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Research Article

Monoidal closedness of the category of ⊤-semiuniform convergence spaces

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Abstract

Lattice-valued semiuniform convergence structures are important mathematical structures in the theory of lattice-valued topology. Choosing a complete residuated lattice L as the lattice background, we introduce a new type of lattice-valued filters using the tensor and implication operations on L, which is called ⊤-filters. By means of ⊤-filters, we propose the concept of ⊤-semiuniform convergence structures as a new lattice-valued counterpart of semiuniform convergence structures. Different from the usual discussions on lattice-valued semiuniform convergence structures, we show that the category of ⊤-semiuniform convergence spaces is a topological and monoidal closed category when L is a complete residuated lattice without any other requirements.

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1. Introduction

In the theory of general topology, the category of topological spaces consisting of topological spaces as objects and continuous maps as morphisms lacks nice categorical properties, such as Cartesian-closedness. In order to overcome this deficiency, convergence spaces via filters were introduced. In the framework of topological spaces, the category of uniform spaces also lacks function spaces, i.e., it is not Cartesian closed. As generalizations of uniform spaces, uniform convergence spaces were proposed. The resulting category is not only Cartesian closed, but also has close relationship with convergence spaces. Considering more nice categorical properties, the notion of semiuniform convergence spaces were introduced by relaxing some axioms in the definition of uniform convergence spaces. In the famous book [18], Preuss presented systematical investigations on convergence spaces and semiuniform convergence spaces.

With the development of lattice-valued topology, convergence structures and semiuniform convergence structures have been generalized to the lattice-valued case, such as Jäger [11–14], Fang [4,5], Yao [25,26], Li [15–17], Flores [7,8], Pang [19–24] and Zhang

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[28–30]. From a categorical aspect, closedness is a very important character of lattice-valued convergence-related structures with L as the lattice background, such as Cartesian-closedness. However, it is usually required that L should be equipped with some restrictions in order to show the closedness of the resulting categories of lattice-valued structures. For examples, the categories of stratified L-generalized convergence spaces [11] and L-uniform convergence spaces [13] are Cartesian-closed whenever L is a complete Heyting algebra and the category of $\top$-convergence spaces [27] is Cartesian-closed whenever L is a complete MV-algebra. This demonstrates that internal feature of L determines the Cartesian-closedness of the lattice-valued structures. So a natural question has risen:

How to generalize the lattice background to ensure the closedness of lattice-valued structures?

Up to now, there have not been further results which can ensure the Cartesian-closedness of lattice-valued structures in a more general lattice background. Besides Cartesian closedness, monoidal closedness is another important categorical property with respect to closedness. In [30], Zhang et al used modified stratified L-filters to introduce stratified L-convergence structures with L a complete residuated lattice and showed the monoidal closedness of the resulting category. Recently, Fang and Fang [6] further proposed the concept of stratified L-semiuniform convergence structures by modified stratified L-filters (L is an integral, commutative unital quantale) and proved that the resulting category is monoidal closed. This motivates us to consider the monoidal closedness of lattice-valued structures by means of modified lattice-valued filters in a more general lattice background.

In many cases, it can be considered similar to show the Cartesian-closedness or the monoidal closedness of a category. However, the requirements on the lattice background L are usually different when dealing with the Cartesian-closedness and the monoidal closedness of lattice-valued filter based structures. Moreover, compared with the lattice-valued fitlers used in showing Cartesion-closedness, it is usually required that some modifications should be equipped on the lattice-valued fitlers in order to construct the tensor product in the definition of monoidal closedness. Recently, Yu and Fang [27] applied $\top$-fitlers [9] to introduce the concept of $\top$-convergence structures and showed that the category of $\top$-convergence spaces is Cartesian closed when L a complete MV-algebra. Jäger and Yue [14] showed the category of $\top$-uniform convergence spaces is Cartesian closed when the lattice background is a commutative and integral quantale which is divisible or is a value quantale. Following the idea on Cartesian-closedness and monoidal closedness of lattice-valued filter based structures, we will first modify $\top$-filters by replacing the $\wedge$-operation by the $*$ on a complete residuated lattice. Then we propose a new kind of lattice-valued semiuniform convergence spaces via modified $\top$-filters, which will be called $\top$-semiuniform convergence spaces. In this approach, we will explore the monoidal closedness of the resulting category where the background lattice is a complete residuated lattice without any other requirements.

This paper is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we introduce a new type of $\top$-filters and make some investigations on its properties. In Section 4, we propose the notion of $\top$-semiuniform convergence spaces by using $\top$-filters. Further, we define the tensor product of $\top$-semiuniform convergence spaces, and prove the category of $\top$-semiuniform convergence spaces is a topological and symmetric monoidal category. In Section 5, we prove that the category of $\top$-semiuniform convergence spaces is a monoidal closed category.

2. Preliminaries

A complete residuated lattice is a triple $(L, \leq, *)$, where $(L, \leq)$ is a complete lattice with the top element $\top$ and the bottom element $\perp$, and $*$ is a commutative, associative binary operation such that

(1) $\top$ is the unit element for $*$, i.e., $\top * a = a$, for $a \in L$;
(2) $a * \left(\bigvee_{j\in J} b_j\right) = \bigvee_{j\in J} a * b_j, \forall a \in L, \{b_j \mid j \in J\} \subseteq L$.

There exists a binary operation $\rightarrow: L \times L \longrightarrow L$ corresponding to $*$, computed by

$$\forall a, b \in L,\ a \rightarrow b = \bigvee\{c \in L \mid a * c \leq b\},$$

called the implication operation on L. Further, $*$ and $\rightarrow$ form an adjoint pair in the sense of $a * b \leq c \Longleftrightarrow b \leq a \rightarrow c$ for all $a, b, c \in L$.

Lemma 2.1 ([2]). *In every complete residuated lattice $(L, \leq, *)$, the following results are valid:*

(1) $\top \rightarrow a = a$, *and* $a \leq b$ *if and only if* $a \rightarrow b = \top$;
(2) $(a \rightarrow b) * (b \rightarrow c) \leq a \rightarrow c$;
(3) $a \rightarrow \bigwedge_{j\in J} b_j = \bigwedge_{j\in J}(a \rightarrow b_j)$;
(4) $\left(\bigvee_{j\in J} a_j\right) \rightarrow b = \bigwedge_{j\in J}(a_j \rightarrow b)$;
(5) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c) = (a * b) \rightarrow c$;
(6) $a * (a \rightarrow b) \leq b$;
(7) $(a \rightarrow c) * (b \rightarrow d) \leq a * b \rightarrow c * d$;
(8) $(a \wedge c) * (b \wedge d) \leq (a * b) \wedge (c * d)$.

Throughout this paper, let L be a complete residuated lattice and X be a nonempty set. An L-subset on X is a map from X to L, and the family of all L-subsets on X will be denoted by L^X, called the L-power set of X. By $\perp_X$ and $\top_X$, we denote the constant L-subsets on X taking the value $\perp$ and $\top$, respectively. We will not distinguish an element $a \in L$ and the constant map $a : X \longrightarrow L$ such that $a(x) = a$ for all $x \in X$.

All algebraic operations on L can be extended to the L-power set L^X in a pointwise way. That is, for all $A, B \in L^X$, $a \in L$ and $x \in X$,

$(A \vee B)(x) = A(x) \vee B(x)$;
$(A \wedge B)(x) = A(x) \wedge B(x)$;
$(A * B)(x) = A(x) * B(x)$, and $(a * A)(x) = a * A(x)$;
$(A \rightarrow B)(x) = A(x) \rightarrow B(x)$, and $(a \rightarrow B)(x) = a \rightarrow B(x)$.

