underline{x}}(M) \to H_{n-1}^{\underline{x}}(0 :_M x_1) \to 0.$$

Lemma 3.6 gives us a long exact sequence

$$\ldots H_n^I(0 :_M x_1) \oplus H_n^J(0 :_M x_1) \to H_n^{I+J}(0 :_M x_1) \to H_{n-1}^{I\cap J}(0 :_M x_1) \to \cdots,$$

where $I = (x_1)$ and $J = (x_2, \ldots, x_n)$. It is clear that $H_i^I(0 :_M x_1) = 0$ for all $i > 1$. By [6, Theorem 3.8], we see that $H_i^{I\cap J}(0 :_M x_1) = 0$ for all $i > 0$. Hence, there is an isomorphism

$$H_i^{(x_2,\ldots,x_n)}(0 :_M x_1) \cong H_i^{(x_1,\ldots,x_n)}(0 :_M x_1)$$

for all $i \geq 2$. This implies that

$$H_n^{(x_1,\ldots,x_n)}(0 :_M x_1) \cong H_n^{(x_2,\ldots,x_n)}(0 :_M x_1) = 0.$$

Consequenlty, there is a short exact sequence

$$0 \to H_n^{\underline{x}}(M) \xrightarrow{x_1} H_n^{\underline{x}}(M) \to H_{n-1}^{\underline{x}}(0 :_M x_1) \to 0$$

which induces another exact sequence

$$0 \to D(H^{\underline{x}}_{n-1}(0 :_M x_1)) \to D(H^{\underline{x}}_n(M)) \xrightarrow{x_1} D(H^{\underline{x}}_n(M)) \to 0.$$

The inductive hypothesis shows that $x_2, \dots, x_n$ is a $D(H^{\underline{x}}_{n-1}(0 :_M x_1))$-coregular sequence. Hence, $x_2, \dots, x_n$ is a $(0 :_{D(H^{\underline{x}}_n(M))} x_1)$-coregular sequence and the assertion follows. □

The following result is a property of coregular sequences.

Corollary 3.8. *Let M be a non-zero Artinian R-module. Let $\underline{x} := x_1, \dots, x_n$ be an M-coregular sequence. Then*

$$\mathrm{Tor}^R_{n+1}(R/\underline{x}, M) = 0.$$

Proof. By Proposition 3.1, there is an isomorphism

$$\mathrm{Tor}^R_{n+1}(R/\underline{x}, M) \cong \mathrm{Tor}^R_1(R/\underline{x}, H^{\underline{x}}_n(M))$$

which induces

$$D(\mathrm{Tor}^R_{n+1}(R/\underline{x}, M)) \cong \mathrm{Ext}^1_R(R/\underline{x}, D(H^{\underline{x}}_n(M))).$$

Combining Corollary 3.5 with Theorem 3.7, we get

$$D(\mathrm{Tor}^R_{n+1}(R/\underline{x}, M)) = 0,$$

and which completes the proof. □

Theorem 3.9. *Let $(R, \mathfrak{m})$ be a local ring and M a non-zero Co-Cohen-Macaulay R-module with $\mathrm{Ndim}M = d > 0$. Assume that $\underline{x} := x_1, \dots, x_n \in \mathfrak{m}$ is an M-coregular sequence. Then $x_1, \dots, x_n$ is a $H^{\mathfrak{m}}_d(M)$-regular sequence.*

Proof. The proof is by induction on n. Let $n = 1$, the short exact sequence

$$0 \to 0 :_M x_1 \to M \xrightarrow{x_1} M \to 0$$

yields the long exact sequence

$$0 \to H^{\mathfrak{m}}_d(0 :_M x_1) \to H^{\mathfrak{m}}_d(M) \xrightarrow{x_1} H^{\mathfrak{m}}_d(M) \to H^{\mathfrak{m}}_{d-1}(0 :_M x_1) \to 0.$$

By [16, Lemma 2.2 (ii)], we see that $\mathrm{Ndim}(0 :_M x_1) = d - 1$ and then $H^{\mathfrak{m}}_d(0 :_M x_1) = 0$ by [4, Proposition 4.8]. It follows from [4, Proposition 4.10] that $H^{\mathfrak{m}}_{d-1}(0 :_M x_1) \neq 0$. This means that x_1 is a $H^{\mathfrak{m}}_d(M)$-regular sequence.

Now, assume that $n > 1$. The short exact sequence

$$0 \to 0 :_M x_1 \to M \xrightarrow{x_1} M \to 0$$

yields the long exact sequence

$$0 \to H_d^{\mathfrak{m}}(M) \xrightarrow{x_1} H_d^{\mathfrak{m}}(M) \to H_{d-1}^{\mathfrak{m}}(0 :_M x_1) \to 0.$$

It should be mentioned that $0 :_M x_1$ is a Co-Cohen-Macaulay R-module with $\mathrm{Ndim}(0 :_M x_1) = d-1$. Note that $x_2, \dots, x_n$ is $0 :_M x_1$-coregular sequence. By the inductive hypothesis, it is a $H_{d-1}^{\mathfrak{m}}(0 :_M x_1)$-regular sequence and the claim follows. □

Proposition 3.10. *Let $(R, \mathfrak{m})$ be a local ring, M a non-zero Co-Cohen-Macaulay R-module with* $\mathrm{Ndim}M = d$*. Assume that R is complete with respect to $\mathfrak{m}$-adic topology. Then* $\mathrm{depth}H_d^{\mathfrak{m}}(M) = d$*.*

Proof. We prove by induction on d. If $d = 0$, then M has finite length. It is clear that $H_0^{\mathfrak{m}}(M) \cong M$ and $\mathrm{depth}H_0^{\mathfrak{m}}(M) = 0$.

