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ON $+\infty$-ω_0-GENERATED FIELD EXTENSIONS

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Abstract. A purely inseparable field extension K of a field k of characteristic $p \neq 0$ is said to be ω_0-generated over k if K/k is not finitely generated, but L/k is finitely generated for each proper intermediate field L. In 1986, Deveney solved the question posed by R. Gilmer and W. Heinzer, which consists in knowing if the lattice of intermediate fields of an ω_0-generated field extension K/k is necessarily linearly ordered under inclusion, by constructing an example of an ω_0-generated field extension where $[k^{p^{-n}} \cap K : k] = p^{2n}$ for all positive integer n. This example has proved to be extremely useful in the construction of other examples of ω_0-generated field extensions (of any finite irrationality degree). In this paper, we characterize the extensions of finite irrationality degree which are ω_0-generated. In particular, in the case of unbounded irrationality degree, any modular extension of unbounded exponent contains a proper subfield of unbounded exponent over the ground field. Finally, we give a generalization, illustrated by an example, of the ω_0-generated to include modular purely inseparable extensions of unbounded irrationality degree.

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1. Introduction

Let α be an infinite cardinal. In universal algebra, an algebra A is said to be a Jónsson α-algebra if A has cardinality α, while each proper subalgebra B of A has cardinality less than α [4, p. 469]. Following this terminology, R. Gilmer and W. Heinzer extended this notion for the first time in [12] to generating sets. Recall that the algebra A is said to be a Jónsson α-generated algebra if A has a generating set of cardinality α, no generating set of smaller cardinality, and each proper subalgebra B of A has a generating set of cardinality less than α. The authors first gave special attention primarily to the cases where $\alpha = \omega_0$ the first infinite cardinal, and where $\alpha = \omega_1$. They then examined separately in [11] a problem of this class for field extensions. Let K be a purely inseparable field extension of a field k of characteristic $p \neq 0$, by analogy, K is said to be ω_0-generated over k (for Jónsson ω_0-generated extension) if K is not finite dimensional over k and yet every proper intermediate field is finite dimensional over k [8]. Moreover, this last condition implies that K/k is countably generated, and hence $[K : k] = \omega_0$. In [11], Robert Gilmer and William Heinzer focused on the question of whether $[k^{p^{-1}} \cap K : k] = p$ is essentially the only possibility for that K to be ω_0-generated over k. More specifically, if K is ω_0-generated over k, must $[k^{p^{-1}} \cap K : k] = p$? In [8], J. K. Deveney constructed an example of an ω_0-generated field extension K/k such that for any positive integer n, $[k^{p^{-n}} \cap K : k] = p^{2n}$. It is easy to verify that K/k is a modular relatively perfect extension of unbounded exponent and of irrationality degree 2. Recall that the irrationality degree of K/k has been defined [13, Definition 2.3] by: $di(K/k) = \sup_{n \in \mathbb{N}}(|B_n|)$ where $|B_n|$ is the cardinality of a minimal generating set B_n of $k^{p^{-n}} \cap K$ over k. Motivated by Dveney's result, for each integer j we have constructed in [6] a purely inseparable extension K/k satisfying:

- Every proper subfield of K/k is finite over k.
- For every positive integer n, $[k^{p^{-n}} \cap K : k] = p^{jn}$.

Improving thus the counterexample of J. K. Deveney, such an extension is also modular and relatively perfect of unbounded exponent, but of irrationality degree j. It's about essentially a form of irreducibility in the sense that K/k cannot be decomposed into $k \longrightarrow K_1 \longrightarrow K$ with each of K_1/k and K/K_1 having unbounded exponent. Furthermore, any extension of finite irrationality degree is composite of finite number of irreducible extensions. We also give a necessary and sufficient condition for an ω_0-generated field extension to be of finite irrationality degree. More specifically, we show that for an ω_0-generated field extension to be of finite irrationality degree it is necessary and sufficient that the minimal intermediate field m of K/k over which K is modular is nontrivial ($m \neq K$). In particular, any modular and ω_0-generated field extension is of finite irrationality degree, and therefore if we take these results into account, it is very probable that the ω_0-generated is related to the extensions of finite irrationality degree. This leads us to study closely the ω_0-generated in the restricted sense. Consistent with this terminology, and with the aim of extending the ω_0-generated to modular purely inseparable extensions of unbounded irrationality degree, we propose another generalization. An extension K/k is said to be j-ω_0-generated if K/k does not admit any intermediate field L of unbounded exponent over k and of irrationality degree less than or equal to j. It is about a form of local irreducibility conditioned by the irrationality degree. If for every integer j, K/k is j-ω_0-generated, K/k will be called $+\infty$-ω_0-generated field extension. We immediately verify that any ω_0-generated field extension is $+\infty$-ω_0-generated, and conversely any $+\infty$-ω_0-generated field extension of finite irrationality degree is ω_0-generated. Moreover, for reasons of noncontradiction, we construct an example of a $+\infty$-ω_0-generated field extension of unbounded irrationality degree.

Throughout this paper, unless otherwise stated, all considered fields are purely inseparable extensions of a common ground field k. They are to be viewed as contained in a common algebraically closed field Ω.

2. Definitions and preliminary results

Let x be an element of K, the least positive integer e such that $x^{p^e} \in k$ is called the **exponent** of x over k, and is denoted by $o(x/k)$. The maximum of the set of exponents of elements of K is called the **exponent** of K over k, if it exists, that is, the smallest integer e (if it exists) such that $K^{p^e} \subseteq k$, where $K^{p^e} = \{a^{p^e} \,|\, a \in K\}$, which will be denoted by $o_1(K/k)$. Otherwise, K/k is said to be of unbounded exponent. If K/k is a finite extension, the irrationality degree of K/k has been defined by $di(K/k) = \min(|G|)$, where G is a generating set of K/k. If K/k is of unbounded exponent, a minimal generating set may not exist [18, Lemma 1.16, Proposition 1.23]; but as for any positive integer n, $k^{p^{-n}} \cap K/k$ has an exponent, according to [18, p. 2, Corollary 1.6], a subset B of $k^{p^{-n}} \cap K/k$ is an r-basis (used as a shortcut for relative p-basis [18, p. 1, Definition 1.2]) of $k^{p^{-n}} \cap K/k$ if and only if B is a minimal generating set of $k^{p^{-n}} \cap K/k$, and consequently, the cardinality of any minimal generating set of $k^{p^{-n}} \cap K/k$ depends only on n, because it is an r-basis, and so it has a unique cardinality by the theory of general dependence [17, p. 132-133, Lemma 6.1, corollary 6.2]. Extending the minimum number of generator of K/k, due to M. F. Becker and S. Mac Lane in [1], which was interesting/valid only in the case when K/k is finite, we have recently defined the irrationality degree of K/k as follows: $di(K/k) = \sup\limits_{n \in \mathbb{N}}(|B_n|)$ where $|B_n|$ is the cardinality of a minimal generating set B_n of $k^{p^{-n}} \cap K$ over k [13, Definition 2.3]. If moreover $di(K/k)$ is finite, then K/k is called a **q-finite extension** [13, Definition 3.1], i.e., there must exist an integer M such that for every positive integer n the field $k^{p^{-n}} \cap K$ is generated by at most M elements over k. It is clear that every finite purely inseparable field extension is in particular q-finite. The converse is true if and only if K/k has an exponent. We will often use the following theorem.

Theorem 2.1 ([13], Theorem 2.7)**.** *For any family $k \subseteq L \subseteq L' \subseteq K$ of purely inseparable extensions, we have $di(L'/L) \leq di(K/k)$.*

We will now highlight the notion of exponents of q-finite extensions (for more details see [13]), by extending some basic definitions and notations given in [7, p. 373].

