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LAGRANGE STABILITY IN TERMS OF TWO MEASURES WITH INITIAL TIME DIFFERENCE FOR SET DIFFERENTIAL EQUATIONS INVOLVING CAUSAL OPERATORS

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Abstract. In this paper, we investigate generalized variational comparison results aimed to study the stability properties in terms of two measures for solutions of Set Differential Equations (SDEs) involving causal operators, taking into consideration the difference in initial conditions. Next, we employ these comparison results in proving the theorems that give sufficient conditions for equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures with initial time difference for the solutions of perturbed SDEs involving causal operators in regard to their unperturbed ones.

1. Introduction

Many researchers were interested in studying set differential equations (SDEs) in the recent decades [2,3,5,8–10,13,14,18,20,23,36,47] due to their unifying properties. Lakshmikantham et al. highlighted these properties in one of the most important resources on this topic [23]. The comprehensiveness of the SDEs is driven from the fact that they encompass the conventional differential and integral equations when the Hukuhara difference and integrals defined on the SDEs are restricted to $\mathbb{R}$; whereas they give us vector differential equations when the restriction is done to $\mathbb{R}^n$ [4, 19, 26].

On the other hand, many well-known differential equations such as integro differential equations [28], impulsive differential equations [22], and differential equations with delay [35], are examples of differential equations involving causal operators. Many research papers dealt with those types of equations. [1, 7–10, 21, 43]

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SDEs with causal operators unifies the fundamental theory of SDEs, including various corresponding dynamical systems. Some relevant works can be found in [5, 8–14, 47]

Although it is never feasible to know the exact solutions of all dynamical systems in practice, their attributes may be determined through a variety of qualitative studies such as stability analysis [2–5, 15, 19, 20, 24, 36], initial time difference (ITD) stability analysis [6, 29, 30, 33, 34, 37, 38, 41–47], practical stability analysis [17, 31, 40, 46], boundedness [2, 6, 11, 16, 32, 37, 38, 40–42], etc.

Many techniques have been used in this process, including the Lyapunov second method [19, 24, 33, 43, 44], variation of parameters [25, 32, 33], "in terms of two measures" methodology [5, 18, 27, 32, 38, 42, 45, 46], and so on.

In this manuscript, we develop generalized variational comparison results aimed to assess a combination of two concepts of stability and other qualitative aspects for SDEs with causal operators that unifies the conceptual framework behind SDEs. Furthermore, we give adequate criteria for equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures with ITD for the solutions of the perturbed forms of these types equations in comparison to their un-perturbed counterparts.

2. Preliminaries

In what follows, we denote the set of all compact non-empty subsets of $\mathbb{R}^n$ by $K(\mathbb{R}^n)$, and the set of all compact and convex non-empty subsets of $\mathbb{R}^n$ by $K_c(\mathbb{R}^n)$.

The Hausdorff metric between any bounded sets A and B in $\mathbb{R}^n$ is defined as

$$D(A,B) = \max\left[\sup_{x\in B} d(x,A)\ ,\ \sup_{y\in A} d(y,B)\right] \tag{1}$$

where

$$d(x,A) = \inf\ \{d(x,y)\ :\ y\in A\} \tag{2}$$

Each of $(K(\mathbb{R}^n), D)$ and $(K_c(\mathbb{R}^n), D)$ forms a complete metric space. The space $K_c(\mathbb{R}^n)$ equipped with the natural addition and non-negative scalar multiplication becomes a semi-linear metric space which can be embedded as a cone into a corresponding Banach space.

The Hausdorff metric satisfies the following properties:

$$\begin{aligned} &(1)\ D(A,B) = D(B,A)\\ &(2)\ D(A+C,B+C) = D(A,B)\\ &(3)\ D(kA,\ kB) = k\ D(A,B)\\ &(4)\ D(A,B) \le D(A,C) + D(C,B) \end{aligned} \tag{3}$$

for any $A,B,C\in K_c(\mathbb{R}^n)$ and $k\in\mathbb{R}_+$, where Minkowski addition of any two non-empty subsets A and B of $\mathbb{R}^n$ is defined by $A+B=\{a+b\ :\ a\in A,\ b\in B\}$ and where scalar multiplication of a value $k\in\mathbb{R}$ and a non-empty subset A of $\mathbb{R}^n$ is defined by $kA=\{ka : a\in A\}$. If $k=-1$, we get $-A=(-1)A=\{-a : a\in A\}$.

In general, $A+(-A)\neq\{0\}$ (unless $A=\{a\}$ is a singleton). To overcome with this implication of Minkowski difference, i.e.

$$A-B=A+(-1)B=\{a-b\ :a\in A,\quad b\in B\} \tag{4}$$

Hukuhara difference between two sets $A,B\in K_c(\mathbb{R}^n)$ is defined as follows:

If there exists a set $C\in K_c(\mathbb{R}^n)$ such that $C+B=A$, then Hukuhara difference exists and we denote it by $A\ominus B$, or simply $A-B$ when there is no confusion with Minkowski difference. i.e. $A\ominus B=C\Leftrightarrow C+B=A$.

An important property of Hukuhara difference is $A-A=\{0\}$ for $A\in K_c(\mathbb{R}^n)$.

Let $U:I\to K_c(\mathbb{R}^n)$ be a given multifunction, where I is an interval of real numbers. U is said to be Hukuhara differentiable at a point $t_0\in I$, if there exists an element $D_HU(t_0)\in K_c(\mathbb{R}^n)$ such that the limits

$$\lim_{h\to 0^+}\frac{U(t_0+h)-U(t_0)}{h}\quad\text{and}\quad\lim_{h\to 0^+}\frac{U(t_0)-U(t_0-h)}{h} \tag{5}$$

both exist in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_HU(t_0)$.

It is implicit in the definition of $D_HU(t_0)$ the exitance of the two differences $U(t_0+h)-U(t_0)$ and $U(t_0)-U(t_0-h)$, for sufficiently small $h>0$.

By embedding $K_c(\mathbb{R}^n)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, we find that if

$$G(t)=G(t_0)+\int_{t_0}^{t}F(s)\,ds,\quad t\in I \tag{6}$$

where $F:I\to K_c(\mathbb{R}^n)$ is integrable in the sense of Bochner, then G is Hukuhara differentiable, i. e. $D_HG(t)$ exits, and the equality $D_HG(t)=F(t)$, a. e. on I, holds.

Also, the Hukuhara integral

$$\int_I F(s)ds=\left[\int_I f(s)ds\ :\ f\ \textit{is a continuous selector of}\ F\right] \tag{7}$$

for any compact set $I\subset\mathbb{R}_+$.

Let $E=C\left[[t_0,\infty),\,K_c(\mathbb{R}^n)\right]$ with norm

$$\sup_{t\in[t_0,\infty)}\frac{D\left[U(t),\theta\right]}{h(t)}<\infty \tag{8}$$

where $U\in E$, θ is the zero element of $\mathbb{R}^n$, which is regarded as a point set; and $h:[t_0,\infty)\to\mathbb{R}_+$ is a continuous map. E equipped with such a norm is a Banach space.

