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PROPERTY OF DEFECT DIMINISHING AND STABILITY

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Abstract. Let Γ be a group and $\mathscr{C}$ a class of groups endowed with bi-invariant metrics. We say that Γ is $\mathscr{C}$-stable if every ε-homomorphism $\Gamma \to G$, $(G, d) \in \mathscr{C}$, is δ_ε-close to a homomorphism, $\delta_\varepsilon \to 0$ when $\varepsilon \to 0$. If $\delta_\varepsilon < C\varepsilon$ for some C we say that Γ is $\mathscr{C}$-stable with a linear rate. We say that Γ has the property of defect diminishing if any asymptotic homomorphism can be changed a little to make errors essentially better. We show that the defect diminishing is equivalent to the stability with a linear rate.

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1. Introduction

The stability of a group Γ (with respect to a class of groups $\mathscr{C}$) means that any almost-homomorphism to $\mathscr{C}$ is close to a homomorphism, see Definition 2.2. In [5] the notion of defect diminishing was introduced, see Definition 3.3 and Definition 3.4. It was shown in [5] that for some classes $\mathscr{C}$ and Γ-modules M the vanishing of the second cohomology $H^2(\Gamma, M)$ implies the defect diminishing and that the defect diminishing implies stability.

In the present paper, we show that (under weaker assumptions) defect diminishing is equivalent to stability with a linear rate for finitely presented groups. It not only provides a more natural proof of Theorem 5.1 of [5] but clarifies the relation between defect diminishing and stability. Particularly, this implies that there are stable groups that do not have defect diminishing. Indeed, O. Becker and J. Mosheiff [2] showed that the rate of stability of $\mathbb{Z}^d, d \geq 2$ is polynomial but not linear (with respect to symmetric groups with normalized Hamming distance) [1]. On the other hand we do not know examples of $(U(n), \|\cdot\|_p)$-stable groups without the defect diminishing. Still, we do believe that such groups should exist. The problem is that the cohomological method is the only method available to show

[1]It is worth mentioning that the stability of any abelian group (with respect to symmetric groups with normalized Hamming distance) was proven by G. Arzhantseva and L. Păunescu in [1].

stability in this case. But this leads to the defect diminishing and stability with linear rate.

2. Stability

Let S be a finite set of symbols. We denote by $F(S)$ the free group on S. Let $R \subseteq F(S)$ be finite and Γ be a finitely presented group $\Gamma = \langle S \mid R \rangle = F(S)/\langle\langle \mathrm{R} \rangle\rangle$ where $\langle\langle \mathrm{R} \rangle\rangle$ is the normal subgroup of $F(S)$ generated by R. Let $\mathscr{C}$ be a class of groups, all equipped with bi-invariant metric. Any map $\phi : S \to G$, for a group $G \in \mathscr{C}$ uniquely determines a homomorphism $F(S) \to G$ that we also denote by ϕ.

Definition 2.1. [5] Let $G \in \mathscr{C}$ and let $\phi, \psi : S \to G$ be maps. The defect of ϕ is defined by:

$$\mathrm{def}_R(\phi) = \max_{r \in R} d_G(\phi(r), 1_G)$$

The distance between ϕ and ψ is defined by:

$$\mathrm{dist}_S(\phi, \psi) = \max_{s \in S} d_G(\phi(s), \psi(s))$$

The homomorphism distance of ϕ is defined by:

$$\mathrm{HomDist}_S(\phi) = \inf_{\pi \in Hom(\Gamma, G)} \mathrm{dist}_S(\phi, \pi \restriction_S)$$

Let $\langle \mathscr{C}^S \rangle = \bigcup_{G \in \mathscr{C}} G^S$ where $G^S = \{\phi : S \to G\}$, that is, $\langle \mathscr{C}^S \rangle$ are all possible maps $\phi : S \to G$ for $G \in \mathscr{C}$.

Definition 2.2. [6] A finitely presented group Γ is called $\mathscr{C}$-stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $\phi \in \langle \mathscr{C}^S \rangle$ the inequality $\mathrm{def}_R(\phi) < \delta$ implies $\mathrm{HomDist}_S(\phi) < \epsilon$. Let us restate it to avoid ambiguity:

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall \phi \in \langle \mathscr{C}^S \rangle \;\; (\mathrm{def}_R(\phi) < \delta \Rightarrow \mathrm{HomDist}_S(\phi) < \epsilon)\,.$$

Remark 2.3. The stability of Γ does not depend on the particular choice of the presentation of the group Γ (see [1]): Tietze transformations preserve stability since the metric is bi-invariant. The stability of a group does depend on the class $\mathscr{C}$.

