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Extended Ostrowski Type Inequalities Involving Conformable Fractional Integrals

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Abstract

The purpose of this work is to establish extended Ostrowski type inequalities involving conformable fractional integrals. We first give an identity for functions whose $\alpha-$fractional derivatives are bounded. After that, two extended Ostrowski type inequalities which involve conformable fractional integrals for functions whose $\alpha-$fractional derivatives are bounded are developed. Additionally, the applications of numerical integration that emerged when investigating these inequalities are given.

Keywords: *Ostrowski inequality; bounded functions; conformable fractional calculus.*
2010 Mathematics Subject Classification: *26D10; 26D15; 26A33.*

1. Introduction

In 1938, Ostrowski established the integral inequality which is one of the fundemental inequalitıes of mathematic as follows (see, [19]): Assume that $\psi:[a,b]\rightarrow\mathbb{R}$ is a differentiable mapping on (a,b) such that the derivative $\psi^{'}:(a,b)\rightarrow\mathbb{R}$ is bounded on (a,b), i.e., $\|\psi'\|_{\infty}=\sup\limits_{\tau\in(a,b)}|\psi'(\tau)|<\infty$. Then, for all $x\in[a,b]$, we have

$$\left|\psi(\varkappa)-\frac{1}{b-a}\int\limits_{a}^{b}\psi(\tau)d\tau\right|\leq\left[\frac{1}{4}+\frac{\left(\varkappa-\frac{a+b}{2}\right)^{2}}{(b-a)^{2}}\right](b-a)\left\|\psi'\right\|_{\infty}. \tag{1.1}$$

The constant $\frac{1}{4}$ is the best possible.
Inequality (1.1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Thus, many researchers and mathematicians worked on new Ostrowski type inequalities for various classes of functions. They also developed the new approach of obtaining bounds for particular quadrature rules by using different versions of Peano kernel..For example, Ostrowski type inequalities for the cases when $\psi'\in L_1$ and $\psi'\in L_p$ were derived by Dragomir and Wang in [9] and [10]. In [6], Cerone et al. presented an original result for twice differentiable functions. A new generalization of Ostrowski's integral inequality for mappings whose derivatives are bounded is provided by Dragomir et al. in [12]. Ostrowki type results for double integrals were provided by Sarikaya in [20] and [21]. In addition to these all studies, researchers have studied Ostrowski type inequalities for different classes of convex functions and mappings whose derivatives are bounded, you can check ([8], [7], [11], [13], [14] and [18]) and the references included there.

2. Definitions and properties of conformable fractional derivative and integral

Recently, the authors have introduced a new simple well-behaved definition of the fractional derivative called the "conformable fractional derivative" depending just on the basic limit definition of the derivative in [17]. Namely, for given a function $\psi:[0,\infty)\rightarrow\mathbb{R}$ the conformable fractional derivative of order $0<\alpha\leq 1$ of f at $\tau>0$ was defined by

$$D_{\alpha}(\psi)(\tau)=\lim_{\varepsilon\rightarrow 0}\frac{\psi\left(\tau+\varepsilon\tau^{1-\alpha}\right)-\psi(\tau)}{\varepsilon}$$

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If ψ is $\alpha-$differentiable in some $(0,a)$, $\alpha>0$, $\lim\limits_{t\to 0^+}\psi^{(\alpha)}(\tau)$ exist, then define

$$\psi^{(\alpha)}(0)=\lim_{\tau\to 0^+}\psi^{(\alpha)}(\tau).$$

Also, note that if ψ is differentiable, then

$$D_\alpha(\psi)(\tau)=\tau^{1-\alpha}\psi'(\tau)$$

where

$$\psi'(\tau)=\lim_{\varepsilon\to 0}\frac{\psi(\tau+\varepsilon)-\psi(\tau)}{\varepsilon}.$$

We can write $\psi^{(\alpha)}(\tau)$ for $D_\alpha(\psi)(\tau)$ to denote the conformable fractional derivatives of ψ of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say ψ is $\alpha-$differentiable.

The following definitions and theorems related to conformable fractional derivative and integral were referred in [1]-[4], [5], [16] and [17].