If K/k is a finite purely inseparable extension. An r-basis $B = \{a_1, a_2, \dots, a_n\}$ of K/k is said to be canonically ordered (Rasala used in [21] the term normal generating sequence) if $o(a_j/k(a_1, a_2, \dots, a_{j-1})) = o_1(K/k(a_1, a_2, \dots, a_{j-1}))$ for $j = 1, 2, \dots, n$. By [6, p. 138, Lemma 1.3], the integer $o(a_j/k(a_1, \dots, a_{j-1}))$ thus defined satisfies $o(a_j/k(a_1, \dots, a_{j-1})) = \inf\{m \in \mathbb{N} | \, di(k(K^{p^m})/k) \leq j - 1\}$. We immediately deduce the result [20, p. 90, Satz 14] which ensures the independence of integers $o(a_i/k(a_1, \dots, a_{i-1}))$, $(1 \leq i \leq n)$, with respect to the choice of canonically ordered r-basis $\{a_1, \dots, a_n\}$ of K/k. In the sequel, we set $o_i(K/k) = o(a_i/k(a_1, \dots, a_{i-1}))$ if $1 \leq i \leq n$, and $o_i(K/k) = 0$ if $i > n$ where $\{a_1, \dots, a_n\}$ is a canonically ordered r-basis of K/k. The invariant $o_i(K/k)$ defined above is called the **i-th exponent** of K/k.

If K/k is q-finite, we denote the intermediate field $k^{p^{-n}} \cap K$ by k_n for all n. By virtue of [7, p. 374, Proposition 6], for each positive integer j, the sequence of natural numbers $(o_j(k_n/k))_{n \geq 1}$ is increasing, and thus $(o_j(k_n/k))_{n \geq 1}$ converges to $+\infty$, or $(o_j(k_n/k))_{n \geq 1}$ becomes constant after a certain rank. One can readily check that, if $(o_j(k_n/k))_{n \geq 1}$ is bounded, then for each $t \geq j$, $(o_t(k_n/k))_{n \geq 1}$ is also bounded (and therefore stationary).

Definition 2.2 ([13], Definition 3.2)**.** Let K/k be a q-finite extension, and j a positive integer. Then the invariant $o_j(K/k) = \lim\limits_{n \to +\infty} (o_j(k_n/k))$ is called the j-th exponent of K/k.

The following result characterizes the exponents of K/k by relating to the behavior of irrationality degree of certain intermediate fields of K/k.

Theorem 2.3 ([13], Lemma 3.1)**.** *Let s be a positive integer $(s \geq 1)$ and K/k a q-finite extension, then $o_s(K/k)$ is finite if and only if there exists a natural number n such that $di(k(K^{p^n})/k) < s$, and we have $o_s(K/k) = \inf\{m \in \mathbb{N} \,|\, di(k(K^{p^m})/k) < s\}$. In particular, $o_s(K/k)$ is infinite if and only if for each $m \in \mathbb{N}$, $di(k(K^{p^m})/k) \geq s$.*

A field k of characteristic p is said to be perfect if $k^p = k$. In the same order of ideas, K/k is said to be relatively perfect if $k(K^p) = K$. We check immediately that:

- If K/L and L/k are relatively perfect, then K/k is also relatively perfect;
- If K/k is relatively perfect, then the same is true for $L(K)/k(L)$;

- For any family $(K_i/k)_{i\in I}$ of relatively perfect extensions, $\prod_i K_i/k$ is also relatively perfect.

Therefore, there exists a unique maximal intermediate field M of K/k where M/k is relatively perfect (for more details see [22, p. 16, Proposition 6]). M is called the **relatively perfect closure** of K/k and is denoted by $rp(K/k)$. The result below makes it possible to reduce the study of properties of exponents of a q-finite extension to a finite extension through the relatively perfect closure.

Theorem 2.4 ([13], Theorem 3.9)**.** *Let K_r/k be the relatively perfect closure of irrationality degree s of a q-finite extension K/k ($di(K_r/k) = s$), then we have:*

(1) *For each $t \leq s$, $o_t(K/k) = +\infty$.*
(2) *For each $t > s$, $o_t(K/k) = o_{t-s}(K/K_r)$.*

In addition, $o_t(K/k)$ is finite if and only if $t > s$.

Here is a list of immediate consequences.

Proposition 2.5 ([13], Proposition 3.10)**.** *Let K and L be two intermediate fields of a q-finite extension M/k. For every $j \in \mathbb{N}^*$, $o_j(L(K)/L) \leq o_j(K/k)$.*

Proposition 2.6 ([13], Proposition 3.11)**.** *Let $k \subseteq L \subseteq L' \subseteq K$ be q-finite extensions. For each $j \in \mathbb{N}^*$, $o_j(L'/L) \leq o_j(K/k)$.*

2.1. Modular extensions. Before we state further preliminaries which we will also need later, we review the following: Let K_1 and K_2 be two intermediate fields between k and K that are k linearly disjoint. For every subfields L_1, L_2 of K_1/k and K_2/k respectively, it is well-known that $L_2(K_1)$ and $L_1(K_1)$ are $k(L_1, L_2)$ linearly disjoint [16, p. 35, Lemma 2.5.3]. In particular, $L_2(K_1) \cap L_1(K_2) = k(L_1, L_2)$. Define a family $\{F_i \,|\, i \in J\}$ of field extensions of k to be linearly disjoint over k if every finite subfamily is linearly disjoint over k [16, p. 36]. It is not hard to see that $k((F_i)_{i\in J}) = \prod_{i\in J} F_i \simeq \otimes_k(\otimes_k F_i)_{i\in J}$ (for additional information about the tensor product see [2, III, p. 42, Definition 5]) if and only if the family $(F_i/k)_{i\in J}$ is k linearly disjoint. Moreover, the properties of linear disjointness of the finite case naturally extend to any linearly disjoint family. In particular, for all $i \in J$, let L_i be a subfield of F_i/k, if $(F_i/k)_{i\in J}$ is k linearly disjoint, by transitivity of linear disjointness, we have $(L_i/k)_{i\in J}$ (respectively, $((\prod_{n\in J} L_n)F_i/k)_{i\in J}$) is k (respectively, $\prod_{n\in J} L_n$) linearly disjoint.

A subset B of K which we will prefer called a **modular r-basis** (M. Weisfeld used the term sub-basis see [25, p. 435]) of K over k if and only if it fulfills the following conditions: $B\cap k=\emptyset$, $K=k(B)$, and, for any finite subset $\{b_1,\dots,b_t\}$ of B, the canonical homomorphism of the tensor product $k(b_1)\otimes_k\dots\otimes_k k(b_t)$ into K is a monomorphism. This is equivalent, by [18, p. 14, Definition 1.21], to for every finite subset $\{b_1,\dots,b_t\}$ of B, $[k(b_1,\dots,b_t):k]=\prod\limits_{i=1}^{t}[k(b_i):k]$, that is, $k(b_1,\dots,b_t)$ is a tensor product over k of the simple extensions $k(b_1),\dots,k(b_t)$.

Recall that K is modular over k if and only if K^{p^n} and k are linearly disjoint over their intersection for all n. Sweedler showed in [23, p. 403, Theorem 1] that if K over k has a finite exponent, then K is modular over k if and only if K can be written as the tensor product of simple extensions of k, that is, K/k has a modular r-basis.

As an immediate consequence of the linear disjointness, we have:

Proposition 2.7. *Let K/k be a purely inseparable extension having a modular r-basis B and $(e_a)_{a\in B}$ a family of integers such that $0\le e_a\le o(a/k)$. Let $L=k((a^{p^{e_a}})_{a\in B})$, then $(B\setminus L)$ and $((a^{p^{e_a}})_{a\in B}\setminus k)$ are two modular r-basis, respectively of K/L and L/k. Furthermore, for each $a\in B$, $o(a/L)=e_a$.*

For each $a\in B$, we put $n_a=o(a/k)$. Consider now the subsets B_1 and B_2 of B defined by $B_1=\{a\in B\,|\,n_a>j\}$, $B_2=B\setminus B_1=\{a\in B\,|\,n_a\le j\}$ (j being a natural number not exceeding $o_1(K/k)$).