Let $Q\in C[E,E]$. Q is said to be a causal map if $U(s)=V(s)$, $t_0\leq s\leq t<\infty$, and $U,V\in E$ then

$$(QU)(s) = (QV)(s), \quad t_0 \leq s \leq t < \infty. \tag{9}$$

Let us consider the following differential equations

$$D_HU = (QU)(t), \quad U(t_0) = U_0 \quad for \quad U_0 \in K_c(\mathbb{R}^n) \quad and \quad t \geq t_0 \geq 0, \tag{10}$$

$$D_HU = (QU)(t), \quad U(\tau_0) = V_0 \quad for \quad V_0 \in K_c(\mathbb{R}^n) \quad and \quad t \geq \tau_0 \geq 0 \tag{11}$$

$$D_HV = (PV)(t), \quad V(\tau_0) = V_0 \quad for \quad V_0 \in K_c(\mathbb{R}^n) \quad and \quad t \geq \tau_0 \tag{12}$$

$$D_HW = (SW)(t), \ W(\tau_0) = V_0 - U_0 \quad for \quad W(\tau_0) = W_0 \in K_c(\mathbb{R}^n) \quad and \quad t \geq \tau_0 \tag{13}$$

where $Q,P,S:E\to E$ are causal operators, and satisfy a local Lipschitz condition on $\mathbb{R}_+ \times S_\rho$ where $S_\rho=\left\{U\in K_c(\mathbb{R}^n) : D\left[U,\tilde{0}\right]<\rho<\infty\right\}$.

It is clear that (10) and (11) are different in the initial time and position. Moreover, if $(PV)(t)$ in (12) is written as $(PV)(t) = (QV)(t) + (RV)(t)$; Then, we consider (12) as the perturbed form corresponding to the unperturbed equation (11) with the perturbation term $(RV)(t)$.

Assuming that $\left(Q\tilde{0}\right)(t)\equiv\tilde{0}$ for $t\geq 0$, and assuming the necessary smoothness of P,Q and R to guarantee the existence and uniqueness of the solution $U(t)=U(t,t_0,U_0)$ of (10) through (t_0,U_0) for all $t\geq t_0$, and those of the solution $V(t)=V(t,\tau_0,V_0)$ of (12) through (τ_0,V_0) for all $t\geq\tau_0$, in addition to their continuous dependence on the initial conditions.

If $U \in C^1[J_1,K_c(\mathbb{R}^n)]$ on $J_1 = [t_0,t_0+T_1]$, then it is said to be a solution of (10) on J_1 if it satisfies (10) on J_1. If U,V and $W \in C^1[J_2,K_c(\mathbb{R}^n)]$ on $J_2 = [t_0,t_0+T_2]$, then these are said to be solutions of (11), (12), (13) on J_2 provided that they satisfy (11), (12), (13) on J_2, respectively.

Now let us define a partial order in the metric space $(K_c(\mathbb{R}^n), D)$. First, we start by defining a cone in $K_c(\mathbb{R}^n)$.

Definition 1. *The subfamily $K \subset K_c(\mathbb{R}^n)$ is said to be a cone in $K_c(\mathbb{R}^n)$ if it consists of sets $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a non-negative n-component vector $u = (u_1,u_2,\ldots,u_n)$ satisfying $u_i \geq 0$ for $i = 1\ldots n$. The subfamily $K^0 \subset K_c(\mathbb{R}^n)$, that consists of sets $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a positive n-component vector $u = (u_1,u_2,\ldots,u_n)$ satisfying $u_i > 0$ for $i = 1\ldots n$, is the nonempty interior of the cone K.*

Definition 2. *For any $U,\ V \in K_c(\mathbb{R}^n)$, if there exists $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K$ and $U = V + Z$ then we say that $U \geq V$ or $V \leq U$. Similarly, if there exists $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K^0$ and $U = V + Z$ then we say that $U > V$ or $V < U$.*

We present below some needed classes to develop the stability results in terms of two measures.

$$\mathbb{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] \ : \ a(u) \ \textit{is strictly increasing in } u \textit{ and } a(0) = 0\} \tag{14}$$

$$\mathbb{L} = \left\{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+] \ : \sigma(u) \ \textit{is strictly decreasing in } u \textit{ and } \lim_{u\to\infty} \sigma(u) = 0 \right\} \tag{15}$$

$$\mathbb{CK} = \left\{ \begin{array}{c} a \in C\left[\mathbb{R}_+^2, \mathbb{R}_+\right] \ : \ a(t,s) \in \mathbb{K} \ for\ each\ t \\ and\ a(t,s) \ is\ continuous\ for\ each\ s \end{array} \right\} \tag{16}$$

$$\Gamma = \left\{h \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+] \ : \inf_{(t,U)} h(t,U) = 0 \right\} \tag{17}$$

$$\Gamma_0 = \left\{h \in \Gamma : \inf_{U} \ h(t,U) = 0 \ , \ for\ each\ t \in \mathbb{R}_+\right\} \tag{18}$$

Next, to introduce a Lyapunov-like function, we present some definitions needed in the qualitative analysis in terms of two measures.

Definition 3. *Let $L \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, then L is said to be*
(i) h-positive definite if there exists a $\rho > 0$ and a $b \in \mathbb{K}$ such that

$$h(t,U) < \rho \ \ implies \ \ b(h(t,U)) \le L(t,U) \tag{19}$$

(ii) h-decrescent if there exists a $\rho > 0$ and a function $a \in \mathbb{K}$ such that

$$h(t,U) < \rho \ \ implies \ \ L(t,U) \le a(h(t,U)) \tag{20}$$

(iii) h-weakly decrescent if there exists a $\rho > 0$ and a function $a \in \mathbb{CK}$ such that

$$h(t,U) < \rho \ \ implies \ \ L(t,U) \le a(t, \ h(t,U)) \tag{21}$$

Definition 4. *Let $h_0, h \in \Gamma$, then we say that h_0 is finer than h if there exists a $\rho > 0$ and a function $\phi \in \mathbb{CK}$ such that*

$$h_0(t,U) \le \rho \ \ implies \ \ h(t,U) \le \phi(t, h_0(t,U)) \tag{22}$$

h_0 is uniformly finer than h if the function ϕ in the above definition is independent of t.

Now, let us introduce the definitions of generalized Dini-like derivatives of L.

Definition 5. *We define the generalized derivative (Dini-like derivatives) for a real-valued function $L \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ as follows:*

$$\begin{aligned} &D_*^+ L(t,s,U) \\ &= \lim_{h\to 0^+} \sup \frac{1}{h} \ \left[L\left(s+h, V\left(t, s+h, U + h\left(Q\tilde{U}\right)(s)\right)\right) - L(s, V(t,s,U))\right] \end{aligned} \tag{23}$$

$$D_{*-}L(t,s,U) = \underset{h\to 0^-}{lim}\, inf \frac{1}{h} \left[L\left(s+h, V\left(t, s+h, U+h\left(Q\tilde{U}\right)(s)\right)\right) - L(s, V(t,s,U)) \right] \tag{24}$$

for $t, s \in \mathbb{R}_+$ *and* $U \in K_c(\mathbb{R}^n)$.

