POLİTEKNİK DERGİSİ

JOURNAL of POLYTECHNIC

ISSN: 1302-0900 (PRINT), ISSN: 2147-9429 (ONLINE)

URL: http://dergipark.org.tr/politeknik

Binormal surfaces of adjoint curves in 3D euclidean space

3-Boyutlu öklid uzayında adjoint eğrilerin binormal yüzeyleri

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To cite to this article: Körpınar T. and Sazak A., "Binormal surfaces of adjoint curves in 3D euclidean space", ***Journal of Polytechnic***, 26(3): 1141-1144, (2023).

Bu makaleye şu şekilde atıfta bulunabilirsiniz: Körpınar T. and Sazak A., "Binormal surfaces of adjoint curves in 3D euclidean space", ***Politeknik Dergisi***, 26(3): 1141-1144, (2023).

Erişim linki (To link to this article): http://dergipark.org.tr/politeknik/archive

DOI: 10.2339/politeknik.1059740

Binormal Surfaces of Adjoint Curves in 3D Euclidean Space

Highlights

- *Frenet-Serret frame formulas were given in 3D Euclidean space.*
- *Definitions of adjoint curve and binormal surface concepts were given.*
- *Some definitions and theorems about curves and surfaces were presented.*
- *Binormal surfaces of adjoint curves were characterized.*
- *With the help of obtained characterizations, new results and theorems were given.*

Graphical Abstract

The first and the second fundamental forms, the mean and the Gaussian curvatures and the principal curvatures of the binormal surface of adjoint curve obtained, respectively, are as follows:

$$\mathbf{I}_\beta = \left(1+t^2\kappa_\gamma^2\right)ds^2+dt^2, \quad \mathbf{II}_\beta = \frac{\kappa_\gamma' t-\tau_\gamma\left(1+\kappa_\gamma^2t^2\right)}{\sqrt{1+t^2\kappa_\gamma^2}}ds^2+\frac{2\kappa_\gamma}{\sqrt{1+t^2\kappa_\gamma^2}}dsdt,$$

$$K_\beta=\frac{-\kappa_\gamma^2}{(1+t^2\kappa_\gamma^2)^2}, \qquad H_\beta=\frac{\kappa_\gamma' t-\tau_\gamma(1+\kappa_\gamma^2t^2)}{\sqrt{(1+t^2\kappa_\gamma^2)^3}},$$

$$k_{\beta_1}=\frac{\kappa_\gamma' t-\tau_\gamma(1+\kappa_\gamma^2t^2)}{2\sqrt{(1+t^2\kappa_\gamma^2)^3}}+\sqrt{\frac{(\kappa_\gamma' t-\tau_\gamma(1+\kappa_\gamma^2t^2))^2+4\kappa_\gamma^2(1+\kappa_\gamma^2t^2)}{4(1+t^2\kappa_\gamma^2)^3}},$$

$$k_{\beta_2}=\frac{\kappa_\gamma' t-\tau_\gamma(1+\kappa_\gamma^2t^2)}{2\sqrt{(1+t^2\kappa_\gamma^2)^3}}-\sqrt{\frac{(\kappa_\gamma' t-\tau_\gamma(1+\kappa_\gamma^2t^2))^2+4\kappa_\gamma^2(1+\kappa_\gamma^2t^2)}{4(1+t^2\kappa_\gamma^2)^3}}.$$

Figure. Characterization equations of the binormal surface

Aim

It is aimed to characterize binormal surfaces associated with adjoint curves in 3D Euclidean space.

Design & Methodology

In addition to the basic definitions and theorems about curves and surfaces, the Frenet-Serret frame formulas are used.

Originality

In this study, new characterizations and results were obtained by associating the previously unrelated adjoint curve and binormal surface concepts.

Findings

Let β be adjoint curve of arc length parametrised curve γ. Then, the binormal surface of β is a flat surface if and only if curvature of γ is vanished.