Proposition 2.8 ([13], Proposition 4.6)**.** *Under the conditions specified above, for any integer $1\le j<o_1(K/k)$, we have $k^{p^{-j}}\cap K=k((a^{p^{n_a-j}})_{a\in B_1},B_2)$.*

Corollary 2.9 ([13], Corollary 4.7)**.** *For every modular extension K/k, and for each positive integer n, $di(k^{p^{-n}}\cap K/k)=di(k^{p^{-1}}\cap K/k)$. In particular, $di(K/k)=di(k^{p^{-1}}\cap K/k)$.*

We have a similar result under weaker hypotheses than that in [18, p. 94, Proposition 3.3], as well as the [9, p. 289, Theorem 3.2].

Proposition 2.10. *Let K_1 and K_2 be subfields of K/k such that $K\simeq K_1\otimes K_2$. If K/K_1 is modular and K_2/k has an exponent, there exists a subset B of K such that $K\simeq K_1\otimes_k(\otimes_k k(\alpha))_{\alpha\in B}$.*

Proof. First, as $K\simeq K_1\otimes_k K_2$, then for each natural number i, for any r-basis C of $k({K_2}^{p^i})/k$, C is also an r-basis of $K_1({K_2}^{p^i})/K_1$. We then choose an r-basis B of K_2/k, as K_2/k has an exponent, then B is a minimal generating set of K_2/k. Let $B_1,\dots,B_n$ be a partition of B obtained by the following procedure: $B_1 = \{x \in B \mid o(x/k) = o_1(K_2/k) = e_1\}$, and for each $1 < i \leq n$, $B_i = \{x \in B \mid o(x/k(B_1,\dots,B_{i-1})) = o_1(K_2/k(B_1,\dots,B_{i-1})) = e_i\}$. By virtue of linear disjointness, for each $i \in \{1\dots,n\}$, for each $x \in B_i$, we also have $o(x/K_1(B_1,\dots,B_{i-1})) = o_1(K/K_1(B_1,\ \dots,B_{i-1})) = e_i$. Taking into account [19, p. 326, Theorem 1], $k({K_2}^{p^{e_i}}) = k({B_1}^{p^{e_i}},\dots,\ {B_{i-1}}^{p^{e_i}})$. Therefore, for each $i \in \{2,\dots,n\}$, the products $(\prod_\alpha (G)^{\alpha p^{e_i}})_G$ where G is a finite subset of elements in $B_1 \cup \dots \cup B_{i-1}$ and the α are suitably chosen, form a linear basis of $k({K_2}^{p^{e_i}})/k$, and by linear disjointness it is also a linear basis of $K_1({K_2}^{p^{e_i}}) = K_1(K^{p^{e_i}})/K_1$. Let M_i denote this basis, and let $x \in B_i$, there exists some unique $c_\alpha \in k$ such that $x = \sum_\alpha c_\alpha y_\alpha$, ($y_\alpha \in M_i$), furthermore the c_α are also unique in K_1. On the other hand, by virtue of modularity, for each $i \in \{1,\dots,n\}$, $K^{p^{e_i}}$ and K_1 are $K_1 \cap K^{p^{e_i}}$ linearly disjoint. As $K_1({K_2}^{p^{e_i}}) = K_1(K^{p^{e_i}})$ and $M_i \subseteq K^{p^{e_i}}$, then M_i is also a linear basis of $K^{p^{e_i}}$ over $K_1 \cap K^{p^{e_i}}$. Taking into account the uniqueness of linear combinations of x in the linear basis M_i, we deduce by identification that the $c_\alpha \in k \cap K^{p^{e_i}}$, and so ${B_i}^{p^{e_i}} \subseteq k \cap K^{p^{e_i}}({K_1}^{p^{e_i}}({B_1}^{p^{e_i}},\dots,{B_{i-1}}^{p^{e_i}}))$ for each $i \in \{1\dots,n\}$. By [18, p. 94, Proposition 3.3], there exists a modular subextension J/k of finite exponent of K/k such that $K \simeq K_1 \otimes_k J$. Thus, the result follows immediately from the Swedleer's theorem. □

In the finite case, the following result generalizes the above proposition.

Proposition 2.11. *Let K_1 and K_2 be two intermediate fields of purely inseparable extension L/k which are k linearly disjoint. Suppose that $di(L/K_1) = di(K_2/k) = n$ and L/K_1 has an exponent. Let s be the smallest integer such that $o_s(K_2/k) = o_n(K_2/k)$. If L/K_1 is modular, there exists a canonically ordered r-basis $\{\alpha_1,\dots,\alpha_n\}$ of $K_1(K_2)/K_1$ verifying*

$$K_1(K_2) \simeq K_1 \otimes k(\alpha_1,\dots,\alpha_s) \otimes_k k(\alpha_{s+1}) \otimes_k \dots \otimes_k k(\alpha_n).$$

Proof. To simplify notation, we set $e_j = o_j(K_2/k)$ for $j = 1,\dots,n$ and $K = K_1(K_2)$. Let $\{\alpha_1,\dots,\alpha_n\}$ be a canonically ordered r-basis of K_2/k. In view of [7, p. 374, Proposition 7], $\{\alpha_1,\dots,\alpha_n\}$ is also a canonically ordered r-basis of K/K_1 and, for each $j \in \{1,\dots,n\}$, $o_j(K/K_1) = e_j$. According to [6, p. 140, Proposition 5.3], for $i = s,\dots,n$, we obtain the structure equations (of α_i with respect to $k(\alpha_1,\dots,\alpha_{s-1})$) of the form ${\alpha_i}^{p^{e_n}} = \sum_{\varepsilon \in \Lambda_{s-1}} C^i_\varepsilon(\alpha_1,\dots,\alpha_{s-1})^{\varepsilon p^{e_n}}$ $(*)$. Here Λ_{s-1} is a suitable multi-index set and the C^i_ε are unique elements of k. Therefore, for all

$i \in \{s \dots, n\}$, the structure equations of α_i with respect to $K_1(\alpha_1, \dots, \alpha_{s-1})$ also defined by the relation $(*)$ above, where the C^i_ε are also unique elements of K_1. As L/K_1 is modular, using the criterion of modularity [7, p. 375, Proposition 10], for each $(i, \varepsilon) \in \{s, \dots, n\} \times \Lambda_{s-1}$, we will have $(C^i_\varepsilon)^{p^{-e_n}} \in L$. Let $F = k((C^i_\varepsilon)^{p^{-e_n}})$ where (i, ε) runs through the set $\{s, \dots, n\} \times \Lambda_{s-1}$ and $H = K_1(F)(\alpha_1, \dots, \alpha_{s-1})$. It's clear that $o_1(F/k) \leq e_n$ and $K \subseteq H \subseteq L$. According to Theorem 2.1 and Proposition 2.6, $n = di(K/K_1) \leq di(H/K_1) \leq di(L/K_1) = n$, and for each $i \in \{s, \dots, n\}$, $e_n = o_i(K/K_1) \leq o_i(H/K_1) \leq e_n$. It follows that $di(H/K_1) = n$, and for each $i \in \{s, \dots, n\}$, $e_n = o_i(H/K_1)$. As $e_{s-1} > e_s = e_n$, by the r-basis completion algorithm [7, p. 374, Proposition 8], there exists elements $b_s, \dots, b_n \in F$ such that $\{\alpha_1 \dots, \alpha_{s-1}, b_s, \dots, b_n\}$ be a canonically ordered r-basis of H/K_1. In particular, we will have:

- For each $i \in \{1, \dots, s-1\}$, $e_i = o_i(H/K_1) = o_i(K_1(\alpha_1, \dots, \alpha_{s-1})/K_1)$ $= o_i(k(\alpha_1, \dots, \alpha_{s-1})/k)$.
- For each $j \in \{s, \dots, n\}$, $e_n = o_j(H/K_1) = o(b_j/K_1(\alpha_1 \dots, \alpha_{s-1}, b_s, \dots, b_{j-1})) \leq o(b_j/k(b_s, \dots, b_{j-1})) \leq o_1(F/k) \leq e_n$, and so $e_n = o_j(H/K_1)$ $= o_j(k(b_s, \dots, b_n)/k)$.