Next, let us introduce the definitions of initial time difference (ITD) equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures, before proceeding with our main results.

Definition 6. *Let* $U(t,t_0,U_0)$ *be any solution of (10) for* $t \geq t_0 \geq 0$, *and let* $\tilde{U}(t,\tau_0,U_0) = U(t-\eta,t_0,U_0)$, *for* $\eta = \tau_0 - t_0$. *The solution* $V(t,\tau_0,V_0)$ *of (12) for* $t \geq \tau_0$ *is said to be*

(i) ITD (h_0,h)*-equi-bounded with respect to the solution* $\tilde{U}$, *if and only if given any* $\alpha > 0$ *and* $\tau_0 \in \mathbb{R}_+$, *there exists* $\beta = \beta(\alpha,\tau_0) > 0$ *such that* $h_0(\tau_0, V_0 - U_0) < \beta$ *implies*

$$h(t, V(t,\tau_0,V_0) - U(t-\eta,t_0,U_0)) < \alpha\ , \quad t \geq \tau_0 \tag{25}$$

(ii) ITD (h_0,h)*-uniformly equi-bounded with respect to the solution* $\tilde{U}$ *if the previous implication in (i) holds for every* $\tau_0 \in \mathbb{R}_+$, *or in otherwords,* $\beta = \beta(\alpha,\tau_0) > 0$ *is independent of* τ_0.

It is worth pointing out that if β *in (ii) satisfy that* $\beta(\cdot,\tau_0) \in \mathbb{K}$, *then the solution* $V(t,\tau_0,V_0)$ *of (12) is ITD* (h_0,h)*-stable with respect to the solution* $\tilde{U}$. *In fact, for* $\varepsilon > 0$ *there exists a continuous function* $\delta = \delta(\varepsilon,\tau_0) > 0$ *in* τ_0, *such that whenever* $\alpha < \delta$, *we have* $\beta = \beta(\alpha,\tau_0) < \varepsilon$.

(iii) ITD (h_0,h)*-equi-attractive in the large with respect to the solution* $\tilde{U}$, *if and only if given any* ε, $\alpha > 0$ *and* $\tau_0 \in \mathbb{R}_+$, *there exists a* $T = T(\tau_0,\varepsilon,\alpha) > 0$ *such that* $h_0(\tau_0, V_0 - U_0) < \alpha$ *implies*

$$h(t, V(t,\tau_0,V_0) - U(t-\eta,t_0,U_0)) < \varepsilon\ , \quad t \geq \tau_0 + T(\tau_0,\varepsilon,\alpha) \tag{26}$$

(iv) ITD (h_0,h)*-uniform equi-attractive in the large with respect to the solution* $\tilde{U}$, *if the previous implication in (iii) holds for every* $\tau_0 \in \mathbb{R}_+$, *or in otherwords,* $T = T(\tau_0,\varepsilon,\alpha) > 0$ *is independent of* τ_0.

(v) ITD (h_0,h)*-Lagrange stable with respect to the solution* $\tilde{U}$, *if and only if it is ITD* (h_0,h)*-equi-bounded and ITD* (h_0,h)*-equi-attractive in the large with respect to the solution* $\tilde{U}$.

(vi) ITD (h_0,h)*-uniform Lagrange stable with respect to the solution* $\tilde{U}$, *if and only if it is ITD* (h_0,h)*-Lagrange stable and both* $\beta = \beta(\alpha,\tau_0) > 0$ *in (i) and* $T = T(\tau_0,\varepsilon,\alpha) > 0$ *in (iii) are independent of* τ_0.

3. ITD Stability Results in Terms of Two Measures

3.1. **ITD Variational Comparison Results.** In what follows, let us present generalized variational comparison results aimed to study the stability properties in terms of two measures for solutions of SDEs involving causal operators, taking into consideration the difference in the initial conditions.

Before that, in order to study the stability properties for the SDEs with causal operators, let us assume that the solutions of the SDEs (10), (11), (12), and (13) exist and that they are unique; additionally, that all the Hukuhara differences exist, so the problem is well-posed.

Theorem 1. *Assume that (i) Both* $L(t,\Omega) \in C\left[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+^N\right]$ *and* $\|W(t,s,\Omega)\|$ *satisfy a local Lipschitz condition in* Ω *for any* t, s*; where* $W(t) = W(t,\tau_0, V_0 - U_0)$ *is the solution of (13) for* $t \geq \tau_0$*,* $\tilde{U}(t,\tau_0,U_0) = U(t-\eta,t_0,U_0)$*, for* $\eta = \tau_0 - t_0$*,* $U(t,t_0,U_0)$ *is any solution of (10) for* $t \geq t_0$*, and* $V(t) = V(t,\tau_0,V_0)$ *is the solution of (12) for* $t \geq \tau_0$*; and let* $\Omega(t) = V(t) - \tilde{U}(t)$*.*

(ii)

$$D_{*-}L(t,s,\Omega) \leq g(t,s,L(s,W(t,s,\Omega))) \tag{27}$$

where

$$\begin{aligned} &D_{*-}L(t,s,\Omega) \\ &= \lim_{\delta\to 0^-}\inf \frac{1}{\delta}\left(L\left(s+\delta, W\left(t, s+\delta, \Omega + \delta\left((PV)(s) - \left(Q\tilde{U}\right)(s)\right)\right)\right)\right. \\ &\left. - L(s,W(t,s,\Omega))\right) \end{aligned} \tag{28}$$

(iii) $g \in C\left[\mathbb{R}_+ \times \mathbb{R}_+^N, \mathbb{R}^N\right]$*,* $g(t,s,u)$ *is quasi-monotone non-decreasing in* u *for any* t, s*; [i.e., if* $u \leq v$*,* $u_i = v_i$ *for some* i *such that* $1 \leq i \leq N$*, then* $g_i(t,s,u) \leq g_i(t,s,v)$*, for* $t, s \in \mathbb{R}_+$ *(In this context, the inequality symbol used in the vectorial inequalities is understood to denote component-wise inequality [39])];*

and $r(t,s,\tau_0,V_0)$ *is the maximal solution of*

$$\frac{du(s)}{ds} = g(t,s,u(s)), \quad u(\tau_0) = u_0 \geq 0 \tag{29}$$

existing for $\tau_0 \leq s \leq t < \infty$*.*

Then, $L(\tau_0, W(t,\tau_0,V_0-U_0)) = u_0$ *implies*

$$L(t,\Omega(t,\tau_0,V_0-U_0)) \leq r_0(t,\tau_0,L(\tau_0,W(t,\tau_0,V_0-U_0))) \tag{30}$$

where $r_0(t,\tau_0,u_0) = r(t,t,\tau_0,u_0)$*.*

Proof. Let us set

$$m(t,s) = L(s,W(t,s,\Omega(s))) \quad for \quad \tau_0 \leq s \leq t. \tag{31}$$