Let $\varphi : X \longrightarrow Y$ be a map. Define $\varphi^{\rightarrow} : L^X \longrightarrow L^Y$ and $\varphi^{\leftarrow} : L^Y \longrightarrow L^X$ by $\varphi^{\rightarrow}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $\varphi^{\leftarrow}(B) = B\circ\varphi$ for $B \in L^Y$, respectively. For a given set X, define a binary map $\mathcal{S}_X(-,-) : L^X \times L^X \longrightarrow L$ by $\mathcal{S}_X(A, B) = \bigwedge_{x\in X}(A(x) \rightarrow B(x))$ for each pair $(A, B) \in L^X \times L^X$. For all $A,\ B \in L^X$, $\mathcal{S}_X(A, B)$ can be interpreted as the degree to which A is a subset of B. It is called subsethood degree [10] or fuzzy inclusion order [2] of L-subsets.

Lemma 2.2 ([2]). *The fuzzy inclusion order $\mathcal{S}_X(-,-)$ satisfies the following properties: $\forall A, B, C, D \in L^X$,*

(1) $A \leq B \Longleftrightarrow \mathcal{S}_X(A, B) = \top$;
(2) $\mathcal{S}_X(A, B) * \mathcal{S}_X(B, C) \leq \mathcal{S}_X(A, C)$;
(3) $\mathcal{S}_X(A, B) * \mathcal{S}_X(C, D) \leq \mathcal{S}_X(A * C, B * D)$;
(4) $\mathcal{S}_X(A, B) \wedge \mathcal{S}_X(C, D) \leq \mathcal{S}_X(A \wedge C, B \wedge D)$;
(5) $A \leq B$ *implies* $\mathcal{S}_X(C, A) \leq \mathcal{S}_X(C, B)$ *and* $\mathcal{S}_X(B, D) \leq \mathcal{S}_X(A, D)$.

Lemma 2.3 ([2]). *Let $\varphi : X \longrightarrow Y$ be a map. Then for $A, B \in L^X, C, D \in L^Y$, it holds that*

$$\mathcal{S}_X(A, B) \leq \mathcal{S}_Y(\varphi^{\rightarrow}(A), \varphi^{\rightarrow}(B)) \text{ and } \mathcal{S}_Y(C, D) \leq \mathcal{S}_X(\varphi^{\leftarrow}(C), \varphi^{\leftarrow}(D)).$$

Definition 2.4 ([3]). A category $\mathbb{C}$ is called a monoidal category if there exist a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ and an object I in $\mathbb{C}$ such that the transformations $a : \otimes \circ (\otimes \times id_{\mathbb{C}}) \longrightarrow \otimes \circ (id_{\mathbb{C}} \times \otimes)$, $l : I \otimes (-) \longrightarrow id_{\mathbb{C}}$ and $r : (-) \otimes I \longrightarrow id_{\mathbb{C}}$ are natural isomorphisms with the components

$$a_{XYZ} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z),$$

$$l_Y : I \otimes Y \longrightarrow Y, r_X : X \otimes I \longrightarrow X,$$

subjected to the following coherence axioms:

(1) The bifunctor $\otimes$ is associative, i.e., Diagram 1 is commutative;
(2) $l_I = r_I$ and Diagram 2 is commutative.

$$\begin{tikzcd}
((X\otimes Y)\otimes Z)\otimes U \arrow[r, "a_{(X\otimes Y)ZU}"] \arrow[d, "a_{XYZ}\otimes id_U"'] & (X\otimes Y)\otimes(Z\otimes U) \arrow[dd, "a_{XY(Z\otimes U)}"] \\
(X\otimes(Y\otimes Z))\otimes U \arrow[d, "a_{X(Y\otimes Z)U}"'] & \\
X\otimes((Y\otimes Z)\otimes U) \arrow[r, "id_X\otimes a_{YZU}"'] & X\otimes(Y\otimes(Z\otimes U))
\end{tikzcd}$$

Diagram 1

$$\begin{tikzcd}
(X\otimes I)\otimes Y \arrow[r, "a_{XIY}"] \arrow[dr, "r_X\otimes id_Y"'] & X\otimes(I\otimes Y) \arrow[d, "id_X\otimes l_Y"] \\
& X\otimes Y
\end{tikzcd}$$

Diagram 2

Definition 2.5 ([3])**.** A monoidal category $\mathbb{C}$ is called symmetric when the natural isomorphism $c : \otimes \longrightarrow \otimes$ whose components $c_{XY} : X \otimes Y \longrightarrow Y \otimes X$ satisfies the subsequent coherence axioms expressed by the commutativity of the following Diagram 3, Diagram 4 and Diagram 5.

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "c_{XY}\otimes id_Z"] \arrow[d, "a_{XYZ}"'] & (Y\otimes X)\otimes Z \arrow[d, "a_{YXZ}"] \\
X\otimes(Y\otimes Z) \arrow[d, "c_{X(Y\otimes Z)}"'] & Y\otimes(X\otimes Z) \arrow[d, "id_Y\otimes c_{XZ}"] \\
(Y\otimes Z)\otimes X \arrow[r, "a_{YZX}"'] & Y\otimes(Z\otimes X)
\end{tikzcd}$$

Diagram 3

$$\begin{tikzcd}
X\otimes I \arrow[r, "c_{XI}"] \arrow[dr, "r_X"'] & I\otimes X \arrow[d, "l_X"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
X\otimes Y \arrow[r, "c_{XY}"] \arrow[dr, equal] & Y\otimes X \arrow[d, "c_{YX}"] \\
& X\otimes Y
\end{tikzcd}$$

Diagram 4 Diagram 5

Definition 2.6 ([1])**.** Let $\mathbf{F} : \mathbb{C} \longrightarrow \mathbb{D}$ and $\mathbf{G} : \mathbb{D} \longrightarrow \mathbb{C}$ be functors. If for each object X in $\mathbb{C}$ and object Y in $\mathbb{D}$, there exists a bijection

$$\omega := \omega_{XY} : [\mathbf{F}X, Y]_{\mathbb{D}} \longrightarrow [X, \mathbf{G}Y]_{\mathbb{C}}$$

such that ω is natural in X and Y, then the triple $\langle \mathbf{F}, \mathbf{G}, \omega \rangle$ is said to be an adjunction from $\mathbb{C}$ to $\mathbb{D}$. In this case, the functor $\mathbf{F}$ is said to be a left adjoint for $\mathbf{G}$ and $\mathbf{G}$ is called a right adjoint for $\mathbf{F}$.

Definition 2.7 ([3])**.** A symmetric monoidal category $\mathbb{C}$ is said to be closed with respect to the tensor operation $\otimes$ if for each object X in $\mathbb{C}$, the functor $(-) \otimes X : \mathbb{C} \longrightarrow \mathbb{C}$ has a right adjoint $[X, -] : \mathbb{C} \longrightarrow \mathbb{C}$.

The class of objects in a category $\mathbb{C}$ is denoted by $|\mathbb{C}|$. For other notions, we refer to [1, 18].

3. $\top$-filters via residuated lattices

In this section, we will propose the concept of $\top$-filters by means of the operations $*$ and $\to$ on a complete residuated lattice and then investigate some of its properties.

Definition 3.1. A nonempty subset $\mathbb{F}$ of L^X is called a $\top$-filter on X provided that

(TF1) If $A \in L^X$ such that $\bigvee_{B\in\mathbb{F}} \mathcal{S}_X(B, A) = \top$, then $A \in \mathbb{F}$;
(TF2) $A_1 * A_2 \in \mathbb{F}$ for all $A_1, A_2 \in \mathbb{F}$;
(TF3) $\bigvee_{x\in X} A(x) = \top$ for all $A \in \mathbb{F}$.

The set of all $\top$-filters on X is denoted by $F_L^{\top}(X)$.

Example 3.2. Given a point $x \in X$, the set $[x]_{\top} \subseteq L^X$, defined by

$$[x]_{\top} := \{A \in L^X \mid A(x) = \top\},$$

is a $\top$-filter on X, which is called the point $\top$-filter of x. In case $X = \{\infty\}$, a single point set, $[\infty]_{\top}$ is the unique $\top$-filter on X.

Actually, the idea of $\top$-filters was originated from [9] in the framework of a complete MV-algebra by using a $\aleph$-condition. Recently, Yu and Fang [27] modified this concept and presented a concrete form by (TF1), (TF3) and

(TF2)* $A_1 \wedge A_2 \in \mathbb{F}$ for all $A_1, A_2 \in \mathbb{F}$.