Theorem 2.1. *Let $\alpha\in(0,1]$ and ψ,φ be $\alpha-$differentiable at a point $\tau>0$. Then*

i. $D_\alpha(a\psi+b\varphi)=aD_\alpha(\psi)+bD_\alpha(\varphi)$, for all $a,b\in\mathbb{R}$,

ii. $D_\alpha(\lambda)=0$, for all constant functions $\psi(t)=\lambda$,

iii. $D_\alpha(\psi\varphi)=\psi D_\alpha(\varphi)+\varphi D_\alpha(\psi)$,

iv. $D_\alpha\left(\dfrac{\psi}{\varphi}\right)=\dfrac{\varphi D_\alpha(\psi)-\psi D_\alpha(\varphi)}{\varphi^2}$.

Definition 2.2 (Conformable fractional integral)**.** *Let $\alpha\in(0,1]$ and $0\le a<b$. A function $\psi:[a,b]\to\mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b\psi(\varkappa)d_\alpha\varkappa:=\int_a^b\psi(\varkappa)\varkappa^{\alpha-1}d\varkappa$$

exists and is finite.

Remark 2.3.

$$I_\alpha^a(\psi)(\tau)=I_1^a\left(t^{\alpha-1}\psi\right)=\int_a^t\frac{\psi(\varkappa)}{\varkappa^{1-\alpha}}d\varkappa,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

Theorem 2.4. *Let $\psi:(a,b)\to\mathbb{R}$ be differentiable and $0<\alpha\le 1$. Then, for all $\tau>a$ we have*

$$I_\alpha^a D_\alpha^a f(\psi)=\psi(\tau)-\psi(a).$$

Theorem 2.5. ***(Integration by parts)*** *Let $\psi,\varphi:[a,b]\to\mathbb{R}$ be two functions such that $\psi\varphi$ is differentiable. Then*

$$\int_a^b\psi(\varkappa)D_\alpha^a(\varphi)(\varkappa)d_\alpha\varkappa=\psi\varphi|_a^b-\int_a^b\varphi(\varkappa)D_\alpha^a(\psi)(\varkappa)d_\alpha\varkappa.$$

Theorem 2.6. *Assume that $\psi:[a,\infty)\to\mathbb{R}$ such that $\psi^{(n)}(\tau)$ is continuous and $\alpha\in(n,n+1]$. Then, for all $\tau>a$ we have*

$$D_\alpha^a I_\alpha^a\psi(\tau)=\psi(\tau).$$

Theorem 2.7. *Let $\alpha\in(0,1]$ and $\psi:[a,b]\to\mathbb{R}$ be a continuous on $[a,b]$ with $0\le a<b$. Then,*

$$|I_\alpha^a(\psi)(\varkappa)|\le I_\alpha^a|\psi|(\varkappa).$$

In [5], Anderson proved Ostrowski's α-fractional inequality using a Motgomery identity as follows:

Theorem 2.8. *Let $a,b,s,\tau\in\mathbb{R}$ with $0\le a<b$, and $\psi:[a,b]\to\mathbb{R}$ be α-fractional differentiable for $\alpha\in(0,1]$. Then,*

$$\left|\psi(t)-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b\psi(\tau)d_\alpha\tau\right|\le\frac{M}{2\alpha(b^\alpha-a^\alpha)}\left[(\tau^\alpha-a^\alpha)^2+(b^\alpha-\tau^\alpha)\right]\tag{2.1}$$

where

$$M=\sup_{\tau\in(a,b)}|D_\alpha\psi(\tau)|<\infty.$$

In recent years, many authors have worked on ineqaulities involving conformable fractional integrals. For example, Anderson presented some important inequalities including conformable integrals such as Hermite-Hadamard, Steffensen and Chebyshev as well as Ostrowski inequality in [5]. Afterwards, Usta et. al attained a new upper bound for Ostrowski inequality (2.1) in [22]. In addition to all these results, a large number of reseachers studied on inequalities involving conformable fractional integrals for various types of functions. For some of them, you can check ([16], [**?**] and [15]) and the references included there.

The main aim of this work is to establish new Ostrowski type inequalities which involve conformable fractional intgerals for functions whose $\alpha-$fractional derivatives are bounded. Also some applications of these results in numeric integration are given.