Then, we have

$$\begin{aligned} m(t,\tau_0) &= L(\tau_0, W(t,\tau_0,\Omega(\tau_0))) = L\left(\tau_0, W\left(t,\tau_0, V(\tau_0) - \tilde{U}(\tau_0)\right)\right) \\ &= L(\tau_0, W(t,\tau_0, V_0 - U_0)) = u_0 \end{aligned} \tag{32}$$

For a sufficiently small positive value δ, we have

$$\begin{aligned} & m(t,s+\delta) - m(t,s) \\ & = L(s+\delta, W(t,s+\delta,\Omega(s+\delta))) - L(s, W(t,s,\Omega(s))) \\ & = L(s+\delta, W(t,s,\Omega(s)) + \delta(SW(t,s,\Omega(s)))(s) + \varepsilon(\delta)) - L(s, W(t,s,\Omega(s))) \end{aligned} \tag{33}$$

where ε stands for error and $\lim_{\delta\to 0^-} \frac{\varepsilon(\delta)}{\delta} = 0$.

Taking into consideration the assumptions in (i) regarding the locally Lipschitz property of $L(t,\Omega)$ and $\|W(t,s,\Omega)\|$ in Ω, it is seen that

$$\begin{aligned} m(t,s+\delta) - m(t,s) \le\ & k(\varepsilon_1(\delta) - \varepsilon_2(\delta)) \\ & + L\left(s+\delta, W\left(t,s,V(s) - \tilde{U}(s)\right) + \delta\left((PV)(s) - \left(Q\tilde{U}\right)(s)\right)\right) \\ & - L\left(s, W\left(t,s,V(s) - \tilde{U}(s)\right)\right) \end{aligned} \tag{34}$$

where $\varepsilon_1, \varepsilon_2$ stand for errors, k stands for Lipschitz constant.

The inequality in the assumption (ii) gives us the following estimation regarding the Dini derivative of $m(t,s)$

$$\begin{aligned} & D_{*-} m(t,s) \\ & \le \lim_{\delta\to 0^-} \inf \frac{1}{\delta} K(\varepsilon_1(\delta) - \varepsilon_2(\delta)) \\ & + \lim_{\delta\to 0^-} \inf \frac{1}{\delta} \quad L\left(s+\delta, W\left(t,s,V(s) - \tilde{U}(s)\right) + \delta\left((PV)(s) - \left(Q\tilde{U}\right)(s)\right)\right) \\ & - \lim_{\delta\to 0^-} \inf \frac{1}{\delta} \quad L\left(s, W\left(t,s,V(s) - \tilde{U}(s)\right)\right) \\ & \le g\left(t,s,L\left(s, W\left(t,s,V(s) - \tilde{U}(s)\right)\right)\right) \\ & = g(t,s,L(s,W(t,s,\Omega(s)))) = g(t,s,m(t,s)) \end{aligned} \tag{35}$$

for $\tau_0 \le s \le t < \infty$.

A comparison result [Theorem 1.7.1] from [26] gives us the following inequality

$$m(t,s) \le r(t,s,\tau_0, L(\tau_0, W(t,\tau_0, V_0 - U_0))) \quad for \quad \tau_0 \le s \le t. \tag{36}$$

Choosing $s=t$ in the right-hand side of the previous inequality, we get

$$\begin{aligned} m\left(t,s\right) &\leq r\left(t,t,\tau_0,L\left(\tau_0,W\left(t,\tau_0,V_0-U_0\right)\right)\right) \\ &= r_0\left(t,\tau_0,L\left(\tau_0,W\left(t,\tau_0,V_0-U_0\right)\right)\right) \end{aligned} \tag{37}$$

which yields the desired estimation in (30) completing the proof. □

Theorem 2. *Under the assumptions of Theorem 1 with* $N=1$ *and* $g\left(t,s,u\right)\equiv 0$, *we have*

$$L\left(t,\Omega\left(t,\tau_0,V_0-U_0\right)\right) \leq L\left(\tau_0,W\left(t,\tau_0,V_0-U_0\right)\right), \quad t\geq\tau_0. \tag{38}$$

Furthermore, we assume

$$D_{*-}L\left(t,s,\Omega\right) \leq -c\left(h\left(s,W\left(t,s,\Omega\right)\right)\right), \quad \tau_0\leq s\leq t<\infty \tag{39}$$

where $c\in\mathbb{K}$ *and* $h\in C\left[\mathbb{R}_+\times K_c\left(\mathbb{R}^n\right),\mathbb{R}_+\right].$

Then, for $t\geq\tau_0$

$$L\left(t,\Omega\left(t,\tau_0,V_0-U_0\right)\right) \leq L\left(\tau_0,W\left(t,\tau_0,V_0-U_0\right)\right) - \int_{\tau_0}^{t} c\left(h\left(s,W\left(t,s,\Omega\left(s\right)\right)\right)\right)ds. \tag{40}$$

Proof. Starting from the statement (35) in the proof of Theorem 1,

$$D_{*-}m\left(t,s\right) \leq g\left(t,s,m\left(t,s\right)\right) \quad for \quad \tau_0\leq s\leq t<\infty. \tag{41}$$

Then, since $g\left(t,s,u\right)\equiv 0$, we get by integrating the two sides of the previous inequality (41), for $s\in\left[\tau_0,t\right]$,

$$\int_{\tau_0}^{t} D_{*-}m\left(t,s\right)ds = L\left(t,W\left(t,t,\Omega\left(t\right)\right)\right) - L\left(\tau_0,W\left(t,\tau_0,\Omega\left(\tau_0\right)\right)\right) \leq 0. \tag{42}$$

Hence, we have

$$L\left(t,\Omega\left(t,\tau_0,V_0-U_0\right)\right) \leq L\left(\tau_0,W\left(t,\tau_0,\mathrm{V}_0-U_0\right)\right) \quad for \quad t\geq\tau_0. \tag{43}$$

Now, let us set

$$M(s,W(t,s,\Omega(s))) \equiv L(s,W(t,s,\Omega(s))) + \int_{\tau_0}^{s} c(h(\xi,W(t,\xi,\Omega(\xi))))d\xi. \tag{44}$$

Then, by taking Dini derivatives of both sides and by assumption (39), we have

$$\begin{aligned} D_{*-}M\left(t,s,\Omega\left(s\right)\right) &= D_{*-}L\left(t,s,\Omega\left(s\right)\right) + c\left(h\left(s,W\left(t,s,\Omega\left(s\right)\right)\right)\right) \\ &\quad - c\left(h\left(\tau_0,W\left(t,\tau_0,\Omega\left(\tau_0\right)\right)\right)\right) \\ &\leq D_{*-}L\left(t,s,\Omega\left(s\right)\right) + c\left(h\left(s,W\left(t,s,\Omega\left(s\right)\right)\right)\right) \\ &\leq -c\left(h\left(s,W\left(t,s,\Omega\left(\mathrm{s}\right)\right)\right)\right) + c\left(h\left(s,W\left(t,s,\Omega\left(s\right)\right)\right)\right) = 0. \end{aligned} \tag{45}$$