Hence, $H = K \simeq K_1 \otimes k(\alpha_1, \dots, \alpha_{s-1}) \otimes_k k(b_s) \otimes_k \cdots \otimes_k k(b_n)$. □

2.2. Equiexponential extensions.

Proposition 2.12. *Let K/k be a purely inseparable extension of exponent e. The following assertions are equivalent:*

(1) *For every r-basis G of K/k, for each $a \in G$, $o(a/k(G \setminus \{a\})) = o(a/k) = e$.*
(2) *There exists an r-basis G of K/k such that for each $a \in G$, $o(a/k(G \setminus \{a\})) = o(a/k) = e$.*
(3) *There exists an r-basis G of K/k verifying $K \simeq \otimes_k(k(a))_{a\in G}$, and for each $a \in G$, $o(a/k) = e$.*
(4) *Any r-basis G of K/k satisfies $K \simeq \otimes_k(k(a))_{a\in G}$ and $o_1(K/k) = e$.*

Proof. We immediately verify that (1) $\Rightarrow$ (2) $\Rightarrow$ (3), so we just have to show that (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Assume that there exists an r-basis G of K/k verifying $K \simeq \otimes_k(\otimes_k k(a))_{a\in G}$, and for each $a \in G$, $o(a/k) = e$. Let B be a finite r-independent (used as shortening of relatively p-independent) subset of K/k, so there exists $G_1 \subseteq G$ such that $B \cup G_1$ is an r-basis of K/k, and therefore $|B| = |G \setminus G_1|$ which we designate by n. Since the exponent of any element of B over k is less than the exponent of K/k, we deduce that $[k(B) : k] \leq \prod_{a\in B} p^{o(a/k)} \leq p^{en}$, and therefore $[K : k(G_1)] \leq [k(B) : k] \leq p^{en}$. But, by virtue of the linear disjointness, $[K : k(G_1)] = [k(G \setminus G_1) : k] = p^{en}$, so $[K : k(G_1)] = [k(B) : k] = \prod_{a \in B} p^{o(a/k)} = p^{en}$. It follows that $k(B) \simeq \otimes_k(\otimes_k k(a))_{a\in B}$. Consequently, any r-basis B_1 of K/k satisfies $K \simeq \otimes_k(\otimes k(a))_{a\in B_1}$ and $o_1(K/k) = e$. On the other hand, condition (1) follows from the fact that if G is an r-basis of K/k, then the same is true for $((ab)_{a\in G\setminus\{b\}} \cup \{b\})$ for every element b of G with the family$((k(ab))_{a\in G\setminus\{b\}}, k(b))$ of subfields of K/k are k linearly disjoint according to condition (4). □

Definition 2.13. An extension that satisfies one of the conditions of the above proposition is called **equiexponential** extension of exponent e.

It is easy to verify the following equivalent conditions:

(1) K/k is equiexponential of exponent e.
(2) There exists an r-basis G of K/k, for every finite subset G_1 of G, we have $k(G_1)/k$ is equiexponential of exponent e.
(3) For any r-basis G of K/k, for any finite subset G_1 of G, we have $k(G_1)/k$ is equiexponential of exponent e.

In particular, any equiexponential extension is modular.

Proposition 2.14. *For any modular relatively perfect extension K/k, for all n, k_n/k is equiexponential of exponent n (recall that $k_n = k^{p^{-n}} \cap K$).*

Proof. From Proposition 2.8, it suffices to show that $k({k_n}^p) = k_{n-1}$. According to the modularity of K/k, K^{p^n} and k are $k \cap K^{p^n}$ linearly disjoint for each $n \geq 1$, and by virtue of transitivity of linear disjointness, $k^{p^{n-1}}(K^{p^n})$ and k are $k^{p^{n-1}}(k \cap K^{p^n})$ linearly disjoint. But K/k is relatively perfect, so $k^{p^{n-1}}(K^{p^n}) = K^{p^{n-1}}$. Therefore $k \cap K^{p^{n-1}} = k^{p^{n-1}}(k \cap K^{p^n})$ or, equivalently, to $k({k_n}^p) = k_{n-1}$. □

As a consequence, in the case of q-finite extensions (notably case of finite extensions) we give a more precise version of the Proposition 2.14.

Proposition 2.15 ([6], p. 147, Proposition 9.4)**.** *Let K/k be a q-finite extension of irrationality degree t which is relatively perfect and modular (respectively, finite extension and equiexponential). Let n and m be two natural numbers such that $n < m$ (respectively, $n < o_1(K/k)$). The following properties are verified:*

- $di(k_m/k_n) = t$.
- *k_m/k_n is equiexponential of exponent $m - n$;*
- $k_n^{p^{-(m-n)}} \cap K = k_m$ *and* $k(k_m^{p^{m-n}}) = k_n$.

In particular, for each positive integer n, we have $[k_n, k] = p^{nt}$.

Corollary 2.16. *If K/k is an equiexponential extension of exponent e, then:*

(1) *For each $i \in \{1, \dots, e\}$, k_i/k and K/k_i are equiexponential of exponent i and $e - i$, respectively.*

(2) *For each $i \in \{1, \dots, e\}$, $k(K^{p^i})/k$ and $K/k(K^{p^i})$ are equiexponential of exponent $e - i$ and i, respectively.*

The above theorem extends [9, p. 292, Theorem 4.4] concerning the homogeneity of modular r-basis of an equiexponential extension (for more details, we refer to [9] and [10]).

Theorem 2.17. *Let $k \subseteq L \subseteq K$ be a purely inseparable extensions such that K/k is equiexponential of exponent e. If K/L is modular, there exists an r-basis G of K/k such that the set $\{a^{p^{o(a/L)}} \mid a \in G$ and $o(a/L) < e\}$ is a modular r-basis of L/k.*

Proof. Since K/L is modular of finite exponent, there exists an r-basis B_1 of K/L such that $K \simeq \otimes_L (\otimes_L L(a))_{a \in B_1}$, (*). To lighten the notation, we set $e_a = o(a/L)$ for each $a \in B_1$ and $C = (a^{p^{e_a}})_{a \in B_1}$. Let B_2 be a subset of L such that B_2 is an r-basis of $L(K^p)/k(K^p)$. Taking into account the transitivity of r-independence, $B_1 \cup B_2$ is also an r-basis of K/k. Now consider the extension M of k obtained by adjoining C and B_2 to k. It's clear that $M \subseteq L$, moreover as K/k is equiexponential, we will have $K \simeq \otimes_k (\otimes_k k(a))_{a \in B_1 \cup B_2}$. By virtue of transitivity of linear disjointness, $K \simeq \otimes_M (\otimes_M M(a))_{a \in B_1}$, (**). In particular, from the relations (*) and (**), for every finite family $\{a_1, \dots, a_n\}$ of elements of B_1, $L(a_1, \dots, a_n) \simeq L(a_1) \otimes_L \cdots \otimes_L L(a_n)$ and $M(a_1, \dots, a_n) \simeq M(a_1) \otimes_M \cdots \otimes_M M(a_n)$. By application of [7, p. 374, Proposition 7], we have successively $[L(a_1, \dots, a_n) : L] = \prod_{i=1}^{n} p^{e_{a_i}}$ and $[M(a_1, \dots, a_n) : M] = \prod_{i=1}^{n} p^{e_{a_i}}$ or, equivalently, to L and K are M linearly disjoint; from whence $L = L \cap K = M$. □

3. $+\infty$-ω_0-generated extensions

3.1. u-sequences.