In this sense, Yu and Fang [27] defined $\top$-convergence spaces and showed the Cartesian-closedness of the resulting category when L is a complete MV-algebra. In this section, we will adopt Definition 3.1 by replacing $\wedge$ by $*$ on a complete residuated lattice and explore its properties as preparations for the sequel sections.

Firstly, let us introduce the concept of bases of $\top$-filters in the sense of Definition 3.1.

Definition 3.3. A nonempty subset $\mathbb{B}$ of L^X is called a $\top$-filter base on X provided that

(B1) $\bigvee_{B\in\mathbb{B}} \mathcal{S}_X(B, B_1 * B_2) = \top$ for all $B_1, B_2 \in \mathbb{B}$;
(B2) $\bigvee_{x\in X} B(x) = \top$ for all $B \in \mathbb{B}$.

In [27], Yu and Fang showed how to generate a $\top$-filter from a $\top$-filter base. The method can also be applied to our case. That is, for a $\top$-filter base $\mathbb{B}$, a $\top$-filter can be generated in the following way:

$$\mathbb{F}_{\mathbb{B}} := \{A \in L^X \mid \bigvee_{B\in\mathbb{B}} \mathcal{S}_X(B, A) = \top\}.$$

In this sense, $\mathbb{B}$ is called a base of $\mathbb{F}_{\mathbb{B}}$. Obviously, a $\top$-filter $\mathbb{F}$ is also a $\top$-filter base and it can be generated by itself as a $\top$-filter base.

Next we present some results on $\top$-filters and $\top$-filter bases and only give some necessary proofs.

Proposition 3.4. *Let $\varphi : X \longrightarrow Y$ be a map, $\mathbb{F} \in F_L^{\top}(X)$, $\mathbb{G} \in F_L^{\top}(Y)$, and $\mathbb{B}_{\mathbb{F}}$ and $\mathbb{B}_{\mathbb{G}}$ be $\top$-filter bases of $\mathbb{F}$ and $\mathbb{G}$, respectively. Then*

(1) *$\varphi^{\Rightarrow}(\mathbb{F})$ generated by the $\top$-filter base $\{\varphi^{\to}(A) \in L^Y \mid A \in \mathbb{F}\}$ is a $\top$-filter on Y, which is called the image of $\mathbb{F}$ under φ. The calculation formula of $\varphi^{\Rightarrow}(\mathbb{F})$ is as follows:*

$$\begin{aligned}\varphi^{\Rightarrow}(\mathbb{F}) &= \{B \in L^Y \mid \bigvee_{A\in\mathbb{F}} \mathcal{S}_Y(\varphi^{\to}(A), B) = \top\}\\ &= \{B \in L^Y \mid \bigvee_{A\in\mathbb{B}_{\mathbb{F}}} \mathcal{S}_Y(\varphi^{\to}(A), B) = \top\}.\end{aligned}$$

(2) *The set $\{\varphi^{\leftarrow}(B) \in L^X \mid B \in \mathbb{G}\}$ is a $\top$-filter base on X if and only if $\bigvee_{y\in\varphi(X)} B(y) = \top$ holds for each $B \in \mathbb{G}$. In this case, it can generate a $\top$-filter by*

$$\begin{aligned}\varphi^{\Leftarrow}(\mathbb{G}) &= \{A \in L^X \mid \bigvee_{B\in\mathbb{G}} \mathcal{S}_X(\varphi^{\leftarrow}(B), A) = \top\}\\ &= \{A \in L^X \mid \bigvee_{B\in\mathbb{B}_{\mathbb{G}}} \mathcal{S}_X(\varphi^{\leftarrow}(B), A) = \top\},\end{aligned}$$

which is called the inverse image of $\mathbb{G}$ *and is denoted by* $\varphi^{\Leftarrow}(\mathbb{G})$.

Proposition 3.5. *Let* $\varphi : X \longrightarrow Y$ *be a constant map with the constant* $\infty \in Y$. *That is,* $\varphi(x) = \infty$ *for each* $x \in X$. *Then* $\varphi^{\Rightarrow}(\mathbb{F}) = [\infty]_{\top}$ *for each* $\mathbb{F} \in F_L^{\top}(X)$.

Proof. Take each $A \in \mathbb{F}$. Then $\varphi^{\rightarrow}(A)(\infty) = \bigvee_{x\in X} A(x) = \top$. By Proposition 3.4 (1), we have

$$\begin{aligned}
B \in \varphi^{\Rightarrow}(\mathbb{F}) &\Longleftrightarrow \bigvee_{A\in\mathbb{F}} \mathcal{S}_Y(\varphi^{\rightarrow}(A), B) = \top \\
&\Longleftrightarrow \bigvee_{A\in\mathbb{F}} \bigwedge_{y\in Y} (\varphi^{\rightarrow}(A)(y) \rightarrow B(y)) = \top \\
&\Longleftrightarrow \bigvee_{A\in\mathbb{F}} (\varphi^{\rightarrow}(A)(\infty) \rightarrow B(\infty)) = \top \\
&\Longleftrightarrow B(\infty) = \top \\
&\Longleftrightarrow B \in [\infty]_{\top},
\end{aligned}$$

which shows $\varphi^{\Rightarrow}(\mathbb{F}) = [\infty]_{\top}$. □

Proposition 3.6. *Let* $\varphi : X \longrightarrow Y$ *be a map,* $\mathbb{F}, \mathbb{G} \in F_L^{\top}(X)$. *Then*

(1) $\mathbb{F} \cap \mathbb{G} \in F_L^{\top}(X)$.
(2) $\varphi^{\Rightarrow}(\mathbb{F} \cap \mathbb{G}) = \varphi^{\Rightarrow}(\mathbb{F}) \cap \varphi^{\Rightarrow}(\mathbb{G})$.

Proof. (1) Obviously.

(2) Since $\varphi^{\Rightarrow}(\mathbb{F}\cap\mathbb{G}) \subseteq \varphi^{\Rightarrow}(\mathbb{F})\cap\varphi^{\Rightarrow}(\mathbb{G})$ is obvious, we only need to show $\varphi^{\Rightarrow}(\mathbb{F})\cap\varphi^{\Rightarrow}(\mathbb{G}) \subseteq \varphi^{\Rightarrow}(\mathbb{F} \cap \mathbb{G})$. Take each $D \in \varphi^{\Rightarrow}(\mathbb{F}) \cap \varphi^{\Rightarrow}(\mathbb{G})$. By Proposition 3.4 (1), we have

$$\begin{aligned}
\top = \top * \top &= \bigvee_{A\in\mathbb{F}} \mathcal{S}_Y\big(\varphi^{\rightarrow}(A), D\big) * \bigvee_{B\in\mathbb{G}} \mathcal{S}_Y\big(\varphi^{\rightarrow}(B), D\big) \\
&= \bigvee_{A\in\mathbb{F}, B\in\mathbb{G}} \mathcal{S}_Y\big(\varphi^{\rightarrow}(A), D\big) * \mathcal{S}_Y\big(\varphi^{\rightarrow}(B), D\big) \\
&\leq \bigvee_{A\in\mathbb{F}, B\in\mathbb{G}} \mathcal{S}_Y\big(\varphi^{\rightarrow}(A), D\big) \wedge \mathcal{S}_Y\big(\varphi^{\rightarrow}(B), D\big) \\
&= \bigvee_{A\in\mathbb{F}, B\in\mathbb{G}} \mathcal{S}_Y\big(\varphi^{\rightarrow}(A) \vee \varphi^{\rightarrow}(B), D\big) \\
&\leq \bigvee_{C\in\mathbb{F}\cap\mathbb{G}} \mathcal{S}_Y\big(\varphi^{\rightarrow}(C), D\big),
\end{aligned}$$

which means $D \in \varphi^{\Rightarrow}(\mathbb{F} \cap \mathbb{G})$, as desired. □

Proposition 3.7. *Let* $\mathbb{F} \in F_L^{\top}(X)$, $\mathbb{G} \in F_L^{\top}(Y)$, *and* $\mathbb{B}_{\mathbb{F}}$ *and* $\mathbb{B}_{\mathbb{G}}$ *be* $\top$*-filter bases of* $\mathbb{F}$ *and* $\mathbb{G}$, *respectively. Then* $\mathbb{B} = \{A \otimes B \in L^{X\times Y} \mid A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}\}$ *is a* $\top$*-filter base on* $X \times Y$, *where* $A \otimes B((x, y)) = A(x) * B(y)$ *for each* $(x, y) \in L^{X\times Y}$.