Thus, $D_{*-}M\left(t,s,\Omega\left(s\right)\right)\leq 0$, in view of (43), gives us for $t\geq\tau_0$,

$$M\left(t,\Omega\left(t,\tau_0,V_0-U_0\right)\right) \leq M\left(\tau_0,W\left(t,\tau_0,V_0-U_0\right)\right). \tag{46}$$

By the definition of M, this implies, for $t \geq \tau_0$,

$$\begin{aligned} &L\left(t, \Omega\left(t, \tau_0, V_0-U_0\right)\right)+\int_{\tau_0}^{t} c\left(h\left(\xi, W\left(t, \xi, \Omega\left(\xi\right)\right)\right)\right) d\xi \\ &\leq L\left(\tau_0, W\left(t, \tau_0, V_0-U_0\right)\right)+\int_{\tau_0}^{\tau_0} c\left(h\left(\xi, W\left(t, \xi, \Omega\left(\xi\right)\right)\right)\right) d\xi \end{aligned} \tag{47}$$

$$L\left(t, \Omega\left(t, \tau_0, V_0-U_0\right)\right)+\int_{\tau_0}^{t} c\left(h\left(\xi, W\left(t, \xi, \Omega\left(\xi\right)\right)\right)\right) d\xi \leq L\left(\tau_0, W\left(t, \tau_0, V_0-U_0\right)\right). \tag{48}$$

Moving the integral term to the right-hand side gives us the desired estimation (40) and this completes the proof. □

3.2. **Main ITD Stability Results in Terms of Two Measures.** Now, let us employ the comparison results in section 3.1 to prove the following theorems giving sufficient conditions for equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures for the solutions of perturbed SDEs involving causal operators in regard to their unperturbed ones.

The next theorem gives sufficient conditions to the ITD (h_0, h)-equi-boundedness of the solution $V\left(t, \tau_0, V_0\right)$ of (12) through $\left(\tau_0, V_0\right)$ for $t \geq \tau_0$ with respect to the solution $\tilde{U}\left(t, \tau_0, U_0\right)=U\left(t-\eta, t_0, U_0\right)$, for $\eta=\tau_0-t_0$, where $U(t)=U\left(t, t_0, U_0\right)$ is the solution of (10) through $\left(t_0, U_0\right)$ for $t \geq t_0$; providing that the solution $V\left(t, \tau_0, V_0\right)$ of (12) is ITD (h_0, h_0)-equi-bounded with respect to $\tilde{U}$.

Theorem 3. *Assume that*

(i) Both $L(t, \Omega) \in C\left[\mathbb{R}_{+} \times K_c\left(\mathbb{R}^n\right), \mathbb{R}_{+}\right]$ and $\|W(t, s, \Omega)\|$ satisfy a local Lipschitz condition in Ω for any t, s; where $W(t)=W\left(t, \tau_0, V_0-U_0\right)$ is the solution of (13) for $t \geq \tau_0$ and

$$\Omega\left(t, \tau_0, V_0-U_0\right)=V(t)-\tilde{U}(t) \quad for \quad t \geq \tau_0 \tag{49}$$

(ii)

$$D_{*-} L(t, s, \Omega) \leq-c(h(s, W(t, s, \Omega(\mathrm{s})))) \text { in } S(h, M) \tag{50}$$

where

$$S(h, M)=\{(t, \Omega) \ : h(t, \Omega)<M \text { for some } h \in \Gamma \text { and } M>0\} \tag{51}$$

and

$$\begin{aligned} &D_{*-} L(t, s, \Omega) \\ &=\lim_{\delta \rightarrow 0^{-}} \inf \frac{1}{\delta}\left(L\left(s+\delta, W\left(t, s+\delta, \Omega+\delta\left((P V)(s)-\left(Q \tilde{U}\right)(s)\right)\right)\right)\right. \\ &\left.-L(s, W(t, s, \Omega))\right) \end{aligned} \tag{52}$$

(iii) For $b \in \mathbb{K}$ and $a_1, a_0 \in \mathbb{CK}$,

$$\begin{aligned} &b\left(h\left(t,\Omega\right)\right)+\int_{\tau_0}^{t} c\left(h\left(s,W\left(t,s,\Omega\left(\mathrm{s}\right)\right)\right)\right)ds \leq L\left(t,\Omega\right) \ \ in\ S\left(h,M\right) \quad and \\ &L\left(t,\Omega\right) \leq a_1\left(t,h\left(t,\Omega\right)\right)+a_0\left(t,h_0\left(t,\Omega\right)\right) \ \ in\ S\left(h,M\right)\cap S\left(h_0,M\right) \end{aligned} \tag{53}$$

(iv) h_0 is finer that h, that is, there exists a function $\phi \in \mathbb{K}$ such that

$$h_0\left(t,\Omega\right) \leq M_0 \quad implies \quad h\left(t,\Omega\right) \leq \phi\left(h_0\left(t,\Omega\right)\right) \tag{54}$$

for some M_0 with $\phi\left(M_0\right) \leq M$;
(v) The solution $V\left(t,\tau_0,V_0\right)$ of (12) for $t \geq \tau_0$ is ITD $\left(h_0,h_0\right)$-equi-bounded with respect to the solution $\tilde{U}\left(t,\tau_0,U_0\right) = U\left(t-\eta,t_0,U_0\right)$, for $\eta = \tau_0 - t_0$.

Then, this implies the ITD $\left(h_0,h\right)$-equi-boundedness of the solution $V\left(t,\tau_0,V_0\right)$ of (12) for $t \geq \tau_0$, with respect to the solution $\tilde{U}$

Proof. We shall show that the solution $V\left(t,\tau_0,V_0\right)$ of (12) for $t \geq \tau_0$ is ITD $\left(h_0,h\right)$-equi-bounded with respect to the solution $\tilde{U}$, that is, given any $\alpha > 0$ and for some $\tau_0 \in \mathbb{R}_+$, there exists $\beta = \beta\left(\alpha,\ \tau_0\right) > 0$ such that $h_0\left(\tau_0,\ V_0 - U_0\right) < \beta$ implies

$$h\left(t,\ V\left(t,\ \tau_0,\ V_0\right) - U\left(t-\eta,\ t_0,\ U_0\right)\right) < \alpha \ for\ t \ \geq \ \tau_0 \tag{55}$$

Assume that (55) is not true, then there exist solutions $\tilde{U}\left(t\right) = U\left(t-\eta,t_0,U_0\right)$, where $U\left(t,t_0,U_0\right)$ is the solution of (10) for $t \geq t_0$; and $V\left(t\right) = V\left(t,\tau_0,V_0\right)$ of (12) for $t \geq \tau_0$, and $t_1 > \tau_0$ such that

$$h_0\left(\tau_0,V_0-U_0\right) < \beta,\ \ h\left(t_1,\Omega\left(t_1\right)\right) = \alpha \ and \ h\left(t,\Omega\left(t\right)\right) \leq \alpha,\ \ for\ \tau_0 \leq t \leq t_1 \tag{56}$$

where $\Omega\left(t\right) = V\left(t\right) - \tilde{U}\left(t\right)$ for $t \geq \tau_0$.