Conclusion

We think that the results we obtained have content that will enrich the studies on differential geometry of curves and surfaces will be valuable for next studies on the subject.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

3-Boyutlu Öklid Uzayında Adjoint Eğrilerin Binormal Yüzeyleri

Araştırma Makalesi / Research Article

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(Geliş/Received : 26.01.2022 ; Kabul/Accepted : 14.03.2022 ; Erken Görünüm/Early View : 07.04.2022)

ÖZ

Bu çalışmada, bağlantılı eğriler yardımıyla tanımlanmış geniş uygulama alanlarına sahip yüzeyler konusuna kendine has bir katkı sunuyoruz. Özelde, bağlantılı eğrilerin önemli örneklerinden biri olan adjoint eğriler ile oluşturulan binormal yüzeyleri inceliyoruz. 3-boyutlu Öklid uzayında Frenet-Serret (FS) çatısı altında adjoint eğrilerin binormal yüzeylerini tanımlayarak, böyle yüzeyleri karakterize ediyoruz. Bu karakterizasyonlar yardımıyla bazı sonuçlar veriyoruz.

Anahtar Kelimeler: Binormal yüzey, adjoint eğri, frenet-serret çatı.

Binormal Surfaces of Adjoint Curves in 3D Euclidean Space

ABSTRACT

In this study, we make a specific contribution to the subject of surfaces with wide application areas defined with the help of associated curves. In particular, we examine the binormal surfaces generated via adjoint curves which are one of the important examples of associated curves. By defining the binormal surfaces of adjoint curves under the Frenet-Serret (FS) frame in 3D Euclidean space, we characterize such surfaces and give some results with the aid of these characterizations.

Keywords: Binormal surface, Adjoint curve, frenet-serret frame

1. INTRODUCTION

Surfaces are one of the meaningful subjects of geometry in terms of having many applications in the fields of engineering and physics, especially petroleum engineering and geophysics. Various surfaces (for example, ruled surfaces) formed as a result of motion of a curve or line in relation to another curve have made significant contributions to bringing geometric solutions to these application areas [1-14].

Associated curves are a subject that has significant contributions in determining the characterization of surfaces to be established with the help of curves and in terms of geometric explanations that can be brought to the subject of a motion in space. Integral curves are one of the common examples of associated curves. The adjoint curves we consider in this study are integral curves determined by integral of binormal vector of a curve [15-22].

In the light of these constructions, the aim of our study is to describe the binormal surfaces of adjoint curves in 3D Euclidean space and to determine their characterizations. First, after some basic reminders, we define the binormal surface of an adjoint curve in the space. Then we get first and second fundamental forms of this surface, its mean and Gaussian curvatures and its principle curvatures. Finally, we present the results obtained from these characterizations.

2. PRELIMINARIES

In this part, we give certain basic formulas and definitions about the space we are working on and the adjoint curves we will be discussing. Later, we present some fundamentals about the characterizations of surfaces.

The Frenet Serret (FS) formulas in 3D Euclidean space are given as

$$\begin{bmatrix} \nabla_{\mathbf{T}}\mathbf{T} \\ \nabla_{\mathbf{T}}\mathbf{N} \\ \nabla_{\mathbf{T}}\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

where κ, τ are curvature and torsion functions of γ, respectively [2]. Let s be arc-length parameter. Then, these formulas are written as

$$\mathbf{T} = \gamma'(s), \quad \mathbf{N} = \frac{\gamma''(s)}{\|\gamma''(s)\|}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

Definition 1. Let $\{\mathbf{T}_\gamma, \mathbf{N}_\gamma, \mathbf{B}_\gamma\}$ be the FS frame of arc length parametrised curve γ. Then, the adjoint curve of γ according to the FS frame is given as [2]

$$\beta(s) = \int_{s_0}^{s} \mathbf{B}_\gamma(s)ds.$$

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Theorem 2. Let $\{\mathbf{T}_\gamma, \mathbf{N}_\gamma, \mathbf{B}_\gamma\}$ be the FS frame of arc length parametrised curve γ, and β be adjoint curve of γ according to the FS frame, also κ_γ and τ_γ be curvature and torsion of γ. Denote by $\{\mathbf{T}_\beta, \mathbf{N}_\beta, \mathbf{B}_\beta\}$ the FS frame of β and by κ_β and τ_β be curvature and torsion of β. Then, following equations hold [2]:

$$\mathbf{T}_\gamma = \mathbf{B}_\beta, \quad \mathbf{N}_\gamma = -\mathbf{N}_\beta, \quad \mathbf{B}_\gamma = \mathbf{T}_\beta,$$
$$\kappa_\beta = \tau_\gamma, \quad \tau_\beta = \kappa_\gamma.$$