Proof. It suffices to verify that $\mathbb{B}$ satisfies (B1) and (B2). Indeed, (B1) Take each $D_1, D_2 \in \mathbb{B}$. Then there exist $A_1, A_2 \in \mathbb{B}_{\mathbb{F}}$ and $B_1, B_2 \in \mathbb{B}_{\mathbb{G}}$ such that $D_1 = A_1 \otimes B_1$ and $D_2 = A_2 \otimes B_2$. This implies

$$\begin{aligned}
&\bigvee_{D \in \mathbb{B}} \mathcal{S}_{X\times Y}(D, D_1 * D_2) \\
&= \bigvee_{A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}} \mathcal{S}_{X\times Y}(A \otimes B, D_1 * D_2) \\
&= \bigvee_{A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}} \mathcal{S}_{X\times Y}(A \otimes B, (A_1 \otimes B_1) * (A_2 \otimes B_2)) \\
&= \bigvee_{A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}} \bigwedge_{(x,y)\in X\times Y} \big(A(x) * B(y) \to A_1(x) * B_1(y) * A_2(x) * B_2(y)\big) \\
&\geq \bigvee_{A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}} \bigwedge_{(x,y)\in X\times Y} \big((A(x) \to A_1(x) * A_2(x)) * (B(y) \to B_1(y) * B_2(y))\big) \\
&\geq \bigvee_{A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}} \big(\mathcal{S}_X(A, A_1 * A_2) * \mathcal{S}_Y(B, B_1 * B_2)\big) \\
&= \bigvee_{A \in \mathbb{B}_{\mathbb{F}}} \mathcal{S}_X(A, A_1 * A_2) * \bigvee_{B \in \mathbb{B}_{\mathbb{G}}} \mathcal{S}_Y(B, B_1 * B_2) \\
&= \top * \top = \top,
\end{aligned}$$

which means (B1) holds.

(B2) Take each $D \in \mathbb{B}$. Then there exist $A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}$ such that $D = A \otimes B$. This implies

$$\begin{aligned}
\bigvee_{(x,y)\in X\times Y} D(x,y) &= \bigvee_{(x,y)\in X\times Y} A \otimes B(x,y) \\
&= \bigvee_{(x,y)\in X\times Y} A(x) * B(y) \\
&= \bigvee_{x \in X} A(x) * \bigvee_{y\in Y} B(y) \\
&= \top * \top = \top,
\end{aligned}$$

which means (B2) holds, as desired. □

Since each $\top$-filter is a $\top$-filter base of itself, it follows from the above proposition that a $\top$-filter on $X \times Y$, denoted by $\mathbb{F} \otimes \mathbb{G}$, can be generated from $\mathbb{F} \in F_L^{\top}(X)$ and $\mathbb{G} \in F_L^{\top}(Y)$ in the following way:

$$\mathbb{F} \otimes \mathbb{G} = \{C \in L^{X\times Y} \mid \bigvee_{A \in \mathbb{F}, B \in \mathbb{G}} \mathcal{S}_{X\times Y}(A \otimes B, C) = \top\}.$$

In this sense, $\mathbb{F} \otimes \mathbb{G}$ is called the tensor product of $\mathbb{F}$ and $\mathbb{G}$. In particular, for each $x \in X$, $[x]_\top \otimes [x]_\top$ is a $\top$-filter on $X \times X$, and it is easy to check that $[x]_\top \otimes [x]_\top = [(x,x)]_\top$.

Proposition 3.8. *Let $\mathbb{F} \in F_L^{\top}(X), \mathbb{G} \in F_L^{\top}(Y)$, $\mathbb{B}_{\mathbb{F}}, \mathbb{B}_{\mathbb{G}}$ be $\top$-filter bases of $\mathbb{F}, \mathbb{G}$, respectively. Then*

$$\mathbb{F} \otimes \mathbb{G} = \{D \in L^{X\times Y} \mid \bigvee_{A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}} \mathcal{S}_{X\times Y}(A \otimes B, D) = \top\}.$$

That is to say, $\{A \otimes B \mid A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}\}$ is also a base of $\mathbb{F} \otimes \mathbb{G}$.

Proof. It suffices to show

$$\mathbb{F} \otimes \mathbb{G} \subseteq \{D \in L^{X\times Y} \mid \bigvee_{A \in \mathbb{B}_{\mathbb{F}}, B \in \mathbb{B}_{\mathbb{G}}} \mathcal{S}_{X\times Y}(A \otimes B, D) = \top\}.$$

Take each $C \in \mathbb{F} \otimes \mathbb{G}$. Then

$$\bigvee_{A \in \mathbb{F}, B \in \mathbb{G}} \mathcal{S}_{X\times Y}(A \otimes B, C) = \top.$$

Since

$$\bigvee_{A_1\in\mathbb{B}_{\mathbb{F}}}\mathcal{S}_X(A_1,A)=\top \text{ and } \bigvee_{B_1\in\mathbb{B}_{\mathbb{G}}}\mathcal{S}_Y(B_1,B)=\top$$

for each $A\in\mathbb{F}$ and $B\in\mathbb{G}$, we have

$$\begin{aligned}
\top &= \bigvee_{A\in\mathbb{F},B\in\mathbb{G}}\mathcal{S}_{X\times Y}(A\otimes B,C)\\
&= \bigvee_{A\in\mathbb{F},B\in\mathbb{G}}\Big(\mathcal{S}_{X\times Y}(A\otimes B,C)*\bigvee_{A_1\in\mathbb{B}_{\mathbb{F}}}\mathcal{S}_X(A_1,A)*\bigvee_{B_1\in\mathbb{B}_{\mathbb{G}}}\mathcal{S}_Y(B_1,B)\Big)\\
&\le \bigvee_{A\in\mathbb{F},B\in\mathbb{G}}\Big(\mathcal{S}_{X\times Y}(A\otimes B,C)*\bigvee_{A_1\in\mathbb{B}_{\mathbb{F}},B_1\in\mathbb{B}_{\mathbb{G}}}\mathcal{S}_{X\times Y}(A_1\otimes B_1,A\otimes B)\Big)\\
&= \bigvee_{A\in\mathbb{F},B\in\mathbb{G}}\ \bigvee_{A_1\in\mathbb{B}_{\mathbb{F}},B_1\in\mathbb{B}_{\mathbb{G}}}\Big(\mathcal{S}_{X\times Y}(A\otimes B,C)*\mathcal{S}_{X\times Y}(A_1\otimes B_1,A\otimes B)\Big)\\
&\le \bigvee_{A_1\in\mathbb{B}_{\mathbb{F}},B_1\in\mathbb{B}_{\mathbb{G}}}\mathcal{S}_{X\times Y}(A_1\otimes B_1,C),
\end{aligned}$$

which means $C\in\{D\in L^{X\times Y}\mid\bigvee_{A\in\mathbb{B}_{\mathbb{F}},B\in\mathbb{B}_{\mathbb{G}}}\mathcal{S}_{X\times Y}(A\otimes B,D)=\top\}$, as desired. □

Proposition 3.9. *Let* $\mathbb{F}$ *be a* $\top$*-filter on* $X\times X$*. Then* $\mathbb{F}^{-1}$ *defined by* $\mathbb{F}^{-1}=\{A^{-1}\in L^{X\times X}\mid A\in\mathbb{F}\}$ *is also a* $\top$*-filter on* $X\times X$*, where* $A^{-1}(x,x')=A(x',x)$ *for each* $(x,x')\in X\times X$*.*

Proof. It is enough to verity that $\mathbb{F}^{-1}$ satisfies (TF1)–(TF3). Indeed,

(TF1) Take each $B\in L^{X\times X}$ such that $\bigvee_{A\in\mathbb{F}^{-1}}\mathcal{S}_{X\times X}(A,B)=\top$. Then

$$\bigvee_{C\in\mathbb{F}}\mathcal{S}_{X\times X}(C,B^{-1})=\bigvee_{A^{-1}\in\mathbb{F}}\mathcal{S}_{X\times X}(A^{-1},B^{-1})=\bigvee_{A\in\mathbb{F}^{-1}}\mathcal{S}_{X\times X}(A,B)=\top,$$

which means $B^{-1}\in\mathbb{F}$, i.e., $B\in\mathbb{F}^{-1}$.