By Theorem 2, we have, for $\tau_0 \leq t \leq t_1$,

$$L\left(t,\Omega\left(t\right)\right) \leq L\left(\tau_0,W\left(t,\tau_0,V_0-U_0\right)\right) - \int_{\tau_0}^{t} c\left(h\left(s,W\left(t,s,\Omega\left(s\right)\right)\right)\right)ds \tag{57}$$

Then, using the assumptions (iii), (56) and (57), we obtain when $t = t_1$,

$$\begin{aligned} & b(\alpha) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(\mathrm{s})))) \, ds \\ & = b(h(t_1, \Omega(t_1))) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(\mathrm{s})))) \, ds \leq L(t_1, \Omega(t_1)) \\ & \leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \\ & \leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \\ & \leq a_1(\tau_0, h(\tau_0, W(t_1, \tau_0, V_0 - U_0))) + a_0(\tau_0, h_0(\tau_0, W(t_1, \tau_0, V_0 - U_0))) \\ & + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \end{aligned} \tag{58}$$

We aim to reach a contradiction to conclude the proof of the theorem. We will use the assumption (v) for this purpose.

Given $0 < \alpha < M$ and that there exists a M_0 with $\phi(M_0) \leq M$.

Choosing $N_1 = N_1(\tau_0, \alpha)$ such that $0 < N_1(\tau_0, \alpha) < M_0$, and

$$h_0(t, \Omega(t)) < N_1 \ \ implies \ \ a_0(t, h_0(t, \Omega(t))) < \frac{b(\alpha)}{2} \ for \ t \geq \tau_0 \tag{59}$$

By assumption (v), corresponding to this N_1, there exists a $\beta_1 = \beta_1(\tau_0, N_1) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \beta_1 \ \ implies \ \ h_0(t, \Omega(t)) < N_1 \ for \ t \geq \tau_0 \tag{60}$$

Thus (59) and (60) give us

$$h_0(\tau_0, V_0 - U_0) < \ \beta_1 \ \ implies \ \ a_0(t, h_0(t, \Omega(t))) < \frac{b(\alpha)}{2} \ for \ t \geq \tau_0 \tag{61}$$

Similarly, we choose $N_2 = N_2(\tau_0, \alpha)$ such that $0 < N_2(\tau_0, \alpha) < M_0$ and

$$h(t, \Omega(t)) < N_2 \ \ implies \ \ a_1(t, h(t, \Omega(t))) < \frac{b(\alpha)}{2} \ for \ t \geq \tau_0 \tag{62}$$

By the assumptions (iv) and (v) also, corresponding to $\phi^{-1}(N_2)$, there exists a $\beta_2 = \beta_2(\tau_0, N_2) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \beta_2 \ \ implies \ \ h_0(t, \Omega(t)) < \phi^{-1}(N_2) \ \ for \ t \geq \tau_0 \tag{63}$$

Since $\phi \in \mathbb{K}$ is strictly monotone increasing; then, we have by taking the composition of ϕ of both sides of the inequality $h_0(t, \Omega(t)) < \phi^{-1}(N_2)$ in (63), with considering (54),

$$h_0(\tau_0, V_0 - U_0) < \beta_2 \;\; implies \quad h(t, \Omega(\mathrm{t})) \leq \phi(h_0(t, \Omega(t))) < \phi\left(\phi^{-1}(N_2)\right) = N_2 \; for \; t \geq \tau_0 \tag{64}$$

So, (62) and (64) give us, for $t \geq \tau_0$,

$$h_0(\tau_0, V_0 - U_0) < \beta_2 \;\; implies \;\; a_1(t, h(t, \Omega(t))) < \frac{b(\alpha)}{2} \tag{65}$$

Let $\beta = \min\{\beta_1, \beta_2\}$, then with this β the following statement holds.

$$\begin{aligned} & h_0(\tau_0, V_0 - U_0) < \beta \;\; implies \\ & a_0(t, h_0(t, \Omega(t))) < \frac{b(\alpha)}{2} \;\; and \;\; a_1(t, h(t, \Omega(t))) < \frac{b(\alpha)}{2} \; for \; t \geq \tau_0 \end{aligned} \tag{66}$$

Hence, when $t = t_1$, using (66), the statement (58) can be written as

$$\begin{aligned} & b(\alpha) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(\mathrm{s})))) ds \\ & = b(h(t_1, \Omega(t_1))) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(\mathrm{s})))) ds \leq L(t_1, \Omega(t_1)) \\ & \leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) ds \\ & \leq L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) ds \\ & \leq a_1(\tau_0, h(\tau_0, W(t_1, \tau_0, V_0 - U_0))) + a_0(\tau_0, h_0(\tau_0, W(t_1, \tau_0, V_0 - U_0))) \\ & + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) ds \\ & < \frac{b(\alpha)}{2} + \frac{b(\alpha)}{2} + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) ds \\ & = b(\alpha) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) ds \end{aligned} \tag{67}$$

This contradiction proves that the solution $V(t, \tau_0, V_0)$ of (12) through (τ_0, V_0) for $t \geq \tau_0$ is ITD (h_0, h)-equi-bounded with respect to the solution $\tilde{U}$.

□

The next theorem gives sufficient conditions to the ITD equi-attractiveness in the large of the solution $V(t, \tau_0, V_0)$ of (12) through (τ_0, V_0) for $t \geq \tau_0$ with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (10) through (t_0, U_0) for $t \geq t_0$; providing that the solution $V(t,\tau_0,V_0)$ of (12) is ITD (h_0,h_0)- equi-attractive in the large with respect to $\tilde{U}$.