The normal vector (unit) field n for a surface ϕ is defined as

$$n = \frac{\phi_s \wedge \phi_t}{\|\phi_s \wedge \phi_t\|},$$

where t time parameter and $\phi_s = \partial\phi / \partial s$, $\phi_t = \partial\phi / \partial t$. Then, the first and the second fundamental forms of the surface are defined as

$$\begin{aligned} \mathbf{I} &= Eds^2 + 2Fdsdt + Gdt^2, \\ \mathbf{II} &= eds^2 + 2fdsdt + gdt^2, \end{aligned} \tag{1}$$

where

$$\begin{aligned} E &= \langle \phi_s, \phi_s \rangle,\ F = \langle \phi_s, \phi_t \rangle,\ G = \langle \phi_t, \phi_t \rangle, \\ e &= \langle \phi_{ss}, n \rangle,\ f = \langle \phi_{st}, n \rangle,\ g = \langle \phi_{tt}, n \rangle. \end{aligned} \tag{2}$$

Also, the mean and Gaussian curvatures are computed by equations

$$H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)},\ K = \frac{eg - f^2}{EG - F^2} \tag{3}$$

and the principal curvatures k_1, k_2 are defined as [3-7]

$$k_1 = H + \sqrt{H^2 - K},\ k_2 = H - \sqrt{H^2 - K}.$$

Theorem 3. A surface is a flat (developable) surface if and only if the Gaussian curvature of the surface vanish [1].

Theorem 4. A surface is a minimal surface if and only if the mean curvature of the surface vanish [1].

Definition 5. The binormal surface of a regular space curve γ is given as $\phi(s,t) = \gamma + t\mathbf{B}$ [5].

Definition 6. A surface is called a Weingarten surface, if this surface satisfies equation $H_s K_t - H_t K_s = 0$, [11].

3. BINORMAL SURFACES OF ADJOINT CURVES WITH THE FS FRAME IN $\mathbb{E}^3$

In this section, we get results and characterizations about binormal surfaces of adjoint curves.

Theorem 7. Let β be adjoint curve of arc length parametrised curve γ. Denote by $\mathbf{I}_\beta$ and $\mathbf{II}_\beta$ be the first and the second fundamental form of the binormal surfaces of β. Then the following states hold:

$$\begin{aligned} \mathbf{I}_\beta &= (1 + t^2\kappa_\gamma^2)ds^2 + dt^2, \\ \mathbf{II}_\beta &= \frac{\kappa_\gamma' t - \tau_\gamma(1 + \kappa_\gamma^2 t^2)}{\sqrt{1 + t^2\kappa_\gamma^2}} ds^2 + \frac{2\kappa_\gamma^2}{\sqrt{1 + t^2\kappa_\gamma^2}} dsdt. \end{aligned}$$

Proof. From Definition 5, the binormal surface of β is written as

$$\phi^\beta(s,t) = \beta + t\mathbf{B}_\beta.$$

Therefore, the following equalities are obtained:

$$\begin{aligned} \phi_s^\beta &= t\kappa_\gamma \mathbf{N}_\gamma + \mathbf{B}_\gamma, \\ \phi_{ss}^\beta &= -t\kappa_\gamma^2 \mathbf{T}_\gamma + (t\kappa_\gamma' - \tau_\gamma)\mathbf{N}_\gamma + \kappa_\gamma \tau_\gamma t \mathbf{B}_\gamma, \\ \phi_t^\beta &= \mathbf{T}_\gamma, \quad \phi_{tt}^\beta = 0, \quad \phi_{st}^\beta = \kappa_\gamma \mathbf{N}_\gamma, \end{aligned}$$

and, from the equalities, the unit standard normal vector field n_β of surface ϕ^β is found as

$$n_\beta = \frac{\phi_s^\beta \times \phi_t^\beta}{\|\phi_s^\beta \times \phi_t^\beta\|} = \frac{\mathbf{N}_\gamma - t\kappa_\gamma \mathbf{B}_\gamma}{\sqrt{1 + t^2\kappa_\gamma^2}}.$$

These equalities are obtained similarly for the binormal surface of γ curve. Then, from Theorem 2 and (2), we obtain

$$\begin{aligned} &E_\beta = 1 + t^2\kappa_\gamma^2, \qquad F_\beta = 0, \qquad G_\beta = 1, \\ &e_\beta = \frac{\kappa_\gamma' t - \tau_\gamma(1 + \kappa_\gamma^2 t^2)}{\sqrt{1 + t^2\kappa_\gamma^2}},\ f_\beta = \frac{\kappa_\gamma}{\sqrt{1 + t^2\kappa_\gamma^2}},\ g_\beta = 0. \end{aligned} \tag{4}$$