Definition 3.1. A sequence $k = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots \subseteq K$ of subfields of a purely inseparable extension K/k is said to be u-sequence (upper sequence) in K over k if for any index i, K_{i+1}/K_i has unbounded exponent.

We tacitly assume henceforth, unless otherwise stated, that K/k is of unbounded exponent. We check that $k = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots \subseteq K$ is a u-sequence if and only if the same holds for $L = L(K_0) \subseteq L(K_1) \subseteq \cdots \subseteq L(K_n) \subseteq \cdots \subseteq K$ for every intermediate field L between k and K that is finite over k. In particular, if K/k is q-finite, then $k = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots \subseteq K$ is a u-sequence if and only if it is the same for $k = K_0 \subseteq rp(K_1/k) \subseteq \cdots \subseteq rp(K_n/k) \subseteq \cdots \subseteq K$.

Proposition 3.2 ([14], Proposition 2.5)**.** *Any decreasing sequence of a q-finite extension is stationary.*

Proof. Let (K_n/k) be a decreasing sequence of subfields of K/k and (F_i/k) the sequence associated with their relatively perfect closure. In view of Theorem 2.1 and [13, Proposition 3.1], the sequence of integers $(di(F_n/k))$ is decreasing, hence stationary starting at rank n_0. We deduce by [13, Corollary 3.7] that $di(F_n/F_{n_0}) = 0$ for all $n \geq n_0$, and so $F_n = F_{n_0}$ for all $n \geq n_0$. By virtue of monotony, for all $n \geq n_0$, $[K_{n+1} : F_{n_0}] \leq [K_n : F_{n_0}]$. In other words, the sequence of integers $([K_n : F_{n_0}])_{n \geq n_0}$ is decreasing, whence stationary from a rank e or, equivalently, to for each $n \geq e$, $[K_n : F_{n_0}] = [K_e : F_{n_0}]$. As for each $n \geq e$, $K_n \subseteq K_e$, then $K_n = K_e$ for every $n \geq e$. □

Corollary 3.3. *In a q-finite extension, any u-sequence is stationary.*

Let K/k be a q-finite extension, we say that K/k has a u-sequence of length n if K can be decomposed into extensions: $k = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$ such that K_{i+1}/K_i has unbounded exponent for each $i \in \{0, \ldots, n-1\}$. Therefore, K/k has a maximal u-sequence, and any u-sequence in K over k may be prolonged to a maximal u-sequence of K/k. It is apparent that a maximal u-sequence presents an irreducible form in the sense that between two consecutive terms there is no proper extension of unbounded exponent, and hence impossible to decompose two consecutive terms into u-sequence of length 2. It should be noted that this form of irreducibility will constitute the subject of what follows.

Proposition 3.4. *In a q-finite extension K/k the length of any u-sequence of K/k is increased by $di(K/k)$. In particular, K/k has a u-sequence of maximal length.*

Proof. We come back to the case where all consecutive terms are relatively perfect in which case the result follows immediately from [13, Proposition 3.8]. □

Remark 3.5. In general, the terms and length of a maximal u-sequence are not unique. However, one can look for other forms of uniqueness, for example one may wonder if a u-sequence of relatively perfect terms and of maximum length preserves the irrationality degree up to a permutation. We do not yet have a precise answer to such a question.

3.2. ω_0-generated extensions. For convenience, we extend slightly the definition of ω_0-generated as follows:

Definition 3.6. A purely inseparable extension K/k of unbounded exponent is called ω_0-generated if L/k has bounded exponent for each proper intermediate field L.

In particular, if K/k is q-finite, then K/k is ω_0-generated if every proper intermediate field is finite dimensional over k and; consequently we return to the definition given separately by J.K Devney in [8], R. Gilmer and W. Heinzer in [11]. We immediately check that:

- Any ω_0-generated extension is relatively perfect.
- K/k is ω_0-generated if and only if $k \longrightarrow K$ is a u-sequence of maximal length and K/k is relatively perfect.
- If K is relatively perfect over k, then for every intermediate field L of K/k of finite exponent, $L(K)/L$ is ω_0-generated if the same holds true for K/k.

The result below ensures the existence of ω_0-generated extensions. More specifically, we have:

Theorem 3.7. *Let K/k be a q-finite extension of unbounded exponent. The set H of subfields of K/k of unbounded exponent ordered by inverse inclusion is inductive (namely, $K_1 \leq K_2$ if and only if $K_2 \subseteq K_1$). In particular, K/k contains an ω_0-generated extension.*

Proof. Immediately follows from Propositions 3.2 and 3.4. □

Without loss of generality, we agree that the definition of an ω_0-generated extension include the extensions of bounded exponent as special cases, since every subfield of an extension of bounded exponent is also of bounded exponent.

Proposition 3.8. *Any q-finite extension is decomposed into a finite number of ω_0-generated extensions.*

Proof. The result is clear if K/k is finite. Otherwise, by Proposition 3.4, K/k has a u-sequence $k = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$ of maximal length n. Necessarily $K_i \subseteq K_{i+1}$ is a u-sequence of maximal length 1. Otherwise K/k admits a u-sequence of length greater than n, a contradiction. Consequently, we are led to prove the result when $k \subseteq K$ is of maximal length. In particular, $rp(K/k)/k$ has no a proper subfield of unbounded exponent. However, according to [13, Proposition 3.1], $K/rp(K/k)$ is finite, and consequently K/k decomposes into a finite number of ω_0-generated extensions. □

Sweedler showed in [23, p. 404, Corollary 2] for any purely inseparable extension K/k, there exists a unique minimal intermediate field m of K/k over which K is modular. Improving [6, p. 148, Theorem 1.4], we have shown in [15, p. 75, Theorem 3.3] that m is not trivial when K/k is q-finite, i.e, $m \neq K$. More precisely, if K/k is of finite irrationality degree and of unbounded exponent, the same is also true for K/m. However, if K/k is of unbounded irrationality degree, we may well lose this property by obtaining $m = K$ (for example see [6, p. 149]).

In the case of modular extensions, the following result shows that the ω_0-generated becomes an intrinsic property exclusively linked to the q-finite extensions.

Theorem 3.9. *For an ω_0-generated extension K/k to be q-finite it is necessary and sufficient that the minimal intermediate field m over which K is modular is nontrivial, i.e., $m \neq K$.*

In the proof, we will need the following result:

Lemma 3.10. *Let K/k be a purely inseparable extension of unbounded exponent and irrationality degree. If K/k is relatively perfect and modular, then K/k contains a proper modular subfield L of unbounded exponent over k.*

Proof. We will build by induction a strictly increasing sequence $(K_n)_{n\geq 1}$ of modular intermediate field of K/k such that for all n, K_n/k has exponent n. As K/k is relatively perfect, according to Proposition 2.14 and Corollary 2.9, for each $n \geq 1$, $di(k^{p^{-n}} \cap K/k) = di(k^{p^{-1}} \cap K/k) = di(K/k)$ and $k^{p^{-n}} \cap K/k$ is equiexponential of exponent n. Let G_1 be an r-basis of $k^{p^{-1}} \cap K/k$, it follows that $k^{p^{-1}} \cap K \simeq \otimes_k(\otimes_k k(a))_{a\in G_1}$. Let us choose an element x of G_1, since G_1 is infinite, there exists a finite subset G'_1 of G_1 such that $x \notin k(G'_1)$, in which case we denote $K_1 = k(G'_1)$. It is clear that K_1/k is modular. We suppose that we have constructed a sequence of extensions $k \subseteq K_1 \subseteq K_2 \subseteq \dots K_n \subseteq K$ such that

(1) For each $i \in \{1, \dots, n\}$, K_i/k is finite modular extension.
(2) For every $i \in \{1, \dots, n\}$, $o_1(K_i/k) = i$.
(3) $x \notin K_n$.