3. Main Results

A new identity including conformable fractional integrals is established in the following Lemma.

Lemma 3.1. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be an $\alpha-$fractional differentiable mapping on (a,b) with $0\le a<b$ and $\alpha\in(0,1]$. Then, we have*

$$\frac{1}{(b^{\alpha}-a^{\alpha})}\int_a^b P(x,t)f^{(\alpha)}(t)d_{\alpha}t=(1-h)f(x)+h\frac{f(a)+f(b)}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b ,f(t)d_{\alpha}t \tag{3.1}$$

for all $h\in[0,1]$ and $\left(a^{\alpha}+h\frac{b^{\alpha}-a^{\alpha}}{2}\right)^{\frac{1}{\alpha}}\le x\le\left(b^{\alpha}-h\frac{b^{\alpha}-a^{\alpha}}{2}\right)^{\frac{1}{\alpha}}$. Here, $P(x,t)$ is defined by

$$P(x,t)=\begin{cases} t^{\alpha}-a^{\alpha}-h\frac{b^{\alpha}-a^{\alpha}}{2} & t\in[a,x) \\ t^{\alpha}-b^{\alpha}+h\frac{b^{\alpha}-a^{\alpha}}{2} & t\in[x,b]. \end{cases}$$

Proof. If we use the elmentary integral operations, owing to the definition of $P(x,t)$, we can write

$$\begin{aligned}\int_a^b P(x,t)f^{(\alpha)}(t)d_{\alpha}t &= \int_a^x\left(t^{\alpha}-a^{\alpha}-h\frac{b^{\alpha}-a^{\alpha}}{2}\right)f^{(\alpha)}(t)d_{\alpha}t+\int_x^b\left(t^{\alpha}-b^{\alpha}+h\frac{b^{\alpha}-a^{\alpha}}{2}\right)f^{(\alpha)}(t)d_{\alpha}t \\ &= \left(t^{\alpha}-a^{\alpha}-h\frac{b^{\alpha}-a^{\alpha}}{2}\right)f(t)\Big|_a^x-\alpha\int_a^x ,f(t)d_{\alpha}t \\ &\quad+\left(t^{\alpha}-b^{\alpha}+h\frac{b^{\alpha}-a^{\alpha}}{2}\right)f(t)\Big|_x^b-\alpha\int_x^b ,f(t)d_{\alpha}t \\ &= \left(x^{\alpha}-a^{\alpha}-h\frac{b^{\alpha}-a^{\alpha}}{2}\right)f(x)+h^{\alpha}\frac{b^{\alpha}-a^{\alpha}}{2}f(a) \\ &\quad+h\frac{b^{\alpha}-a^{\alpha}}{2}f(b)-\left(x^{\alpha}-b^{\alpha}+h\frac{b^{\alpha}-a^{\alpha}}{2}\right)f(x)-\alpha\int_a^b ,f(t)d_{\alpha}t.\end{aligned}$$

Thus, the proof is completed. □

We give a new inequlity for functions whose $\alpha-$fractional derivatives are bounded in the following theorem.

Theorem 3.2. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be an $\alpha-$fractional differentiable mapping on (a,b) with $0\le a<b$ and $\alpha\in(0,1]$. If $f^{(\alpha)}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\left\|f^{(\alpha)}\right\|_{\infty}=\sup_{t\in(a,b)}\left|f^{(\alpha)}(t)\right|<\infty$, we have the inequality*

$$\left|\frac{(1-h)}{\alpha}f(x)+h\frac{f(a)+f(b)}{2\alpha}-\frac{1}{(b^{\alpha}-a^{\alpha})}\int_a^b ,f(t)d_{\alpha}t\right|\le\frac{b^{\alpha}-a^{\alpha}}{\alpha}\left[\frac{2h(h-1)+1}{4\alpha}+\frac{1}{\alpha}\left(\frac{x^{\alpha}-\frac{a^{\alpha}+b^{\alpha}}{2}}{b^{\alpha}-a^{\alpha}}\right)^2\right]\left\|f^{(\alpha)}\right\|_{\infty} \tag{3.2}$$

for all $h\in[0,1]$ and $\left(a^{\alpha}+h\frac{b^{\alpha}-a^{\alpha}}{2}\right)^{\frac{1}{\alpha}}\le x\le\left(b^{\alpha}-h\frac{b^{\alpha}-a^{\alpha}}{2}\right)^{\frac{1}{\alpha}}$.