Assume that $d > 0$ and $x_1, \dots, x_d$ is a maximal M-coregular sequence in $\mathfrak{m}$. By the similar method in the proof of Theorem 3.7, one has

$$\mathrm{depth}H_d^{\mathfrak{m}}(M)/x_1 H_d^{\mathfrak{m}}(M) = \mathrm{depth}H_{d-1}^{\mathfrak{m}}(0 :_M x_1) = d-1.$$

Combining the assumption with [4, Proposition 4.6], we see that $H_d^{\mathfrak{m}}(M)$ is finitely generated R-module. Hence, [3, Proposition 1.2.10 (d)] shows that

$$\mathrm{depth}H_d^{\mathfrak{m}}(M)/x_1 H_d^{\mathfrak{m}}(M) = \mathrm{depth}H_d^{\mathfrak{m}}(M) - 1,$$

and which completes the proof. □

Corollary 3.11. *Let $(R, \mathfrak{m})$ be a local ring, M a non-zero Co-Cohen-Macaulay R-module with* $\mathrm{Ndim}M = d$*. Assume that R is complete with respect to $\mathfrak{m}$-adic topology and $x_1, \dots, x_d \in \mathfrak{m}$ is an M-coregular sequence such that $\mathfrak{m} = (x_1, \dots, x_d)$. Then* $\mathrm{fd}_R M = \mathrm{depth}R$*.*

Proof. Using Auslander-Buchsbaum formula, Proposition 3.3 (ii) and Proposition 3.10, we get the claim. □

Theorem 3.12. *Let $(R, \mathfrak{m})$ be a local ring and M a non-zero Artinian R-module with* $\mathrm{Ndim}M = d$*. Assume that R is complete with respect to I-adic topology. Then the following statements hold:*

(i) $\dim H_d^I(M) \le d$;

(ii) *If $H_d^I(M) \neq 0$, then* $\mathrm{depth}H_d^I(M) \ge \min\{2, d\}$.

Proof. We use induction on d. Let $d = 0$, then M has finite length and $H_0^I(M) \cong M$. Consequently, one gets $\dim H_0^I(M) = 0$ and $\mathrm{depth}H_0^I(M) = 0$.

Let $d > 0$. By [4, Proposition 4.4 and Corollary 4.5], we may assume that there exists $x \in \mathfrak{m}$ which is an M-coregular element. It was shown in [16, Lemma 2.2] that $\mathrm{Ndim}(0 :_M x) = d - 1$. The short exact sequence

$$0 \to 0 :_M x \to M \xrightarrow{x} M \to 0$$

induces a long exact sequence

$$0 \to H_d^I(M) \xrightarrow{x} H_d^I(M) \to H_{d-1}^I(0 :_M x) \to \cdots .$$

Since $H_d^I(M)/xH_d^I(M)$ is isomorphic to a submodule of $H_{d-1}^I(0 :_M x)$, it follows from the hypothesis that

$$\dim H_d^I(M)/xH_d^I(M) \leq d - 1.$$

In view of [6, Theorem 5.3] and the hypothesis, we can conclude that $H_d^I(M)$ is a finitely generated R-module. This implies that

$$\dim H_d^I(M)/xH_d^I(M) = \dim H_d^I(M) - 1$$

and then $\dim H_d^I(M) \leq d$.

It induces from the Nakayama's Lemma and the assumption that $H_d^I(M)/xH_d^I(M) \neq 0$. This indicates that $H_{d-1}^I(0 :_M x) \neq 0$. Consequently, if $d = 1$, then we see that $\mathrm{depth} H_1^I(M) = 1$. We consider the cases where $d \geq 2$. By the inductive hypothesis, one can claim that $\mathrm{depth} H_{d-1}^I(0 :_M x) \geq 1$. Let y be a $H_{d-1}^I(0 :_M x)$-regular element. It is easy to check that y is a $H_d^I(M)/xH_d^I(M)$-regular element. Thus, we can assert that $\mathrm{depth} H_d^I(M) \geq \min\{2, d\}$. □

Corollary 3.13. *Let $(R, \mathfrak{m})$ be a local ring, M a non-zero Co-Cohen-Macaulay R-module with* $\mathrm{Ndim} M = d$. *Assume that R is complete with respect to $\mathfrak{m}$-adic topology. Then $H_d^{\mathfrak{m}}(M)$ is a Cohen-Macaulay R-module of dimension d.*

Proof. Combining Proposition 3.10 with Theorem 3.12, the assertion follows. □

Corollary 3.14. *Let $(R, \mathfrak{m})$ be a local ring, M a non-zero Artinian R-module with* $\mathrm{Ndim} M = d \leq 2$. *Assume that R is complete with respect to I-adic topology and $H_d^I(M) \neq 0$. Then $H_d^I(M)$ is Cohen-Macaulay.*

Acknowledgement. The author would like to thank the referee for the valuable suggestions and comments.

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