Interesting examples $\mathscr{C} = \{(G_n, d_n) \mid n \in \mathbb{N}\}$ are:

(1) $G_n = U(n)$, the group of Unitary $n \times n$ matrix. The metric d_n is induced by the normalized Hilbert-Schmidt norm $\|A\|_{HS} = \sqrt{\frac{1}{n}\,\mathrm{tr}(A^*A)}$ ($d_n(A, B) = \|A - B\|$).

(2) $G_n = U(n)$, the metric d_n is induced by the Schatten p-norm $\|A\|_p = (\mathrm{tr}\,|T|^p)^{\frac{1}{p}}$, where $|T| = \sqrt{T^*T}$. Note that if $p = 2$ then $\|A\|_2 = \|A\|_{\mathrm{Frob}}$.

(3) $G_n = U(n)$, the metric d_n is induced by the operator norm $\|A\|_{op} = \sup_{\|v\|=1} \|Av\|$ also known as Schatten ∞-norm.

(4) $G_n = \mathrm{Sym}(n)$, the symmetric group of n elements. d_n is the normalized Hamming distance: $d_n(\alpha, \beta) = \frac{1}{n}|\{j \mid \alpha(j) \neq \beta(j)\}|$.

2.1. Rate of stability. The rate of stability is, roughly speaking, the dependence of ϵ and δ in Definition 2.2. See [2] for details. To make this precise we define the function $D_{(S,R)} : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$D_{(S,R)}(\delta) = \sup_{\phi \in \langle \mathscr{C}^S \rangle} \{\mathrm{HomDist}_S(\phi) \mid \mathrm{def}_R(\phi) < \delta\}.$$

The function $D_{(S,R)}$ is monotone increasing and depends on the presentation of the group Γ, but we show now that this dependence is just linear.

The following lemma is a reformulation of Definition 2.2. The analogue of the lemma is used as the definition of stability in [3].

Lemma 2.4. $\lim_{\delta \to 0^+} D_{(S,R)}(\delta) = 0$ *if and only if* Γ *is* $\mathscr{C}$*-stable.*

Following O. Becker and J. Mosheiff, we define the rate stability D_Γ of the group Γ as a class of functions (see Definition 2.7).

Definition 2.5. Let $f, g : (0, \delta_0] \to \mathbb{R}^+$ be monotone nondecreasing functions. Write $f \preceq g$ if $f(\delta) \leq g(C\delta) + C\delta$ for some $C > 0$ and all $\delta \in (0, \delta_0]$ for some $\delta_0 > 0$. We define the equivalence relation $\sim$ by saying that $f \sim g$ if and only if $f \preceq g$ and $g \preceq f$ (notice that the relation $\preceq$ is reflexive and transitive). Let $[f]$ denote the class of f with regard to this equivalence relation. Clearly, $\preceq$ defines a partial order on equivalence classes: $[f] \preceq [g]$ if and only if $f \preceq g$.

Note that if $f \preceq \mathrm{id}$ then $f(\delta) \leq M\delta$ for some M. Here id is an identical function: $\mathrm{id}(\delta) = \delta$.

Proposition 2.6. [2] *Let* $\Gamma = \langle S \mid R \rangle$ *be a finitely presented group. If* $\Gamma = \langle S' \mid R' \rangle$ *is another finite presentation of* Γ*. Then* $D_{(S,R)} \sim D_{(S',R')}$.

Definition 2.7. Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group. The rate stability D_Γ of the group Γ is the equivalence class $D_\Gamma = \left[D_{(S,R)}\right]$.

Proposition 2.6 implies that the rate of stability D_Γ of the finitely presented group Γ does not depend on the presentation of Γ.

By the definition of $\sim$ the rate of stability D_Γ of a group Γ can not be faster then linear. The following lemma shows that it is not just by definition of $\sim$ but rather a natural phenomenon for non-free groups.