Proof. Taking the absolute value of both sides of the equality (3.1) and later using boundedness of the function $f^{(\alpha)}$, it is found that

$$\begin{aligned}\left|(1-h)f(x)+h\frac{f(a)+f(b)}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b ,f(t)d_{\alpha}t\right| &= \left|\frac{1}{(b^{\alpha}-a^{\alpha})}\int_a^b P(x,t)f^{(\alpha)}(t)d_{\alpha}t\right| \\ &\le \frac{1}{(b^{\alpha}-a^{\alpha})}\int_a^b |P(x,t)|\left|f^{(\alpha)}(t)\right|d_{\alpha}t \\ &\le \left\|f^{(\alpha)}\right\|_{\infty}\frac{1}{(b^{\alpha}-a^{\alpha})}\int_a^b |P(x,t)|d_{\alpha}t.\end{aligned}$$

It is observed that

$$\int_a^b |P(x,t)|\, d_\alpha t = \int_a^x \left| t^\alpha - a^\alpha - h\frac{b^\alpha - a^\alpha}{2} \right| d_\alpha t + \int_x^b \left| t^\alpha - b^\alpha + h\frac{b^\alpha - a^\alpha}{2} \right| d_\alpha t$$
$$= \int_a^{\left(a^\alpha + h^\alpha \frac{b^\alpha - a^\alpha}{2}\right)^{\frac{1}{\alpha}}} \left(h\frac{b^\alpha - a^\alpha}{2} + a^\alpha - t^\alpha \right) d_\alpha t + \int_{\left(a^\alpha + h^\alpha \frac{b^\alpha - a^\alpha}{2}\right)^{\frac{1}{\alpha}}}^x \left(t^\alpha - a^\alpha - h\frac{b^\alpha - a^\alpha}{2} \right) d_\alpha t$$
$$+ \int_x^{\left(b^\alpha - h^\alpha \frac{b^\alpha - a^\alpha}{2}\right)^{\frac{1}{\alpha}}} \left(b^\alpha - h\frac{b^\alpha - a^\alpha}{2} - t^\alpha \right) d_\alpha t + \int_{\left(b^\alpha - h^\alpha \frac{b^\alpha - a^\alpha}{2}\right)^{\frac{1}{\alpha}}}^b \left(t^\alpha - b^\alpha + h\frac{b^\alpha - a^\alpha}{2} \right) d_\alpha t.$$

Calculating the above four integrals, the desired inequality (3.2) can be obtained. □

Now, we deal with new results for mappings whose $\alpha-$fractional derivatives are element of L_p.

Theorem 3.3. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be an $\alpha-$fractional differentiable mapping on (a,b) with $0 \le a < b$ and $\alpha \in (0,1]$. If $f^{(\alpha)} \in L_p[a,b]$, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, one has the inequality*

$$\left| \frac{(1-h)}{\alpha} f(x) + h\frac{f(a)+f(b)}{2\alpha} - \frac{1}{b^\alpha - a^\alpha} \int_a^b , f(t) d_\alpha t \right| \le \left\| f^{(\alpha)} \right\|_p \frac{1}{\alpha \left[\alpha(q+1)\right]^{\frac{1}{q}} (b^\alpha - a^\alpha)} \left[2\left(h\frac{b^\alpha - a^\alpha}{2} \right)^{q+1} + \left(x^\alpha - a^\alpha - h\frac{b^\alpha - a^\alpha}{2} \right)^{q+1} + \left(b^\alpha - x^\alpha - h\frac{b^\alpha - a^\alpha}{2} \right)^{q+1} \right]^{\frac{1}{q}}. \tag{3.3}$$

for all $h \in [0,1]$ and $\left(a^\alpha + h\frac{b^\alpha - a^\alpha}{2}\right)^{\frac{1}{\alpha}} \le x \le \left(b^\alpha - h\frac{b^\alpha - a^\alpha}{2}\right)^{\frac{1}{\alpha}}$. Here, $\left\| f^{(\alpha)} \right\|_p = \left(\int_a^b \left| f^{(\alpha)}(x) \right|^p dx \right)^{\frac{1}{p}}$.