Hence, the first and the second fundamental forms of the binormal surfaces of β are obtained as

$$\begin{aligned} \mathbf{I}_\beta &= (1 + t^2\kappa_\gamma^2)ds^2 + dt^2, \\ \mathbf{II}_\beta &= \frac{\kappa_\gamma' t - \tau_\gamma(1 + \kappa_\gamma^2 t^2)}{\sqrt{1 + t^2\kappa_\gamma^2}} ds^2 + \frac{2\kappa_\gamma}{\sqrt{1 + t^2\kappa_\gamma^2}} dsdt. \end{aligned}$$

Corollary 8. Let β be adjoint curve of arc length parametrised curve γ. Denote by H_β, K_β be the mean and the Gaussian curvature of binormal surface of β, respectively. Then the following states hold:

$$\begin{aligned} K_\beta &= \frac{-\kappa_\gamma^2}{(1 + t^2\kappa_\gamma^2)^2}, \\ H_\beta &= \frac{\kappa_\gamma' t - \tau_\gamma(1 + \kappa_\gamma^2 t^2)}{\sqrt{(1 + t^2\kappa_\gamma^2)^3}}. \end{aligned} \tag{5}$$

Proof. From equalities (4), we get

$$\begin{aligned} K_\beta &= \frac{e_\beta g_\beta - f_\beta^2}{E_\beta G_\beta - F_\beta^2} = \frac{-\kappa_\gamma^2}{(1 + t^2\kappa_\gamma^2)^2}, \\ H_\beta &= \frac{E_\beta g_\beta - 2F_\beta f_\beta + G_\beta e_\beta}{2\left(E_\beta G_\beta - F_\beta^2\right)} = \frac{\kappa_\gamma' t - \tau_\gamma(1 + \kappa_\gamma^2 t^2)}{2\sqrt{(1 + t^2\kappa_\gamma^2)^3}}. \end{aligned}$$

Theorem 9. Let β be adjoint curve of arc length parametrised curve γ. Then, the binormal surface of β is minimal if and only if

$$\kappa_\gamma' = \frac{\tau_\gamma}{t} + \kappa_\gamma^2 \tau_\gamma t. \tag{6}$$

Proof. The proof is straightforwardly done with the help of Theorem 4 and Corollary 8.

Theorem 10. Let β be adjoint curve of arc length parametrised curve γ. Then, the binormal surface of β is a flat surface if and only if it has vanishing curvature of γ.

Proof. From Theorem 3 and Corollary 8, the proof is plainly obtained.

Corollary 11. Let β be adjoint curve of arc length parametrised curve γ. In this case, the following conditions are provided:

i. The binormal surface of β is minimal if and only if
$$\kappa_\gamma = \frac{1}{t}\tan\left(\int \tau_\gamma ds + tc\right),$$
ii. Let the torsion of γ be constant. Then, the binormal surface of β is minimal if and only if
$$\kappa_\gamma = \frac{1}{t}\tan(\tau_\gamma s + tc),$$
iii. Let the curvature of γ be constant. Then, the binormal surface of β is minimal if and only if it has vanishing the torsion of curve γ,
iv. Let the binormal surface of β be flat. Then, the binormal surface of β is minimal if and only if this surface is a plane.

Here, c is an integration constant.

Proof. *i.* By solving the differential equation (6), we obtain
$$\kappa_\gamma = \frac{1}{t}\tan\left(\int \tau_\gamma ds + tc\right).$$
ii. Let the torsion of γ be constant. Similarly, we get
$$\kappa_\gamma = \frac{1}{t}\tan\,(\tau_\gamma s + tc).$$
iii. Let the curvature of γ be constant and be the binormal surface of β is minimal. From the equation (6), we get
$$\tau_\gamma\left(\frac{1}{t} + \kappa_\gamma^2 t\right) = 0,$$
and, therefore it's obtained $\tau_\gamma = 0$. Conversely, let be $\tau_\gamma = 0$. Then, it is plainly seen that the binormal surface of β is minimal.

iv. Since the binormal surface of β is flat, $\kappa_\gamma = 0$. Let the binormal surface of β be minimal. Then, from the equation (6), $\tau_\gamma = 0$. From Theorem 2, $\tau_\beta = \kappa_\beta = 0$. Therefore the binormal surface of β is a plane. Conversely, let the binormal surface of β be a plane. Then it's straightforwardly obtained $H_\beta = 0$. Hence the binormal surface of β is minimal.