(TF2) It follows immediately from $(A*B)^{-1}=A^{-1}*B^{-1}$ for each $A,B\in L^{X\times X}$.

(TF3) Take each $A\in\mathbb{F}^{-1}$, i.e., $A^{-1}\in\mathbb{F}$. Then

$$\bigvee_{(x,y)\in X\times X}A(x,y)=\bigvee_{(x,y)\in X\times X}A^{-1}(y,x)=\bigvee_{(y,x)\in X\times X}A^{-1}(y,x)=\top,$$

as desired. □

Proposition 3.10. *Let* $\varphi:X\longrightarrow U$ *and* $\psi:Y\longrightarrow V$ *be two maps,* $p_X:X\times Y\longrightarrow X, p_Y:X\times Y\longrightarrow Y$ *be the projection maps, and* $\mathbb{F}\in F_L^{\top}(X)$*,* $\mathbb{G}\in F_L^{\top}(Y)$*,* $\mathbb{K}\in F_L^{\top}(X\times Y)$*,* $\mathbb{H}\in F_L^{\top}(X\times X)$*. Then*

(1) $(\varphi\times\psi)^{\Rightarrow}(\mathbb{F}\otimes\mathbb{G})=\varphi^{\Rightarrow}(\mathbb{F})\otimes\psi^{\Rightarrow}(\mathbb{G})$*;*

(2) $p_X^{\Rightarrow}(\mathbb{F}\otimes\mathbb{G})=\mathbb{F}$*,* $p_Y^{\Rightarrow}(\mathbb{F}\otimes\mathbb{G})=\mathbb{G}$*;*

(3) $p_X^{\Rightarrow}(\mathbb{K})\otimes p_Y^{\Rightarrow}(\mathbb{K})\subseteq\mathbb{K}$*;*

(4) $\big((\varphi\times\varphi)^{\Rightarrow}(\mathbb{H})\big)^{-1}=(\varphi\times\varphi)^{\Rightarrow}(\mathbb{H}^{-1})$*,*

where $\varphi\times\psi:X\times Y\longrightarrow U\times V$ *is defined by* $\varphi\times\psi(x,y)=(\varphi(x),\psi(y))$ *for each* $(x,y)\in X\times Y$*.*

Proof. For (1), let $\mathbb{B}_1=\{(\varphi\times\psi)^{\rightarrow}(A\otimes B)\mid A\in\mathbb{F},B\in\mathbb{G}\}$ and $\mathbb{B}_2=\{\varphi^{\rightarrow}(A)\otimes\psi^{\rightarrow}(B)\mid A\in\mathbb{F},B\in\mathbb{G}\}$. By Propositions 3.4 and 3.8, we know $\mathbb{B}_1$ is a $\top$-filter base of $(\varphi\times\psi)^{\Rightarrow}(\mathbb{F}\otimes\mathbb{G})$ and $\mathbb{B}_2$ is a $\top$-filter base of $\varphi^{\Rightarrow}(\mathbb{F})\otimes\psi^{\Rightarrow}(\mathbb{G})$. Since $(\varphi\times\psi)^{\rightarrow}(A\otimes B)=\varphi^{\rightarrow}(A)\otimes\psi^{\rightarrow}(B)$ holds, we get $\mathbb{B}_1=\mathbb{B}_2$. This means $(\varphi\times\psi)^{\Rightarrow}(\mathbb{F}\otimes\mathbb{G})=\varphi^{\Rightarrow}(\mathbb{F})\otimes\psi^{\Rightarrow}(\mathbb{G})$.

For (2), it follows immediately from $p_X^{\rightarrow}(A\otimes B)=A$ and $p_Y^{\rightarrow}(A\otimes B)=B$ for each $A\in L^X$ and $B\in L^Y$.

For (3), take each $D\in p_X^{\Rightarrow}(\mathbb{K})\otimes p_Y^{\Rightarrow}(\mathbb{K})$. Then

$$\begin{aligned}
\top &= \bigvee_{A,B\in\mathbb{K}}\mathcal{S}_{X\times Y}(p_X^{\rightarrow}(A)\otimes p_Y^{\rightarrow}(B),D)\\
&\le \bigvee_{A,B\in\mathbb{K}}\mathcal{S}_{X\times Y}(A*B,D)\\
&\le \bigvee_{C\in\mathbb{K}}\mathcal{S}_{X\times Y}(C,D),
\end{aligned}$$

which means $D \in \mathbb{K}$. This shows $p_X^{\Rightarrow}(\mathbb{K}) \otimes p_Y^{\Rightarrow}(\mathbb{K}) \subseteq \mathbb{K}$.

For (4), take each $D \in L^{U\times U}$. Then

$$\begin{aligned}
D \in \big((\varphi\times\varphi)^{\Rightarrow}(\mathbb{H})\big)^{-1} &\Longleftrightarrow D^{-1} \in (\varphi\times\varphi)^{\Rightarrow}(\mathbb{H}) \\
&\Longleftrightarrow \bigvee_{A\in\mathbb{H}} \mathcal{S}_{U\times U}\big((\varphi\times\varphi)^{\rightarrow}(A), D^{-1}\big) = \top \quad \text{(by Proposition 3.4)} \\
&\Longleftrightarrow \bigvee_{A\in\mathbb{H}} \mathcal{S}_{U\times U}\big((\varphi\times\varphi)^{\rightarrow}(A^{-1}), D\big) = \top \\
&\Longleftrightarrow \bigvee_{B\in\mathbb{H}^{-1}} \mathcal{S}_{U\times U}\big((\varphi\times\varphi)^{\rightarrow}(B), D\big) = \top \\
&\Longleftrightarrow D \in (\varphi\times\varphi)^{\Rightarrow}(\mathbb{H}^{-1}), \quad \text{(by Proposition 3.4)}
\end{aligned}$$

which shows $\big((\varphi\times\varphi)^{\Rightarrow}(\mathbb{H})\big)^{-1} = (\varphi\times\varphi)^{\Rightarrow}(\mathbb{H}^{-1})$. □

Proposition 3.11. *Let* $\mathbb{F},\mathbb{G} \in F_L^{\top}(X)$ *and* $\mathbb{H} \in F_L^{\top}(Y)$. *Then*

$$(\mathbb{F}\cap\mathbb{G})\otimes\mathbb{H} = (\mathbb{F}\otimes\mathbb{H})\cap(\mathbb{G}\otimes\mathbb{H}).$$