Proof. If we use Hölder's inequality for integrals after taking modulus in both sides of the identity (3.1), then we find that

$$\left| (1-h) f(x) + h\frac{f(a)+f(b)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b , f(t) d_\alpha t \right| = \left| \frac{1}{(b^\alpha - a^\alpha)} \int_a^b P(x,t) f^{(\alpha)}(t) d_\alpha t \right|$$
$$\le \frac{1}{(b^\alpha - a^\alpha)} \int_a^b |P(x,t)| \left| f^{(\alpha)}(t) \right| d_\alpha t$$
$$\le \frac{1}{(b^\alpha - a^\alpha)} \left(\int_a^b |P(x,t)|^q d_\alpha t \right)^{\frac{1}{q}} \left(\int_a^b \left| f^{(\alpha)}(t) \right|^p d_\alpha t \right)^{\frac{1}{p}}$$
$$\le \left\| f^{(\alpha)} \right\|_p \frac{1}{(b^\alpha - a^\alpha)} \left(\int_a^b |P(x,t)|^q d_\alpha t \right)^{\frac{1}{q}}.$$

Calculating the above $\alpha-$fractional integral by using the properties of conformable fractional integral, the required inequality (3.3) is attained. □

4. Applications to Numerical Integration

Int his section, we deal with applications of the integral inequalities developed in the previous section, to obtain estimates of composite quadrature rules. In other words, we examine recent approaches to estimations of the composite quadrature formula for functions whose first derivatives are bounded.

Assume that $I_z : a = \varkappa_0 < \varkappa_1 < ... < \varkappa_{z-1} < x_z = b$ is a partition of the interval $[a,b]$, $h \in [0,1]$ and $\left(x_i^\alpha + h\frac{k_i}{2}\right)^{\frac{1}{\alpha}} \le \xi_i \le \left(x_{i+1}^\alpha - h\frac{k_i}{2}\right)^{\frac{1}{\alpha}}$ for $i = 0,...,z-1$. Define the quadrature

$$S_\alpha(f,,\xi,I_z) := \frac{(1-h)}{\alpha} \sum_{i=0}^{z-1} k_i f(\xi_i) + \frac{h}{\alpha} \sum_{i=0}^{z-1} \frac{f(x_i) + f(x_{i+1})}{2} k_i \tag{4.1}$$

where $k_i = \varkappa_{i+1}^\alpha - \varkappa_i^\alpha$ for $i = 0, ..., z-1$.

Theorem 4.1. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be an $\alpha-$fractional differentiable mapping on (a,b) with $0\le a<b$ and $\alpha\in(0,1]$. If $f^{(\alpha)}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\left\|f^{(\alpha)}\right\|_\infty=\sup_{t\in(a,b)}\left|f^{(\alpha)}(t)\right|<\infty$,, then we possess the representation*

$$\int_a^b f(t)d_\alpha t=S_\alpha(f,,\xi,I_z)+R_\alpha(f,,\xi,I_z)$$

where $S_\alpha(f,,\xi,I_z)$ is as defined in (4.1) and the remainder satisfies the estimations:

$$|R_\alpha(f,,\xi,I_z)| \le \left[\frac{2h(h-1)+1}{4\alpha^2}\sum_{i=0}^{z-1}k_i^2+\frac{1}{\alpha^2}\sum_{i=0}^{z-1}\left(\xi_i^\alpha-\frac{x_i^\alpha+x_{i+1}^\alpha}{2}\right)^2\right]\left\|f^{(\alpha)}\right\|_\infty \tag{4.2}$$

for all $h\in[0,1]$ and $\left(x_i^\alpha+h\frac{k_i}{2}\right)^{\frac{1}{\alpha}}\le\xi_i\le\left(x_{i+1}^\alpha-h\frac{k_i}{2}\right)^{\frac{1}{\alpha}}$ for $i=0,...,z-1$.