Lemma 2.8. [2] *Let* $\Gamma = \langle S \mid R\rangle$ *be a finitely presented group with* $R \neq \emptyset$, $R \neq \{1_\Gamma\}$ *and* $\mathscr{C}$ *is the class of symmetric groups with the normalized Hamming distance. Then there exists* $C > 0$ *and* $\delta_0 > 0$ *such that* $C\delta \leq D_{(S,R)}(\delta)$ *for all* $\delta \in (0, \delta_0]$.

By O. Becker and J. Mosheiff [2] if $\mathscr{C}$ is symmetric group with Hamming distance and $d = 2, 3, 4, \dots$ then $\left[\delta^{1/b}\right] \preceq D_{\mathbb{Z}^d} \preceq \left[\delta^{1/c}\right]$ for any $b < 2$ and some $c = c_d$, depending on d.

3. Property of defect diminishing

In this section we give the definition of the property of defect diminishing and a proof of the main theorem.

Definition 3.1. An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$, such that:

(i) $A \in \mathcal{U}$ and $A \subset B$ implies $B \in \mathcal{U}$,
(ii) $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$,
(iii) $A \notin \mathcal{U}$ if, and only if $\mathbb{N} \setminus A \in \mathcal{U}$.

We say that $\mathcal{U}$ is non-principal if $\{n\} \notin \mathcal{U}$ for every $n \in \mathbb{N}$. The existence of non-principal ultrafilters on $\mathbb{N}$ is ensured by the axiom of choice. We fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Given a bounded sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers we denote the limit along the ultrafilter by $\lim\limits_{n\to\mathcal{U}} x_n \in (-\infty, \infty)$. Formally, the limit is the unique $x \in \mathbb{R}$ such for all $\epsilon > 0$ we have $\{n \in \mathbb{N} : \mid x_n - x \mid < \epsilon\} \in \mathcal{U}$. For more information on ultrafilters and ultralimits see [4] appendix B.

We will use the notation Landau, let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers, we denote by $x_n = O_{\mathcal{U}}(y_n)$ if there exists $C > 0$ such that $\{n \mid x_n \leq Cy_n\} \in \mathcal{U}$. We denote by $x_n = o_{\mathcal{U}}(y_n)$ if there is a third sequence of positive real numbers ε_n such that $\lim\limits_{n\to\mathcal{U}} \varepsilon_n = 0$ and $x_n = \varepsilon_n y_n$.

Definition 3.2. [5] A sequence of maps $\phi_n : S \to G_n$, for $(G_n, d_n) \in \mathscr{C}$ is called an asymptotic homomorphism to $\mathscr{C}$ if

$$\lim_{n\to\mathcal{U}} \mathrm{def}_R(\phi_n) = 0.$$

Definition 3.3. Let $\phi_n : S \to G_n$ with $G_n \in \mathscr{C}$ be an asymptotic homomorphism, we say that an asymptotic homomorphism $\phi'_n : S \to G_n$ diminishes the defect of $(\phi_n)_{n\in\mathbb{N}}$ if:

(a) $\mathrm{dist}_S(\phi_n, \phi'_n) = O_{\mathcal{U}}(\mathrm{def}_R(\phi_n))$,
(b) $\mathrm{def}_R(\phi'_n) = o_{\mathcal{U}}(\mathrm{def}_R(\phi_n))$.

We say that $(\phi_n)_{n\in\mathbb{N}}$ has the property of defect diminishing if there is an asymptotic homomorphism $(\phi'_n)_{n\in\mathbb{N}}$ that diminishes the defect of $(\phi_n)_{n\in\mathbb{N}}$.

Definition 3.4. The group Γ has the property of defect diminishing (with respect to $\mathscr{C}$) if every asymptotic homomorphism to $\mathscr{C}$ has the property of defect diminishing.

Theorem 3.5. *Let $\Gamma = \langle S \mid R\rangle$ be a finitely presented group and $\mathscr{C}$ a class of groups such that each $(G, d) \in \mathscr{C}$ is a complete metric space. Then the group Γ has the property of defect diminishing if and only if $D_{(S,R)} \preceq \mathrm{id}$.*

Corollary 3.6. *Let Γ be a finitely presented group and $\mathscr{C}$ a class of groups such that each $(G, d) \in \mathscr{C}$ is a complete metric space. The group Γ has the property of defect diminishing if and only if $D_\Gamma \preceq [\mathrm{id}]$.*

Corollary 3.7. *The property of defect diminishing does not depend on the particular choice of the presentation of the group Γ.*

Proof. If $\mathscr{C}$ is a class of groups such that each $(G, d) \in \mathscr{C}$ is a complete metric space, the proof follows from the Proposition 2.6 and Theorem 3.5. General case may be proved directly similarly to Proposition 2.6. □

For the proof of Theorem 3.5 we need the following proposition.