Theorem 4. *Assume that*

(i) Both $L(t,\Omega)\in C\left[\mathbb{R}_+\times K_c(\mathbb{R}^n),\mathbb{R}_+\right]$ *and* $\|W(t,s,\Omega)\|$ *satisfy a local Lipschitz condition in* Ω *for any* t,s; *where* $W(t)=W(t,\tau_0,V_0-U_0)$ *is the solution of (13) for* $t\geq\tau_0$ *and*

$$\Omega(t,\tau_0,V_0-U_0)=V(t)-\tilde{U}(t)\ \ for\ t\geq\tau_0 \tag{68}$$

(ii)

$$D_{*-}L(t,s,\Omega)\leq -c\left(h\left(s,W\left(t,s,\Omega\left(\mathrm{s}\right)\right)\right)\right)\ \ in\ S(h,M) \tag{69}$$

where

$$S(h,M)=\{(t,\Omega)\ :h(t,\Omega)<M\ \text{for some}\ h\in\Gamma\ \text{and}\ M>0\} \tag{70}$$

and

$$\begin{aligned} &D_{*-}L(t,s,\Omega)\\ &=\lim_{\delta\to0^-}\inf\frac{1}{\delta}\left(L\left(s+\delta,W\left(t,s+\delta,\Omega+\delta\left((PV)(s)-\left(Q\tilde{U}\right)(s)\right)\right)\right)\right.\\ &\left.-L(s,W(t,s,\Omega))\right)\end{aligned} \tag{71}$$

(iii) For $b\in\mathbb{K}$ *and* $a_1,a_0\in\mathbb{CK}$,

$$\begin{aligned} &b(h(t,\Omega))+\int_{\tau_0}^{t}c\left(h\left(s,W\left(t,s,\Omega\left(\mathrm{s}\right)\right)\right)\right)ds\leq L(t,\Omega)\ \ in\ S(h,M)\ \ and\\ &L(t,\Omega)\leq a_1(t,h(t,\Omega))+a_0(t,h_0(t,\Omega))\ \ in\ S(h,M)\cap S(h_0,M)\end{aligned} \tag{72}$$

(iv) h_0 *is finer that* h, *that is, there exists a function* $\phi\in\mathbb{K}$ *such that*

$$h_0(t,\Omega)\leq M_0\ \ \ implies\ \ \ h(t,\Omega)\leq\phi(h_0(t,\Omega)) \tag{73}$$

for some M_0 *with* $\phi(M_0)\leq M$;

(v) The solution $V(t,\tau_0,V_0)$ *of (12) for* $t\geq\tau_0$ *is ITD* (h_0,h_0)*-equi-attractive in the large with respect to the solution* $\tilde{U}(t,\tau_0,U_0)=U(t-\eta,t_0,U_0)$, for $\eta=\tau_0-t_0$.

Then, this implies the ITD (h_0,h)*-equi-attractiveness in the large of the solution* $V(t,\tau_0,V_0)$ *of (12) with respect to the solution* $\tilde{U}$.

Proof. We shall show that the solution $V(t,\tau_0,V_0)$ of (12) for $t\geq\tau_0$ is ITD (h_0,h)-equi-attractive in the large with respect to the solution $\tilde{U}$, that is, given any $\varepsilon,\alpha>0$ and $\tau_0\in\mathbb{R}_+$, there exists a $T=T(\tau_0,\varepsilon,\alpha)>0$ such that $h_0(\tau_0,V_0-U_0)<\alpha$ implies

$$h(t,V(t,\tau_0,V_0)-U(t-\eta,t_0,U_0))<\varepsilon\ ,\quad t\geq\tau_0+T(\tau_0,\varepsilon,\alpha) \tag{74}$$

Assume that (74) is not true, then there exist solutions $\tilde{U}(t) = U(t-\eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is the solution of (10) for $t \geq t_0$; and $V(t) = V(t, \tau_0, V_0)$ of (12) for $t \geq \tau_0$, and a sequence $\{t_k\}$, $t_k \geq \tau_0 + T$ and $\lim_{k\to\infty} t_k = \infty$ such that

$$h_0(\tau_0, V_0 - U_0) < \alpha, \quad h(t_k, \Omega(t_k)) \geq \varepsilon \; for \; t_k \geq \tau_0 + T \tag{75}$$

where $\Omega(t) = V(t) - \tilde{U}(t)$ for $t \geq \tau_0$.

By Theorem 2, we have, for $t \geq \tau_0$,

$$L(t, \Omega(t)) \leq L(\tau_0, W(t, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t} c(h(s, W(t, s, \Omega(s))))\, ds \tag{76}$$

Then, using the assumptions (iii), (75) and (76), we obtain

$$\begin{aligned} b(\varepsilon) &+ \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(\mathrm{s}))))\, ds \\ &\leq b(h(t_k, \Omega(t_k))) + \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(\mathrm{s}))))\, ds \leq L(t_k, \Omega(t_k)) \\ &\leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(s))))\, ds \\ &\leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) + \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(s))))\, ds \\ &\leq a_1(\tau_0, h(\tau_0, W(t_k, \tau_0, V_0 - U_0))) + a_0(\tau_0, h_0(\tau_0, W(t_k, \tau_0, V_0 - U_0))) \\ &+ \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(s))))\, ds \end{aligned} \tag{77}$$

We aim to reach a contradiction to conclude the proof of the theorem. We will use the assumption (v) for this purpose.

Given $0 < \varepsilon < M$ and that there exists a M_0 with $\phi(M_0) \leq M$.

Choosing $N_1 = N_1(\tau_0, \varepsilon)$ such that $0 < N_1(\tau_0, \varepsilon) < M_0$, and

$$h_0(t, \Omega(t)) < N_1 \;\; implies \;\; a_0(t, h_0(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \; for \; t \geq \tau_0 \tag{78}$$

By assumption (v), corresponding to this N_1, there exists a α_1 and a $T_1 = T_1(\tau_0, N_1, \alpha_1) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \alpha_1 \;\; implies \;\; h_0(t, \Omega(t)) < N_1 \; for \; t \geq \tau_0 + T_1 \tag{79}$$

Thus (78) and (79) give us

$$h_0(\tau_0, V_0 - U_0) < \alpha_1 \;\; implies \;\; a_0(t, h_0(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \; for \; t \geq \tau_0 + T_1 \tag{80}$$

Similarly, we choose $N_2 = N_2(\tau_0, \varepsilon)$ such that $0 < N_2(\tau_0, \varepsilon) < M_0$ and

$$h(t, \Omega(t)) < N_2 \ \ implies \ \ a_1(t, h(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \ \ for \ t \geq \tau_0 \tag{81}$$

By the assumptions (iv) and (v) also, corresponding to $\phi^{-1}(N_2)$, there exists a α_2 and a $T_2 = T_2(\tau_0, N_2, \alpha_2) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \alpha_2 \ \ implies \ \ h_0(t, \Omega(t)) < \phi^{-1}(N_2) \ \ for \ t \geq \tau_0 + T_2 \tag{82}$$

Since $\phi \in \mathbb{K}$ is strictly monotone increasing; then, we have by taking the composition of ϕ of both sides of the inequality $h_0(t, \Omega(t)) < \phi^{-1}(N_2)$ in (82), with considering (73),

$$\begin{aligned} &h_0(\tau_0, V_0 - U_0) < \alpha_2 \ \ implies \\ &h(t, \Omega(\mathrm{t})) \leq \phi(h_0(t, \Omega(t))) < \phi\left(\phi^{-1}(N_2)\right) = N_2 \ for \ t \geq \tau_0 + T_2 \end{aligned} \tag{83}$$