Proof. Take each $D \in (\mathbb{F}\otimes\mathbb{H})\cap(\mathbb{G}\otimes\mathbb{H})$. Then

$$\begin{aligned}
\top = \top * \top &= \bigvee_{A\in\mathbb{F},H_1\in\mathbb{H}} \mathcal{S}_{X\times Y}(A\otimes H_1, D) * \bigvee_{B\in\mathbb{G},H_2\in\mathbb{H}} \mathcal{S}_{X\times Y}(B\otimes H_2, D) \\
&= \bigvee_{A\in\mathbb{F},B\in\mathbb{G},H_1,H_2\in\mathbb{H}} \mathcal{S}_{X\times Y}(A\otimes H_1, D) * \mathcal{S}_{X\times Y}(B\otimes H_2, D) \\
&\le \bigvee_{A\in\mathbb{F},B\in\mathbb{G},H_1,H_2\in\mathbb{H}} \mathcal{S}_{X\times Y}(A\otimes H_1, D) \wedge \mathcal{S}_{X\times Y}(B\otimes H_2, D) \\
&= \bigvee_{A\in\mathbb{F},B\in\mathbb{G},H_1,H_2\in\mathbb{H}} \mathcal{S}_{X\times Y}\big((A\otimes H_1)\vee(B\otimes H_2), D\big) \\
&\le \bigvee_{A\in\mathbb{F},B\in\mathbb{G},H_1*H_2\in\mathbb{H}} \mathcal{S}_{X\times Y}\big((A\otimes H_1)\vee(B\otimes H_2), D\big) \\
&\le \bigvee_{A\in\mathbb{F},B\in\mathbb{G},H_1*H_2\in\mathbb{H}} \mathcal{S}_{X\times Y}\big((A\otimes(H_1*H_2))\vee(B\otimes(H_1*H_2)), D\big) \\
&\le \bigvee_{A\in\mathbb{F},B\in\mathbb{G},H\in\mathbb{H}} \mathcal{S}_{X\times Y}\big((A\otimes H)\vee(B\otimes H), D\big) \\
&= \bigvee_{A\in\mathbb{F},B\in\mathbb{G},H\in\mathbb{H}} \mathcal{S}_{X\times Y}\big((A\vee B)\otimes H, D\big) \\
&\le \bigvee_{C\in\mathbb{F}\cap\mathbb{G},H\in\mathbb{H}} \mathcal{S}_{X\times Y}\big(C\otimes H, D\big),
\end{aligned}$$

which implies $D \in (\mathbb{F}\cap\mathbb{G})\otimes\mathbb{H}$. This shows $(\mathbb{F}\otimes\mathbb{H})\cap(\mathbb{G}\otimes\mathbb{H}) \subseteq (\mathbb{F}\cap\mathbb{G})\otimes\mathbb{H}$. The inverse holds obviously. Thus, we obtain $(\mathbb{F}\otimes\mathbb{H})\cap(\mathbb{G}\otimes\mathbb{H}) = (\mathbb{F}\cap\mathbb{G})\otimes\mathbb{H}$. □

4. $\top$-semiuniform convergence spaces

In this section, we introduce the notion of $\top$-semiuniform convergence spaces and verify the category of $\top$-semiuniform convergence spaces is a topological and symmetric monoidal category.

Definition 4.1. A set $\Lambda \subseteq F_L^{\top}(X\times X)$ is called a $\top$-semiuniform convergence structure on X provided that

(TSUC1) $[x]_{\top}\otimes[x]_{\top} \in \Lambda$;
(TSUC2) $\mathbb{F}\in\Lambda$ and $\mathbb{F}\subseteq\mathbb{G}$ imply $\mathbb{G}\in\Lambda$;
(TSUC3) $\mathbb{F}\in\Lambda$ implies $\mathbb{F}^{-1}\in\Lambda$.

For a $\top$-semiuniform convergence structure Λ on X, the pair (X,Λ) is called a $\top$-semiuniform convergence space.

A map $\varphi : (X, \Lambda_X) \longrightarrow (Y, \Lambda_Y)$ between $\top$-semiuniform convergence spaces is called uniformly continuous provided that $\mathbb{F} \in \Lambda_X$ implies $(\varphi \times \varphi)^{\Rightarrow}(\mathbb{F}) \in \Lambda_Y$ for each $\mathbb{F} \in F^{\top}_L(X \times X)$.

It is easy to check that all $\top$-semiuniform convergence spaces as objects and all uniformly continuous maps as morphisms form a category, which is denoted by $\top$-**SUConv**.

Let $\Lambda(X)$ denote the fibre of X, i.e.,

$$\Lambda(X) := \{\Lambda_X \mid \Lambda_X \text{ is a } \top\text{-semiuniform convergence structure on } X\}.$$

We can define an order on $\Lambda(X)$ by for each $\Lambda^X_1, \Lambda^X_2 \in \Lambda(X)$, $\Lambda^X_1 \leq \Lambda^X_2$ if and only if $id_X : (X, \Lambda^X_1) \longrightarrow (X, \Lambda^X_2)$ is uniformly continuous. In this case, we call Λ^X_1 is finer than Λ^X_2 or Λ^X_2 is coarser than Λ^X_1.

Example 4.2. (1) We define a set $\Lambda^X_{ind} = F^{\top}_L(X \times X)$. This is the coarsest $\top$-semiuniform convergence structure on X, which is called the indiscrete $\top$-semiuniform convergence structure on X.

(2) We define a set $\Lambda^X_{dis} = \{[x]_\top \otimes [x]_\top \mid x \in X\}$. This is the finest $\top$-semiuniform convergence structure on X, which is called the discrete $\top$-semiuniform convergence structure on X.

(3) For each single point set $X = \{\infty\}$, there is a unique $\top$-filter $[\infty]_\top \otimes [\infty]_\top$ on $X \times X$. So there is a unique $\top$-semiuniform convergence structure on $\{\infty\}$, which is denoted by $\Lambda_{\{\infty\}}$. Concretely, $\Lambda_{\{\infty\}} = \{[\infty]_\top \otimes [\infty]_\top\}$.

Theorem 4.3. *The category* $\top$-**SUConv** *is a topological category over* **Set**.

Proof. *Existence of initial structures.* Given a source $\{\varphi_i : X \longrightarrow (X_i, \Lambda_{X_i})\}_{i \in I}$ in $\top$-**SUConv**, define a set $\Lambda_X \subseteq F^{\top}_L(X \times X)$ by

$$\Lambda_X = \{\mathbb{F} \in F^{\top}_L(X \times X) \mid \forall i \in I, (\varphi_i \times \varphi_i)^{\Rightarrow}(\mathbb{F}) \in \Lambda_{X_i}\}.$$

Now let us prove that Λ_X is the initial structure w.r.t. the source $\{\varphi_i : X \longrightarrow (X_i, \Lambda_{X_i})\}_{i \in I}$. It is easy to show that Λ_X is a $\top$-semiuniform convergence structure on X. Next it suffices to verify Λ_X is the unique $\top$-semiuniform convergence structure on X such that for each object (Y, Λ_Y) in $\top$-**SUConv** and for each map $\psi : Y \longrightarrow X$, the map $\psi : (Y, \Lambda_Y) \longrightarrow (X, \Lambda_X)$ is uniformly continuous if and only if the composite map $\varphi_i \circ \psi : (Y, \Lambda_Y) \longrightarrow (X_i, \Lambda_{X_i})$ is uniformly continuous for each $i \in I$. The necessity is straightforward. Conversely, if the composite map $\varphi_i \circ \psi : (Y, \Lambda_Y) \longrightarrow (X_i, \Lambda_{X_i})$ is uniformly continuous for each $i \in I$, then for each $\mathbb{G} \in \Lambda_Y$, it follows that

$$((\varphi_i \times \varphi_i) \circ (\psi \times \psi))^{\Rightarrow}(\mathbb{G}) = ((\varphi_i \circ \psi) \times (\varphi_i \circ \psi))^{\Rightarrow}(\mathbb{G}) \in \Lambda_{X_i}.$$

By the definition of Λ_X, we get $(\psi \times \psi)^{\Rightarrow}(\mathbb{G}) \in \Lambda_X$, which means the map $\psi : (Y, \Lambda_Y) \longrightarrow (X, \Lambda_X)$ is uniformly continuous. Suppose that there is another $\top$-semiuniform convergence structure Λ'_X on X such that for each object (Y, Λ_Y) in $\top$-**SUConv** and for each map $\psi : Y \longrightarrow X$, the map $\psi : (Y, \Lambda_Y) \longrightarrow (X, \Lambda'_X)$ is uniformly continuous if and only if the composite map $\varphi_i \circ \psi : (Y, \Lambda_Y) \longrightarrow (X_i, \Lambda_{X_i})$ is uniformly continuous for each $i \in I$. Let $\psi = id_X : (X, \Lambda_X) \longrightarrow (X, \Lambda'_X)$. Since $\varphi_i \circ id_X = \varphi_i : (X, \Lambda_X) \longrightarrow (X_i, \Lambda_{X_i})$ is uniformly continuous for each $i \in I$, we conclude that ψ is uniformly continuous. That is to says $\Lambda_X \subseteq \Lambda'_X$. On the other hand, it follows from the uniform continuity of $id_X : (X, \Lambda'_X) \longrightarrow (X, \Lambda'_X)$ that the map $\varphi_i : (X, \Lambda'_X) \longrightarrow (X_i, \Lambda_{X_i})$ is uniformly continuous for each $i \in I$. By the definition of Λ_X, we obtain $\Lambda'_X \subseteq \Lambda_X$. This shows $\Lambda'_X = \Lambda_X$.