Proposition 3.8. *If $\Gamma = \langle S \mid R\rangle$ has the property of defect diminishing then there exists $M, \varepsilon \in \mathbb{R}^+$ such that for all $G \in \mathscr{C}$ and $\phi \in G^S$ with $\mathrm{def}_R(\phi) < \epsilon$ there exists $\psi \in G^S$ such that:*

(1) $\mathrm{def}_R(\psi) < \frac{1}{2}\,\mathrm{def}_R(\phi)$.
(2) $\mathrm{dist}_S(\phi, \psi) < M\,\mathrm{def}_R(\phi)$.

Proof. Suppose that the conclusion of the proposition is false. Then for every $n \in \mathbb{N}$ there is $\phi_n \in (G_n)^S$ with $G_n \in \mathscr{C}$ and $\mathrm{def}_R(\phi_n) < \frac{1}{n}$, such that every $\psi \in (G_n)^S$ with $\mathrm{def}_R(\psi) < \frac{1}{2}\,\mathrm{def}_R(\phi_n)$ satisfies $\mathrm{dist}_S(\phi_n, \psi) \geq n\,\mathrm{def}_R(\phi_n)$.

So we have an asymptotic homomorphism $(\phi_n)_{n\in\mathbb{N}}$ that does not have the property of defect diminishing. Therefore, Γ does not have the property of defect diminishing. □

Proof of Theorem 3.5. Suppose that $D_{(S,R)} \preceq \mathrm{id}$, that is, there exists $M > 0$ and $\delta_0 > 0$ such that $\forall 0 < \delta < \delta_0$ we have that $D_{(S,R)}(\delta) < M\delta$. Let $(\phi_n)_{n\in\mathbb{N}}$ be an asymptotic homomorphism and $\epsilon_n = \mathrm{def}_R(\phi_n)$. By the definition of asymptotic homomorphism $\lim\limits_{n\to\mathcal{U}} \epsilon_n = 0$. Let $X = \{n \mid \epsilon_n < \delta_0\}$. For $n \in X$ we have that

$\mathrm{HomDist}_S(\phi_n) < M\epsilon_n$ by Definition 2.1 and there is a $\pi_n \in Hom(\Gamma, G_n)$ that complies $\mathrm{dist}_S(\phi_n, \pi_n \restriction_S) < M\,\mathrm{def}_R(\phi_n)$. Define $\phi'_n = \pi_n$ for $n \in X$ and $\phi'_n = \phi_n$ for $n \notin X$. Then ϕ'_n diminishing the defect of ϕ_n as $X \in \mathcal{U}$.

Suppose that the group Γ has the property of defect diminishing. We apply Proposition 3.8. Let $M, \varepsilon \in \mathbb{R}^+$ be as in Proposition 3.8. Let $\phi \in G^S$ be with $\mathrm{def}_R(\phi) < \varepsilon$. Inductively we may construct a sequence of maps $\phi_j \in G^S$, $\phi_0 = \phi$, such that $\mathrm{def}_R(\phi_j) < \frac{1}{2}\mathrm{def}_R(\phi_{j-1}) < \frac{\varepsilon}{2^j}$ and $\mathrm{dist}_S(\phi_j, \phi_{j-1}) < M\,\mathrm{def}_R(\phi_{j-1}) < M\,\mathrm{def}_R(\phi)\frac{1}{2^{j-1}}$. It follows that $(\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. The Cauchy sequence $(\phi_n)_{n\in\mathbb{N}}$ of points in G^S has a limit that is also in G^S, this limit we denote by ϕ_∞. We can check that ϕ_∞ is a homomorphism and $\mathrm{dist}_S(\phi, \phi_\infty) < 2M\,\mathrm{def}_R(\phi)$. It follows that $D_{(S,R)}(\delta) < 2M\delta$ for $\delta < \varepsilon$. Therefore, $D_{(S,R)} \preceq \mathrm{id}$. □

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