Let G_{n+1} be an r-basis of $k^{p^{-n-1}} \cap K/k$, from Proposition 2.12, $k^{p^{-n-1}} \cap K \simeq \otimes_k(\otimes_k k(a))_{a\in G_{n+1}}$. As $o_1(K_n/k) = n$, we deduce that $K_n \subseteq k^{p^{-n-1}} \cap K$. But K_n/k is finite and G_{n+1} is infinite, therefore there exists a finite subset G'_{n+1} of G_{n+1} such that $K_n \subseteq k(G'_{n+1})$. If $x \notin k(G'_{n+1})$, then $K_{n+1} = k(G'_{n+1})$ is suitable. If $x \in k(G'_{n+1})$, since $k^{p^{-n-1}} \cap K \simeq \otimes_k(\otimes_k k(a))_{a\in G'_{n+1}} \otimes_k (\otimes_k k(a))_{a\in G_{n+1}\setminus G'_{n+1}}$, $x \notin k(G_{n+1} \setminus G'_{n+1})$. Otherwise, as $k(G'_{n+1})$ and $k(G_{n+1} \setminus G'_{n+1})$ are k linearly disjoint, then $x \in k(G'_{n+1}) \cap k(G_{n+1} \setminus G'_{n+1}) = k$, a contradiction. Let y be an element of $G_{n+1} \setminus G'_{n+1}$, ($y$ exists because G_{n+1} is infinite and G'_{n+1} is finite). Let $K_{n+1} = K_n(y)$, it is immediately verified that

- $x \notin K_{n+1}$, since $K_{n+1} \subseteq k(G_{n+1} \setminus G'_{n+1})$ and $x \notin k(G_{n+1} \setminus G'_{n+1})$.
- K_{n+1}/k is finite and $o_1(K_{n+1}/k) = o(y/k) = n+1$.
- $K_{n+1} \simeq K_n \otimes_k k(y)$, (application of the transitivity of linear disjointness of $k(G'_{n+1})$ and $k(G_{n+1} \setminus G'_{n+1})$), and as K_n/k is modular, by [5, p. 55, Lemma 3.4], K_{n+1}/k is modular.

Hence, K_{n+1}/k is suitable, and so $L = \bigcup_{i \geq 1} K_i$ is modular [24, p. 40, Proposition 1.2] and of unbounded exponent over k with $x \notin L$. □

Proof of Theorem 3.9. The necessary condition immediately follows from [15, p. 75, Theorem 3.3]. Conversely, let m be the minimal intermediate field over which K is modular. Since K/k is ω_0-generated and $m \neq K$, m/k has an exponent e, and from Lemma 3.10, K/m will be q-finite. In the following, for every n, we set $K_n = m^{p^{-e-n}} \cap K$ and $di(K/m) = l$. Let G_n be an r-basis of K_n/m, taking into account Proposition 2.14 and Corollary 2.9, $|G_n| = l$ and $o_1(K_n/m) = e+n$. Moreover, we have $k({K_n}^{p^e}) = k(m^{p^e}, {G_n}^{p^e}) = k({G_n}^{p^e})$, so $di(k({K_n}^{p^e})/k) \leq l$ and $o_1(k({K_n}^{p^e})/k) \geq o_1(m({K_n}^{p^e})/m) = n$. In particular, the extension $H = \bigcup k({K_n}^{p^e})$ of k has unbounded exponent, but as K/k is ω_0-generated, we get $K = H$. However, by virtue of [13, Proposition 2.3], $di(H/k) = \sup_{n \in \mathbb{N}}(di(K_n/k)) \leq l$, it follows that K/k is q-finite. □

Corollary 3.11. *Any ω_0-generated modular extension is q-finite.*

In the following subsection we extend the notion of ω_0-generated extension.

3.3. Generalization of an ω_0-generated extension.

Definition 3.12. Let j be a positive integer. A purely inseparable extension K/k of unbounded exponent is said to be j-ω_0-generated if K/k has no proper intermediate field of unbounded exponent and of irrationality degree less than or equal to j.

In other words, any proper intermediate field of K/k whose irrationality degree does not exceed j strictly has an exponent.

Definition 3.13. A purely inseparable extension K/k is called $+\infty$-ω_0-generated if K/k is j-ω_0-generated for all j.

Remark 3.14. By Theorem 3.9, any modular ω_0-generated extension is of finite irrationality degree. This is no longer the case for $+\infty$-ω_0-generated extension. Indeed, in Theorem 3.20, we exhibit an example of a modular $+\infty$-ω_0-generated extension of infinite irrationality degree. The construction requires the following results.

Theorem 3.15. *Given a purely inseparable extension K/k which is relatively perfect and modular, and let L be a proper intermediate field of K/k. If K/L is modular and $[L:k]<\infty$, then for every integer $n>e=o_1(L/k)$, $k^{p^{-n}}\cap K/k(L^{p^{e-1}})$ is modular. In particular, $K/k(L^{p^{e-1}})$ is also modular.*

For the proof of this theorem, we will use the following results. Firstly, for all non-negative integer n, consider $K_n=k^{p^{-e-n}}\cap K$ and $L_n=L^{p^{-n}}\cap K$.

Lemma 3.16. *Under the same assumptions of the above theorem, for each positive integer n, there exists two subfields N and M of K_n/k such that:*

- *$L\subseteq k(N^{p^n})$, with N/k is finite.*
- *$K_n\simeq M\otimes_k N\simeq(M\otimes_k L)\otimes_L N$. Moreover, M/k and N/k are equiexponential of exponent $n+e$.*
- *$L(M)/L(M^p)$ and $L({L_{n+e}}^p)/L(M^p)$ are $L(M^p)$ linearly disjoint.*
- *$L_{n+e}/L(M)$ is modular with $di(L_{n+e}/L(M))=di(K_n/M)=di(N/k)=di(K_n/L(M))$.*

Proof. Since L/k has an exponent e, $L\subseteq k^{p^{-e}}\cap K$; from whence $L\to L^{p^{-n}}\cap K\to K_n\to L_{n+e}$. Let G be an r-basis of K_n/k. As K/k is relatively perfect and modular, then according to Proposition 2.14, K_n/k is equiexponential of exponent $n+e$. In particular, $K_n\simeq\otimes_k(\otimes_k k(a))_{a\in G}$, and therefore $K_0=k({K_n}^{p^n})\simeq\otimes_k(\otimes_k k(a^{p^n}))_{a\in G}$. But L/k is finite and $L\subseteq K_0$, so there exists a finite subset G_1 of G such that $L\subseteq k({G_1}^{p^n})$. Let us denote the relative complement of G_1 in G by G_2, ($G_2=G\setminus G_1$), and consider the extensions N and M of k obtained, respectively, by adjoining G_1 and G_2 to k. It is immediately verified that:

- $K_n\simeq M\otimes_k N\simeq(M\otimes_k L)\otimes_L N$.
- M and N are equiexponential of exponent $n+e$.