Fiber-smallness. The class of all $\top$-semiuniform convergence structure on X denoted by $\Lambda(X)$ is a set since $\Lambda(X) \subseteq \{0,1\}^{\mathcal{P}(F^{\top}_L(X \times X))}$.

Terminal separator property. For a single point set $X = \{\infty\}$, there is exactly one $\top$-semiuniform convergence structure Λ_∞ on X (See Example 4.2 (3)). □

In the sequel, we aim to show that the category of $\top$-semiuniform convergence spaces is a symmetric monoidal category. To this end, we will first construct a tensor operation between $\top$-semiuniform convergence spaces to define a bifunctor from $\top$-**SUConv** $\times$ $\top$-**SUConv** to $\top$-**SUConv**.

Given two $\top$-semiuniform convergence spaces (X, Λ_X) and (Y, Λ_Y), we define a set $\Lambda_X \otimes \Lambda_Y \subseteq F_L^{\top}((X \times Y) \times (X \times Y))$ as follows:

$$\Lambda_X \otimes \Lambda_Y = \{\mathbb{H} \in F_L^{\top}((X \times Y) \times (X \times Y)) \mid (p_X \times p_X)^{\Rightarrow}(\mathbb{H}) \in \Lambda_X, (p_Y \times p_Y)^{\Rightarrow}(\mathbb{H}) \in \Lambda_Y\},$$

where $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are projection maps.

Proposition 4.4. *$\Lambda_X \otimes \Lambda_Y$ defined above is a $\top$-semiuniform convergence structure on $X \times Y$.*

Proof. It suffices to show that $\Lambda_X \otimes \Lambda_Y$ satisfies (TSUC1)–(TSUC3). Indeed,

(TSUC1) Take each $(x, y) \in X \times Y$. Then it follows from Proposition 3.10 that

$$(p_X \times p_X)^{\Rightarrow}([(x, y)]_{\top} \otimes [(x, y)]_{\top}) = [x]_{\top} \otimes [x]_{\top}$$

and

$$(p_Y \times p_Y)^{\Rightarrow}([(x, y)]_{\top} \otimes [(x, y)]_{\top}) = [y]_{\top} \otimes [y]_{\top}.$$

Since $[x]_{\top} \otimes [x]_{\top} \in \Lambda_X$ and $[y]_{\top} \otimes [y]_{\top} \in \Lambda_Y$, we have

$$(p_X \times p_X)^{\Rightarrow}([(x, y)]_{\top} \otimes [(x, y)]_{\top}) \in \Lambda_X$$

and

$$(p_Y \times p_Y)^{\Rightarrow}([(x, y)]_{\top} \otimes [(x, y)]_{\top}) \in \Lambda_Y.$$

By the definition of $\Lambda_X \otimes \Lambda_Y$, we get $[(x, y)]_{\top} \otimes [(x, y)]_{\top} \in \Lambda_X \otimes \Lambda_Y$.

(TSUC2) Obviously.

(TSUC3) Take each $\mathbb{H} \in F_L^{\top}((X \times Y) \times (X \times Y))$ such that $\mathbb{H} \in \Lambda_X \otimes \Lambda_Y$. By the definition of $\Lambda_X \otimes \Lambda_Y$, we have

$$(p_X \times p_X)^{\Rightarrow}(\mathbb{H}) \in \Lambda_X$$

and

$$(p_Y \times p_Y)^{\Rightarrow}(\mathbb{H}) \in \Lambda_Y.$$

Then it follows from Proposition 3.10 that

$$(p_X \times p_X)^{\Rightarrow}(\mathbb{H}^{-1}) = ((p_X \times p_X)^{\Rightarrow}(\mathbb{H}))^{-1} \in \Lambda_X$$

and

$$(p_Y \times p_Y)^{\Rightarrow}(\mathbb{H}^{-1}) = ((p_Y \times p_Y)^{\Rightarrow}(\mathbb{H}))^{-1} \in \Lambda_Y.$$

This shows $\mathbb{H}^{-1} \in \Lambda_X \otimes \Lambda_Y$, as desired. □

By the above proposition, we can get an object map $\otimes : \top$-**SUConv** $\times$ $\top$-**SUConv**$\longrightarrow$$\top$-**SUConv** defined by for every two $\top$-semiuniform convergence spaces (X, Λ_X) and (Y, Λ_Y),

$$(X, \Lambda_X) \otimes (Y, \Lambda_Y) = (X \times Y, \Lambda_X \otimes \Lambda_Y).$$

We call $(X \times Y, \Lambda_X \otimes \Lambda_Y)$ the tensor product of (X, Λ_X) and (Y, Λ_Y) and $\otimes$ the tensor operation. Further, we will show the tensor operation $\otimes$ is a bifunctor.

Proposition 4.5. *The tensor operation $\otimes : \top$-**SUConv** $\times$ $\top$-**SUConv**$\longrightarrow$$\top$-**SUConv** is a bifunctor.*

Proof. By Proposition 4.4, we only need to show that for every two uniformly continuous maps $\varphi : (X, \Lambda_X) \longrightarrow (Y, \Lambda_Y)$ and $\psi : (U, \Lambda_U) \longrightarrow (V, \Lambda_V)$, the map

$$\varphi \otimes \psi := \varphi \times \psi : (X \times U, \Lambda_X \otimes \Lambda_U) \longrightarrow (Y \times V, \Lambda_Y \otimes \Lambda_V)$$

is uniformly continuous. Take each $\mathbb{H} \in F_L^{\top}((X \times U) \times (X \times U))$ such that

$$\mathbb{H} \in \Lambda_X \otimes \Lambda_U.$$

For clarity, let $p_X : X \times U \longrightarrow X$, $p_U : X \times U \longrightarrow U$, $q_Y : Y \times V \longrightarrow Y$ and $q_V : Y \times V \longrightarrow V$ denote the projection maps, respectively. Then

$$(\varphi \times \varphi) \circ (p_X \times p_X) = (q_Y \times q_Y) \circ ((\varphi \times \psi) \times (\varphi \times \psi))$$

and

$$(\psi \times \psi) \circ (p_U \times p_U) = (q_V \times q_V) \circ ((\varphi \times \psi) \times (\varphi \times \psi)).$$

By the definition of $\Lambda_X \otimes \Lambda_U$, we also have $(p_X \times p_X)^{\Rightarrow}(\mathbb{H}) \in \Lambda_X$ and $(p_U \times p_U)^{\Rightarrow}(\mathbb{H}) \in \Lambda_U$. Since φ and ψ are uniformly continuous, we get

$$\big((q_Y \times q_Y) \circ ((\varphi \times \psi) \times (\varphi \times \psi))\big)^{\Rightarrow}(\mathbb{H}) = ((\varphi \times \varphi) \circ (p_X \times p_X))^{\Rightarrow}(\mathbb{H}) \in \Lambda_Y$$

and

$$\big((q_V \times q_V) \circ ((\varphi \times \psi) \times (\varphi \times \psi))\big)^{\Rightarrow}(\mathbb{H}) = ((\psi \times \psi) \circ (p_U \times p_U))^{\Rightarrow}(\mathbb{H}) \in \Lambda_V.$$