Theorem 12. Let β be adjoint curve of arc length parametrised curve γ. Then, the binormal surface of β is a Weingarten surface if and only if
$$\begin{aligned}&2\kappa_\gamma^3 t[(1+\kappa_\gamma^2 t^2)(t\kappa_\gamma'' - 2t^2\kappa_\gamma\tau_\gamma\kappa_\gamma' - \tau_\gamma'\kappa_\gamma^2 t^2\\ &-\tau_\gamma') - 3t^2\kappa_\gamma\kappa_\gamma'(t\kappa_\gamma' - \tau_\gamma - \tau_\gamma\kappa_\gamma^2 t^2)] = (\kappa_\gamma'\\ &-2\kappa_\gamma' t^2\kappa_\gamma^2 + \tau_\gamma\kappa_\gamma^4 t^3 + \tau_\gamma\kappa_\gamma^4 t)(\kappa_\gamma'\kappa_\gamma^2 t^2 - \kappa_\gamma').\end{aligned}$$

Proof. From Definition 5, if the binormal surface of β is a Weingarten surface, then

$$(H_\beta)_s(K_\beta)_t - (H_\beta)_t(K_\beta)_s = 0.$$

The result can be readily found by obtaining partial derivatives depending on the *t* and *s* parameters in the equation (6).

Corollary 13. Let β be adjoint curve of arc length parametrised curve γ and $\kappa_\gamma \neq 0$ be a constant. Then, the binormal surface of β is a Weingarten surface if and only if the torsion of γ is constant.

Proof. Let β be adjoint curve of arc length parametrised curve γ and $\kappa_\gamma \neq 0$ be a constant. Let the binormal surface of β be a Weingarten surface. Taking $\kappa_\gamma' = 0$ in Theorem 12, it is obtained
$$2\kappa_\gamma^3 t\tau_\gamma'\left(1+\kappa_\gamma^2 t^2\right)^2 = 0,$$
and, hence $\tau_\gamma' = 0$.

Corollary 14. Let β be adjoint curve of arc length parametrised curve γ. Let k_{β_1}, k_{β_2} be principal curvatures of binormal surfaces of β, respectively. Then the following expressions hold:
$$k_{\beta_1} = \frac{\kappa_\gamma' t - \tau_\gamma(1+\kappa_\gamma^2 t^2)}{2\sqrt{(1+t^2\kappa_\gamma^2)^3}} + \sqrt{\frac{(\kappa_\gamma' t - \tau_\gamma(1+\kappa_\gamma^2 t^2))^2 + 4\kappa_\gamma^2(1+\kappa_\gamma^2 t^2)}{4(1+t^2\kappa_\gamma^2)^3}},$$
$$k_{\beta_2} = \frac{\kappa_\gamma' t - \tau_\gamma(1+\kappa_\gamma^2 t^2)}{2\sqrt{(1+t^2\kappa_\gamma^2)^3}} - \sqrt{\frac{(\kappa_\gamma' t - \tau_\gamma(1+\kappa_\gamma^2 t^2))^2 + 4\kappa_\gamma^2(1+\kappa_\gamma^2 t^2)}{4(1+t^2\kappa_\gamma^2)^3}}.$$

Proof. The proof is easily done with the aid of Corollary 8.

4. CONCLUSION

In this study, after defining the geometric expression of a binormal surface generated by means of the adjoint of a curve, we gave certain characterizations for this surface. We determined the first and the second fundamental forms, the mean and the Gaussian curvatures, and principal curvatures for this binormal surface. In addition, we obtained some results by examining the cases of such a surface as a minimal, flat and Weingarten surface in a related way.

Considering the place of associated curves and ruled surfaces in geometry and their contributions to physics and engineering, we predict that consequences of our investigation will contribute to future studies on this topic. In our next study, we intend to examine the relationship of a different associated curve sample with normal surfaces determined by the normal vector field.

DECLARATION OF ETHICAL STANDARDS

The authors of this article declare that the materials and methods used in their studies do not require ethical committee approval and/or legal-specific permission.

AUTHORS' CONTRIBUTIONS

Talat KÖRPINAR: Contributed equally to obtaining the equations and analyzing the results.