In particular, for each $x\in G_2$, $o(x/L(G_2\setminus\{x\}))=n+e$, and consequently if there exists $x\in G_2$ such that $x\in L({L_{n+e}}^p)(G_2\setminus\{x\})$, we will have $n+e=o(x/L(G_2\setminus\{x\}))\leq o_1(L({L_{n+e}}^p)(G_2\setminus\{x\})/L(G_2\setminus\{x\}))\leq o_1(L({L_{n+e}}^p)/L)=n+e-1$, a contradiction; from whence G_2 is r-independent in L_{n+e}/L or, equivalently, to $L(M)$ and $L({L_{n+e}}^p)$ are linearly disjoint over $L(M^p)$. Therefore there exists a subset G_3 of L_{n+e} such that $G_2 \cup G_3$ is an r-basis of $L_{n+e}/L({L_{n+e}}^p)$, so $G_2 \cup G_3$ is a minimal generating set of L_{n+e}/L. Since K/L is modular and relatively perfect, $L_{n+e} \simeq \otimes_L(\otimes_L L(a))_{a\in G_2\cup G_3} \simeq (L \otimes_k M) \otimes_L (\otimes_L L(a))_{a\in G_3}) \simeq M \otimes_k (\otimes_L L(a))_{a\in G_3})$. Hence, $L_n(M) \simeq M \otimes_k (\otimes_L L(a^{p^e}))_{a\in G_3} \simeq (M \otimes_k L) \otimes_L (\otimes_L L(a^{p^e}))_{a\in G_3} \subseteq K_n$ and $K_n \simeq M \otimes_k N \simeq (M \otimes_k L) \otimes_L N \subseteq L_{n+e}$. Firstly, as N/k is equiexponential of exponent $n + e$ and $L \subseteq k(N^{p^n})$, we will have $|G_1| = di(N/k) = di(N/k(N^p)) = di(N/k(N^{p^n})) \leq di(N/L) \leq di(N/k)$, and thus $di(N/L) = |G_1|$. On the other hand, by virtue of Theorem 2.1 and [13, Corollary 2.5], we have $|G_3| = di(L_n(M)/L(M)) \leq di(K_n/L(M)) = di(N/L)$ and $di(K_n/L(M)) \leq di(L_{n+e}/L(M)) = |G_3|$, (namely $K_n \subseteq L_{n+e}$). As a result, $|G_3| = |G_1| = di(N/k)$. □

As $K_n \simeq M \otimes_k N \simeq (M \otimes_k L) \otimes_L N$ and K_n/k are equiexponential of exponent $n + e$, it is immediately verified that:

- For each $i \in \{1, \ldots, n\}$, $k({K_n}^{p^i}) = K_{n-i} = k(M^{p^i}) \otimes_k k(N^{p^i})$, so $M({K_n}^{p^i}) = M(K_{n-i}) = M \otimes_k k(N^{p^i})$. In particular, for each $i \in \{1, \ldots, n\}$, $M(K_i)/M$ is equiexponential of exponent $e + i$ and $di(N/k) = di(M(K_i)/M)$.
- $L_{n+e}/L(M)$ is equiexponential of exponent $n + e$.

In the following we set $di(N/k) = j$, and denote by s the largest integer such that $o_s(L/k) = o_1(L/k) = e$.

Lemma 3.17. *Under the above conditions, for every positive integer n, we have:*

(1) *$di(M({K_n}^{p^i})/L(M)) = di(N/k)$, for each $i \in \{0, \ldots, n-1\}$.*

(2) *$di(M({K_n}^{p^n})/L(M)) = di(M(K_0)/L(M)) = j - s$.*

In particular, for each $r \in \{j - s + 1, \ldots, j\}$, $o_r(K_n/L(M)) = o_{j-s+1}(K_n/L(M)) = n$.

Proof. Let $\{\alpha_1, \ldots, \alpha_m\}$ be a canonically ordered r-basis of L/k, hence $k \to k(\alpha_1, \ldots, \alpha_s) \to L \to K_0 \to K_n$. Let B be an r-basis of $M(K_0)/M(L)$, therefore $M(K_0) = M(\alpha_1, \ldots, \alpha_m, B)$. But $L(M) \simeq L \otimes_k M$, then $M(\alpha_1, \ldots, \alpha_s)/M$ is equiexponential of exponent e. We complete the system $\{\alpha_1, \ldots, \alpha_s\}$ into an r-basis of $M(K_0)/M$ by a subset C of K_0 [7, p. 374, Proposition 8]. In particular, we will have $|B| = di(M(K_0)/L(M)) \leq di(M(K_0)/M(\alpha_1, \ldots, \alpha_s)) = |C| = j - s$. Moreover, for each $r \in \{s + 1, \ldots, m\}$, $o(\alpha_r/k(\alpha_1, \ldots, \alpha_s)) < e$, thus by applying the r-basis completion algorithm [7, p. 374, Proposition 8], we have $M(K_0) = M(\alpha_1, \ldots, \alpha_s, B)$, so B is an r-basis of $M(K_0)/M(\alpha_1, \ldots, \alpha_s)$, and therefore $|B| = j - s$, whence $di(M(K_0)/M(L)) = |B| = j - s$. Similarly, we have $L(M)({K_n}^{p^{n-1}}) = K_1(M)$ and $L(M)({K_n}^{p^n}) = M(K_0)$. As $K_n \simeq L(M) \otimes_L N$, then $M(K_n{}^{p^{n-1}}) \simeq L(M) \otimes_L L(N^{p^{n-1}})$, it follows that $di(M(L)(K_n{}^{p^{n-1}})/M(L)) = di(M(K_1)/M(L)) = di(L(N^{p^{n-1}})/L)$. But N/k is equiexponential of exponent $n+e$ and $L \subseteq k(N^{p^n})$, by virtue of Theorem 2.1 and [13, Corollary 2.5], we will have $j = di(k(N^{p^{n-1}})/k(N^{p^n})) \leq di(L(N^{p^{n-1}})/L) \leq di(k(N^{p^{n-1}})/k) = j$. As a result, $di(M(K_1)/M(L)) = j$, whence, according to Theorem 2.3, for each $r \in \{j-s+1, \dots, j\}$, $o_r(K_n/L(M)) = o_{j-s+1}(K_n/L(M)) = n$. □

Proof of Theorem 3.15. Throughout this demonstration, we will use the previous notations. First, we briefly recap some useful results: for every positive integer n, we have

(1) $K_n \subseteq L_{n+e}$.
(2) $L_{n+e}/L(M)$ is modular with $di(L_{n+e}/L(M)) = j = di(K_n/L(M))$.
(3) $K_n \simeq L(M) \otimes_L N$.

By virtue of Proposition 2.11, there exists a canonically ordered r-basis $\{a_1, \dots, a_j\}$ of $K_n/L(M)$ such that $K_n \simeq L(M) \otimes_L L(a_1, \dots, a_{j-s}) \otimes_L L(a_{j-s+1}) \otimes_L \cdots \otimes_L L(a_j)$, and so for each $i \in \{j-s+1, \dots, j\}$, $a_i{}^{p^n} \in L$. Let $\{\alpha_1, \dots, \alpha_m\}$ be an r-basis of L/k, hence $K_n = M(\alpha_1, \dots, \alpha_m, a_1, \dots, a_j)$. As $o(\alpha_i/k) \leq e$ for each $i \in \{1, \dots, m\}$ and K_n/M is equiexponential of exponent $n+e$, then by the r-basis completion algorithm and Lemma 3.16, we have $K_n \simeq M(a_1, \dots, a_j) \simeq M \otimes_k k(a_1) \otimes_k \cdots \otimes_k k(a_j)$. But K_n/k and K_n/M are equiexponential of exponent $n+e$, therefore $k(a_{j-s+1}{}^{p^n}, \dots, a_j{}^{p^n})/k$ is equiexponential of exponent e. On the other hand, $k(a_{j-s+1}{}^{p^n}, \dots, a_j{}^{p^n}) \subseteq L$, thus by completing this system to a canonically ordered r-basis of L/k, we get $k(L^{p^{e-1}}) = k(a_{j-s+1}{}^{p^{n+e-1}}, \dots, a_j{}^{p^{n+e-1}})$. Accordingly, by virtue of Proposition 2.7, we will have $K_n \simeq M \otimes k(a_1) \otimes_k \cdots \otimes_k k(a_j) \simeq (M \otimes_k k(L^{p^{e-1}})) \otimes_{k(L^{p^{e-1}})} k(L^{p^{e-1}})(a_1) \otimes_{k(L^{p^{e-1}})} \cdots \otimes_{k(L^{p^{e-1}})} k(L^{p^{e-1}})(a_j)$ with M/k is modular. According to [5, p. 55, Lemma 3.4], we deduce that $K_n/k(L^{p^{e-1}})$ is also modular. □