Proof. If we reconsider the inequalities of Theorem 2.1 on the interval $\left[\varkappa_i^\alpha,\varkappa_{i+1}^\alpha\right]$ for $i=0,\ ...,\ z-1$, then we have

$$\begin{aligned}&\left|\frac{(1-h)}{\alpha}f(\xi_i)\left(\varkappa_{i+1}^\alpha-\varkappa_i^\alpha\right)+h\frac{f(\varkappa_i)+f(\varkappa_{i+1})}{2\alpha}\left(\varkappa_{i+1}^\alpha-\varkappa_i^\alpha\right)-\int_{\varkappa_i}^{\varkappa_{i+1}},f(t)d_\alpha t\right|\\ &\le \left[\frac{2h(h-1)+1}{4\alpha^2}\left(\varkappa_{i+1}^\alpha-\varkappa_i^\alpha\right)^2+\frac{1}{\alpha^2}\left(\xi_i^\alpha-\frac{x_i^\alpha+x_{i+1}^\alpha}{2}\right)^2\right]\left\|f^{(\alpha)}\right\|_\infty\end{aligned}$$

for all $h\in[0,1]$ and $\left(x_i^\alpha+h\frac{\varkappa_{i+1}^\alpha-\varkappa_i^\alpha}{2}\right)^{\frac{1}{\alpha}}\le\xi_i\le\left(x_{i+1}^\alpha-h\frac{\varkappa_{i+1}^\alpha-\varkappa_i^\alpha}{2}\right)^{\frac{1}{\alpha}}$ for $i=0,...,z-1$.
Applying the triangle inequality for integrals to the resulting inequalities after summing over i from 0 to $z-1$, the estimations (4.2) can be easily obtained. □

Remark 4.2. *If we choose $\alpha=1$ in (4.2), we possess the representation*

$$\int_a^b f(t)dt=S_1(f,,\xi,I_z)+R_1(f,,\xi,I_z)$$

where $S_1(f,,\xi,I_z)$ is as defined in (4.1) and the remainder satisfies the estimations:

$$|R_1(f,,\xi,I_z)| \le \left[\frac{2h(h-1)+1}{4}\sum_{i=0}^{z-1}k_i^2+\sum_{i=0}^{z-1}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\|f'\|_\infty \tag{4.3}$$

for all $h\in[0,1]$ and $x_i+h\frac{k_i}{2}\le\xi_i\le x_{i+1}-h\frac{k_i}{2}$ for $i=0,...,z-1$. Here, $k_i=\varkappa_{i+1}-\varkappa_i$ for $i=0,...,z-1$. The estimation (4.3) was proved by Dragomir et al. in [12].

Remark 4.3. *Under the same assumption of Theorem 4.1 with $\xi_i^\alpha=\frac{\varkappa_i^\alpha+\varkappa_{i+1}^\alpha}{2}$, we have the representation*

$$\int_a^b f(t)d_\alpha t=S_{\alpha,M}(f,I_z)+R_{\alpha,M}(f,I_z)$$

where the remainder term satisfies the inequality

$$\left|R_{\alpha,M}(\psi,I_z)\right|\le\frac{2h(h-1)+1}{4\alpha^2}\left\|f^{(\alpha)}\right\|_\infty\sum_{i=0}^{z-1}k_i^2 \tag{4.4}$$

for $k_i=\varkappa_{i+1}^\alpha-\varkappa_i^\alpha$, $i=0,...,z-1$.

Remark 4.4. *Under the same assumption of Theorem 4.1 with $h=0$, we have the representation*

$$\int_a^b f(t)d_\alpha t=S_\alpha(f,,\xi,I_z)+R_\alpha(f,,\xi,I_z)$$

where $S_\alpha(f,,\xi,I_z)$ is as defined

$$S_\alpha(f,\xi,I_z)=\frac{1}{\alpha}\sum_{i=0}^{z-1}k_if(\xi_i)$$

and the remainder term satisfies the estimation

$$|R_\alpha(f,,\xi,I_z)|\le\frac{1}{\alpha^2}\left\|f^{(\alpha)}\right\|_\infty\sum_{i=0}^{z-1}\left(\xi_i^\alpha-\frac{x_i^\alpha+x_{i+1}^\alpha}{2}\right)^2$$

for $\xi_i\in[x_i,x_{i+1}]$, $i=0,...,z-1$. This estimation is a numeric application of the Ostrowski type inequality given by Anderson in [5].

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