So, (81) and (83) give us, for $t \geq \tau_0 + T_2$,

$$h_0(\tau_0, V_0 - U_0) < \alpha_2 \ \ implies \ \ a_1(t, h(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \tag{84}$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$, and $T = \max\{T_1, T_2\}$, then,

$$T{=}T(T_1, T_2) = T(\tau_0, N_1, \alpha_1, N_2, \alpha_2) = T(\tau_0, \varepsilon, \alpha) \tag{85}$$

Therefore, with these α, T the following statement holds.

$$\begin{aligned} &h_0(\tau_0, V_0 - U_0) < \alpha \ \ implies \\ &a_0(t, h_0(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \ and \ a_1(t, h(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \ for \ t \geq \tau_0 + T \end{aligned} \tag{86}$$

Hence, when $t = t_1$, using (86), the statement (77) can be written as

$$\begin{aligned}
& b(\varepsilon) + \int_{\tau_0}^{t_1} c(h(s, W(t_k, s, \Omega(\mathrm{s})))) \, ds \\
& \leq b(h(t_k, \Omega(t_k))) + \int_{\tau_0}^{t_1} c(h(s, W(t_k, s, \Omega(\mathrm{s})))) \, ds \leq L(t_k, \Omega(t_k)) \\
& \leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t_1} c(h(s, W(t_k, s, \Omega(s)))) \, ds \\
& \leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) + \int_{\tau_0}^{t_1} c(h(s, W(t_k, s, \Omega(s)))) \, ds \\
& \leq a_1(\tau_0, h(\tau_0, W(t_k, \tau_0, V_0 - U_0))) + a_0(\tau_0, h_0(\tau_0, W(t_k, \tau_0, V_0 - U_0))) \\
& + \int_{\tau_0}^{t_1} c(h(s, W(t_k, s, \Omega(s)))) \, ds \\
& < \frac{b(\varepsilon)}{2} + \frac{b(\varepsilon)}{2} + \int_{\tau_0}^{t_1} c(h(s, W(t_k, s, \Omega(s)))) \, ds \\
& = b(\varepsilon) + \int_{\tau_0}^{t_1} c(h(s, W(t_k, s, \Omega(s)))) \, ds
\end{aligned} \tag{87}$$

This contradiction proves the ITD (h_0, h)-equi-attractiveness in the large of the solution $V(t, \tau_0, V_0)$ of (12) for $t \geq \tau_0 + T(\tau_0, \varepsilon, \alpha)$ with respect to the solution $\tilde{U}$. □

The next theorem gives sufficient conditions to the ITD (h_0, h)-Lagrange stability of the solution $V(t, \tau_0, V_0)$ of (12) through (τ_0, V_0) for $t \geq \tau_0$ with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (10) through (t_0, U_0) for $t \geq t_0$; providing that the solution $V(t, \tau_0, V_0)$ of (12) is ITD (h_0, h_0)- Lagrange stable with respect to $\tilde{U}$.

Theorem 5. *Assume that*

(i) Both $L(t, \Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ and $\|W(t, s, \Omega)\|$ satisfy a local Lipschitz condition in Ω for any t, s; where $W(t) = W(t, \tau_0, V_0 - U_0)$ is the solution of (13) for $t \geq \tau_0$ and

$$\Omega(t, \tau_0, V_0 - U_0) = V(t) - \tilde{U}(t) \quad \textit{for } t \geq \tau_0 \tag{88}$$

(ii)

$$D_{*-}L(t, s, \Omega) \leq -c(h(s, W(t, s, \Omega(\mathrm{s})))) \quad \textit{in } S(h, M) \tag{89}$$

where

$$S(h, M) = \{(t, \Omega) \ : h(t, \Omega) < M \text{ for some } h \in \Gamma \text{ and } M > 0\} \tag{90}$$

and

$$D_{*-}L\left(t,s,\Omega\right)$$
$$= \lim_{\delta\to 0^-}\inf\frac{1}{\delta}\left(L\left(s+\delta,W\left(t,s+\delta,\Omega+\delta\left(\left(PV\right)(s)-\left(Q\tilde{U}\right)(s)\right)\right)\right)\right. \tag{91}$$
$$\left. -L\left(s,W\left(t,s,\Omega\right)\right)\right)$$

(iii) For $b\in\mathbb{K}$ *and* $a_1,a_0\in\mathbb{CK}$,

$$b\left(h\left(t,\Omega\right)\right)+\int_{\tau_0}^{t}c\left(h\left(s,W\left(t,s,\Omega\left(\mathrm{s}\right)\right)\right)\right)ds\leq L\left(t,\Omega\right)\ \ in\ S\left(h,M\right)and$$
$$L\left(t,\Omega\right)\leq a_1\left(t,h\left(t,\Omega\right)\right)+a_0\left(t,h_0\left(t,\Omega\right)\right)\ \ in\ S\left(h,M\right)\cap S\left(h_0,M\right) \tag{92}$$

(iv) h_0 *is finer that* h*, that is, there exists a function* $\phi\in\mathbb{K}$ *such that*

$$h_0\left(t,\Omega\right)\leq M_0\quad implies\quad h\left(t,\Omega\right)\leq\phi\left(h_0\left(t,\Omega\right)\right) \tag{93}$$

for some M_0 *with* $\phi\left(M_0\right)\leq M$;

(v) The solution $V\left(t,\tau_0,V_0\right)$ *of (12) for* $t\geq\tau_0$ *is ITD* $\left(h_0,h_0\right)$*-Lagrange stable with respect to the solution* $\tilde{U}\left(t,\tau_0,U_0\right)=U\left(t-\eta,t_0,U_0\right)$, for $\eta=\tau_0-t_0$.

Then, this implies the ITD $\left(h_0,h\right)$*-Lagrange stability of the solution* $V\left(t,\tau_0,V_0\right)$ *of (12) for* $t\geq\tau_0$ *with respect to the solution* $\tilde{U}$.

Proof. The ITD $\left(h_0,h_0\right)$-Lagrange stability of the solution $V\left(t,\tau_0,V_0\right)$ of (12) for $t\geq\tau_0$ with respect to the solution $\tilde{U}$ gives us by definition the ITD $\left(h_0,h_0\right)$-equi-boundedness and the ITD $\left(h_0,h_0\right)$-equi-attractiveness in the large of the solution $V\left(t,\tau_0,V_0\right)$ of (12) for $t\geq\tau_0$ with respect to the solution $\tilde{U}$. Hence, by applying Theorem 3 and Theorem 4 respectively, we obtain the ITD $\left(h_0,h\right)$-equi-boundedness and the ITD $\left(h_0,h\right)$-equi-attractiveness in the large of the solution $V\left(t,\tau_0,V_0\right)$ of (12) with respect to the solution $\tilde{U}$. That is to say it is ITD $\left(h_0,h\right)$-Lagrange stable with respect to the solution $\tilde{U}$, by definition. □

4. Conclusions

In this manuscript, we have presented sufficient conditions for ITD equiboundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures for the solutions of perturbed SDEs involving causal operators in regard to their unperturbed ones, and proved the sufficiency of these conditions using ITD variational comparison results.

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