Lemma 3.18. *Let K/k be an equiexponential extension of exponent $n > 1$ and L a proper intermediate field of $k^{p^{-1}} \cap K/k$. If $k \not\subseteq K^p$, there exists an extension K'/K satisfying the conditions below:*

(1) *$di(K/k) = di(K'/k)$,*
(2) *K'/k is equiexponential of exponent $n+1$,*
(3) *K'/L is not modular.*

Proof. If K/L is not modular, then $K' = K(B^{p^{-1}})$, where B is an r-basis of K/k, is suitable. If K/L is modular, according to Theorem 2.17, there exists an r-basis G of K/k such that $G_1 = \{(a^{p^{o(a/L)}})_{a \in G} |\, o(a/L) < n\}$ is also a modular r-basis of

L/k and $K \simeq \otimes_L(\otimes_L L(a))_{a\in G}$. Since $L \subseteq k^{p^{-1}} \cap K$ and K/k is equiexponential of exponent n, for every $a \in G$, we have $o(a/k^{p^{-1}} \cap K) = n-1 \leq o(a/L) \leq o_1(K/k) = n$. It follows that $G_1 = \{a \in G \text{ such that } o(a/L) = n-1\}$, and consequently $K \simeq \otimes_L(\otimes_L L(a))_{a\in G_1} \otimes_L (\otimes_L L(a))_{a\in G\setminus G_1}$. Necessarily $G \setminus G_1$ and G_1 are nonempty, otherwise $k^{p^{-1}} \cap K = L$ or $L = k$. However this contradicts the fact that L is a proper subfield of $k^{p^{-1}} \cap K/k$. Let $\alpha \in G \setminus G_1$ and $\beta \in G_1$. As $k \not\subseteq K^p$, there exists $t \in k$ such that $t \notin K^p$. We then set $G' = (a^{p^{-1}})_{a\in G\setminus\{\beta\}} \cup \{t^{p^{-1}}\alpha^{p^{-1}} + \beta^{p^{-1}}\}$ and $K' = k(G')$. It is easily verified that

- K'/k is equiexponential of exponent $n+1$.
- $K \subseteq K'$ and $di(K/k) = di(K'/k)$.

Suppose that K'/L is modular. As $\alpha^{p^{n-1}} \notin L$ or, equivalently, to $(1, \alpha^{p^{n-1}})$ is linearly independent over L, then it is remains in particular linearly independent over $L \cap K'^{p^n}$. We complete this system to a linear basis B of K'^{p^n} over $L \cap K'^{p^n}$. Since K'^{p^n} and L are $L \cap K'^{p^n}$ linearly disjoint by virtue of modularity, B is also a linear basis of $L(K'^{p^n})$ over L. But $(\alpha^{p^{-1}}t^{p^{-1}} + \beta^{p^{-1}})^{p^n} = t^{p^{n-1}}\alpha^{p^{n-1}} + \beta^{p^{n-1}}$ and $(\alpha^{p^{-1}}t^{p^{-1}} + \beta^{p^{-1}})^{p^n}$ is written uniquely as a sum of elements of B, by identification we will have $t^{p^{n-1}} \in k \cap K'^{p^n}$, and so $t^{p^{-1}} \in k^{p^{-1}} \cap K' = k^{p^{-1}} \cap K \subseteq K$, this contradicts the fact that $t \notin K^p$. It follows that K'/L is not modular. □

Lemma 3.19. *Let Ω be an algebraic closure of a field k of characteristic $p > 0$ and H the set of intermediate fields of Ω/k that are finite over k. If k is countable, the same is also true for Ω and H.*

Proof. Consider the equivalence relation $\sim$ on Ω defined by $\alpha \sim \beta$ if and only if $irr(\alpha, k) = irr(\beta, k)$ where $irr(\alpha, k)$ and $irr(\beta, k)$ are respectively the minimal polynomials over k of α and β. Let E be a system of coset representatives for $\Omega/\sim$ (we can choose the elements of E among the roots of all the irreducible monic polynomials in such a way that each polynomial will be identified by one and only one root, that is, by an element of E). Since the roots of a polynomial are finite, for every $a \in E$, $|\overline{a}|$ is finite. Similarly, we have $k[X]$ is countable, in particular E is also countable, and consequently $\Omega = \bigcup_{a\in E} \overline{a}$ is countable [3, III, p. 49, Corollary 3]. In the sequel, we shall denote for every positive integer n, $H_n = \{L \in H$ such that L/k is generated by at most n elements of $\Omega\}$. It's clear that the mapping:

$$\begin{array}{rcl} \Omega^n & \longrightarrow & H_n, \\ (\alpha_1, \ldots, \alpha_n) & \longmapsto & k(\alpha_1, \ldots, \alpha_n), \end{array}$$

is surjective, so $|H_n| \leq |\Omega^n|$, and consequently H_n is countable. Since $H = \bigcup_{n\geq 1} H_n$, H/k is countable [3, III, p. 49, Corollary 3]. □

We now have all the necessary tools to construct a $+\infty$-ω_0-generated extension of unbounded irrationality degree. For this, we consider a countable field k of characteristic $p > 0$ and of unbounded imperfection degree (cardinality of a p-base of k, which is equal to $di(k/k^p)$), and let $((X_i)_{i\in\mathbb{N}^*}, t)$ be a p-independent subset of k. We set $M_1 = k((X_i{}^{p^{-1}})_{i\in\mathbb{N}^*})$ and $M_2 = k((X_i{}^{p^{-2}})_{i\in\mathbb{N}^*})$. Let E be the set of proper intermediate fields of M_1/k. By virtue of Lemma 3.19, E can be presented as $E = (L_n)_{n\geq 3}$. By repeated application of Lemma 3.18, we construct a sequence of increasing extensions $(M_n/k)_{n\geq 3}$ satisfying:

(1) M_n/L_n is not modular.
(2) M_n/k is equiexponential of exponent n.

Finally let $K = \bigcup_{i\in\mathbb{N}^*} M_n$.

Theorem 3.20. *The extension K/k above is modular and $+\infty$-ω_0-generated of unbounded irrationality degree (By Theorem 3.9, this extension is not ω_0-generated).*

For the proof we will use in addition the following result:

Lemma 3.21 ([6], p. 155, Lemma 2.6)**.** *Let $k \subseteq S \subset K$ be purely inseparable extensions such that K/k is modular. If L is an intermediate field of S/k over which S is modular, then K/L is also modular, (in particular, the same is true for K/S).*

Proof of Theorem 3.20. Firstly, by construction K/k is relatively perfect of unbounded irrationality degree. Let S be a proper intermediate field of K/k of irrationality degree j over k. Suppose that S/k is of unbounded exponent. By virtue of Theorem 3.7, S/k contains an ω_0-generated extension that is denoted by S'. In particular S'/k is relatively perfect. Let L' be the minimal intermediate field of S'/k over which S' is modular, so L'/k is finite from Theorem 3.9. If we set $L = L'^{p^{-1}} \cap S'$, then S'/L is modular [6, p. 144, Proposition 6.4]. Thus, in view of Lemma 3.21 above, K/L is also modular. By using Theorem 3.15, for every integer $n > o_1(L/k)$, the same is also true for $k_n/k(L^{p^{e-1}})$ where $e = o_1(L/k)$ and $k_n = k^{p^{-n}} \cap K$. In addition, $k(L^{p^{e-1}})$ belongs to E (because it is finite and of exponent 1 over k), therefore there exists a natural number t such that $L_t = k(L^{p^{e-1}})$. But $k_n = M_n$ for every $n \geq 3$, so M_n/L_t is modular for every integer $n > \sup(o_1(L/k), t)$, and a fortiori $L_t(M_n^{p^{n-t}})/L_t$ is modular. It follows that M_t/L_t is also modular, which contradicts the construction of M_n. □

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