Ahmet SAZAK: Contributed equally to obtaining the equations and analyzing the results, and wrote the manuscript.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

REFERENCES

[1] Do Carmo M., "***Differential Geometry of Curves and Surfaces***", Prentice-Hall, Englewood Cliffs, (1976).

[2] Nurkan S. K., Güven I. A. and Karacan M. K., "Characterizations of adjoint curves in Euclidean 3-space", ***Proc Natl Acad Sci. India Sect A Phys Sci.***, 89: 155-161, (2019).

[3] Kaymanli G. U., Okur S. and Ekici C., "The Ruled Surfaces Generated By Quasi Vectors", ***IV. International Scientific and Vocational Studies Congress - Science and Health***, November, (2019).

[4] Lopez R., "Differential geometry of curves and surfaces in Lorentz-Minkowski space", ***MiniCourse taught at IME-USP***, Brasil, (2008).

[5] Dreibelbis D., "Singularities of the Gauss map and the binormal surface", ***Adv. Geom***., 3: 453-468, (2003).

[6] Lopez R., Sipus Z. M., Gajcic L. P., Protrka I., "Harmonic evolutes of B-scrolls with constant mean curvature in Lorentz-Minkowski space", ***International Journal of Geometric Methods in Modern Physics***, 16(5), (2019).

[7] Aydemir I. and Orbay K. "The Ruled Surfaces Generated By Frenet Vectors of Timelike Ruled Surface in the Minkowski Space R_1^3", ***World Applied Science Journal***, 6(5): 692-696, (2009).

[8] Protrka I., "Harmonic Evolutes of Timelike Ruled Surfaces in Minkowski Space", ***18th Scientific-Professional Colloquium on Geometry and Graphics***, September, (2015).

[9] Protrka I., "The harmonic evolute of a helicoidal surfaces in Minkowski 3-space", ***Proceedings of the Young Researcher Workshop on Differential Geometry in Minkowski Space***, Granada, Spain, 133-142, (2017).

[10] Sarıoglugil A. and Tutar A., "On Ruled Surface in Euclidean Space E^3", ***Int. J. Contemp. Math. Sci.,*** 2(1): 1-11, (2007).

[11] Kühnel W., "Ruled W-surfaces", ***Arch. Math. (Basel)***, 62: 475-480, (1994).

[12] Ünlütürk Y. and Ekici C., "Parallel Surfaces of Spacelike Ruled Weingarten Surfaces in Minkowski 3-space", ***New Trends in Mathematical Sciences***, 1(1): 85-92, (2013).

[13] Karadağ H. B., Kılıç E. and Karadağ M., "On the developable ruled surfaces Kinematically generated in Minkowski 3-Space", ***Kuwait Journal of Science***, 41(1): 21-34, (2014).

[14] Altın M., Kazan A. and Karadağ H. B., "Ruled surfaces in E3 with density", ***Honam Mathematical Journal***, 41(4): 683-695, (2019).

[15] Senturk G. Y. and Yuce S., "Characteristic properties of the ruled surface with Darboux frame in E^3", ***Kuwait J. Sci.***, 42(2): 14-33, (2015).

[16] Choi J. H. and Kim Y. H., "Associated curves of a Frenet curve and their applications", ***Appl. Math. Comput.***, 218 (18): 9116-9124, (2012).

[17] Körpınar T., Sarıaydın M.T., and Turhan E., "Associated curves according to Bishop frame in Euclidean 3-space", ***Adv. Model. Optim.***, 15(3): 713-717, (2013).

[18] Macit N. and Düldül M., "Some new associated curves of a Frenet curve in E3 and E4", ***Turkish J. Math.***, 38: 1023-1037, (2014).

[19] Yılmaz S., "Characterizations of some associated and special curves to type-2 Bishop frame in E^3", ***Kırklareli University J. Eng. Sci.***, 1: 66-77, (2015).

[20] Karadağ M. and Sivridağ A.İ., "On a Study of the Quaternionic Lorentzian Curve", ***Journal of Polytechnic***, 21(4): 937-940, (2018).

[21] Gündüz H., Kazan A. and Karadağ H.B., "Rotational surfaces generated by cubic hermitian and cubic bezier curves", ***Journal of Polytechnic***, 22(4): 1075-1082, (2019).

[22] Karadağ M., "A New Approach for the Pseudo-Quaternionic Lorentzian Evolute and Involute Curves", ***Journal of Polytechnic***, 1(1): 1-7, (2021)