This means

$$((\varphi \times \psi) \times (\varphi \times \psi))^{\Rightarrow}(\mathbb{H}) \in \Lambda_Y \otimes \Lambda_V.$$

Thus, $\varphi \otimes \psi =: \varphi \times \psi : (X \times U, \Lambda_X \otimes \Lambda_U) \longrightarrow (Y \times V, \Lambda_Y \otimes \Lambda_V)$ is uniformly continuous. □

By the bifunctor $\otimes$, we can construct a transformation

$$a : \otimes \circ (\otimes \times id_{\top\text{-}\mathbf{SUConv}}) \longrightarrow \otimes \circ (id_{\top\text{-}\mathbf{SUConv}} \times \otimes)$$

by for each $(X, \Lambda_X), (Y, \Lambda_Y), (Z, \Lambda_Z) \in |\top\text{-}\mathbf{SUConv}|$,

$$a_{XYZ} : \big((X \times Y) \times Z, (\Lambda_X \otimes \Lambda_Y) \otimes \Lambda_Z\big) \longrightarrow \big(X \times (Y \times Z), \Lambda_X \otimes (\Lambda_Y \otimes \Lambda_Z)\big).$$

Concretely, for each $((x, y), z) \in (X \times Y) \times Z$,

$$a_{XYZ}(((x, y), z)) = (x, (y, z)).$$

For preparations, we first present the following projection maps in advance.

$$p_{XY}^{(XY)Z} : (X \times Y) \times Z \longrightarrow X \times Y, \ \ p_Z^{(XY)Z} : (X \times Y) \times Z \longrightarrow Z,$$

$$p_X^{X(YZ)} : X \times (Y \times Z) \longrightarrow X, \ \ p_{YZ}^{X(YZ)} : X \times (Y \times Z) \longrightarrow Y \times Z,$$

$$p_X^{XY} : X \times Y \longrightarrow X, \ \ p_Y^{XY} : X \times Y \longrightarrow Y,$$

$$p_Y^{YZ} : Y \times Z \longrightarrow Y, \ \ p_Z^{YZ} : Y \times Z \longrightarrow Z.$$

Proposition 4.6. *The transformation*

$$a : \otimes \circ (\otimes \times id_{\top\text{-}\mathbf{SUConv}}) \longrightarrow \otimes \circ (id_{\top\text{-}\mathbf{SUConv}} \times \otimes)$$

is a natural isomorphism.

Proof. It is easy to check that the transformation a is natural. Here it remains to show that for each $(X, \Lambda_X), (Y, \Lambda_Y), (Z, \Lambda_Z)$ in $|\top\text{-}\mathbf{SUConv}|$, the map

$$a_{XYZ} : \big((X \times Y) \times Z, (\Lambda_X \otimes \Lambda_Y) \otimes \Lambda_Z\big) \longrightarrow \big(X \times (Y \times Z), \Lambda_X \otimes (\Lambda_Y \otimes \Lambda_Z)\big)$$

is an isomorphism. To this end, we need to show

(1) a_{XYZ} is a bijection.

(2) $a_{XYZ} : \big((X \times Y) \times Z, (\Lambda_X \otimes \Lambda_Y) \otimes \Lambda_Z\big) \longrightarrow \big(X \times (Y \times Z), \Lambda_X \otimes (\Lambda_Y \otimes \Lambda_Z)\big)$ is uniformly continuous.

(3) $a_{XYZ}^{-1} : \big(X \times (Y \times Z), \Lambda_X \otimes (\Lambda_Y \otimes \Lambda_Z)\big) \longrightarrow \big((X \times Y) \times Z, (\Lambda_X \otimes \Lambda_Y) \otimes \Lambda_Z\big)$ is uniformly continuous.

For (1), it is straightforward.

For (2), take each $\mathbb{H} \in F_L^\top\big(((X\times Y)\times Z)\times((X\times Y)\times Z)\big)$ such that $\mathbb{H}\in(\Lambda_X\otimes\Lambda_Y)\otimes\Lambda_Z$. By the definition of $(\Lambda_X\otimes\Lambda_Y)\otimes\Lambda_Z$ and $\Lambda_X\otimes\Lambda_Y$, we obtain

$$(p_{XY}^{(XY)Z}\times p_{XY}^{(XY)Z})^{\Rightarrow}(\mathbb{H})\in\Lambda_X\otimes\Lambda_Y$$

and

$$(p_Z^{(XY)Z}\times p_Z^{(XY)Z})^{\Rightarrow}(\mathbb{H})\in\Lambda_Z,$$

which is equivalent to

$$\big((p_X^{XY}\times p_X^{XY})\circ(p_{XY}^{(XY)Z}\times p_{XY}^{(XY)Z})\big)^{\Rightarrow}(\mathbb{H})\in\Lambda_X,$$

$$\big((p_Y^{XY}\times p_Y^{XY})\circ(p_{XY}^{(XY)Z}\times p_{XY}^{(XY)Z})\big)^{\Rightarrow}(\mathbb{H})\in\Lambda_Y$$

and

$$(p_Z^{(XY)Z}\times p_Z^{(XY)Z})^{\Rightarrow}(\mathbb{H})\in\Lambda_Z.$$

Since

$$(p_X^{XY}\times p_X^{XY)})\circ(p_{XY}^{(XY)Z}\times p_{XY}^{(XY)Z})=(p_X^{X(YZ)}\times p_X^{X(YZ)})\circ(a_{XYZ}\times a_{XYZ}),$$

$$(p_Y^{XY}\times p_Y^{XY})\circ(p_{XY}^{(XY)Z}\times p_{XY}^{(XY)Z})=(p_Y^{YZ}\times p_Y^{YZ})\circ(p_{YZ}^{X(YZ)}\times p_{YZ}^{X(YZ)})\circ(a_{XYZ}\times a_{XYZ})$$

and

$$p_Z^{(XY)Z}\times p_Z^{(XY)Z}=(p_Z^{YZ}\times p_Z^{YZ})\circ(p_{YZ}^{X(YZ)}\times p_{YZ}^{X(YZ)})\circ(a_{XYZ}\times a_{XYZ}),$$

we get

$$\big((p_X^{X(YZ)}\times p_X^{X(YZ)})\circ(a_{XYZ}\times a_{XYZ})\big)^{\Rightarrow}(\mathbb{H})\in\Lambda_X,$$

$$\big((p_Y^{YZ}\times p_Y^{YZ})\circ(p_{YZ}^{X(YZ)}\times p_{YZ}^{X(YZ)})\circ(a_{XYZ}\times a_{XYZ})\big)^{\Rightarrow}(\mathbb{H})\in\Lambda_Y$$

and

$$\big((p_Z^{YZ}\times p_Z^{YZ})\circ(p_{YZ}^{X(YZ)}\times p_{YZ}^{X(YZ)})\circ(a_{XYZ}\times a_{XYZ})\big)^{\Rightarrow}(\mathbb{H})\in\Lambda_Z,$$

By the definition of $\Lambda_X\otimes(\Lambda_Y\otimes\Lambda_Z)$, we have

$$(a_{XYZ}\times a_{XYZ})^{\Rightarrow}(\mathbb{H})\in\Lambda_X\otimes(\Lambda_Y\otimes\Lambda_Z).$$

This shows a_{XYZ} is uniformly continuous.

For (3), it is similar to (2). □

Proposition 4.7. *Let $p_X: X\times Y\longrightarrow X$ and $p_Y: X\times Y\longrightarrow Y$ be projection maps. Then $p_X:(X\times Y,\Lambda_X\otimes\Lambda_Y)\longrightarrow(X,\Lambda_X)$ and $p_Y:(X\times Y,\Lambda_X\otimes\Lambda_Y)\longrightarrow(Y,\Lambda_Y)$ are uniformly continuous.*

Proof. Take each $\mathbb{H}\in F_L^\top((X\times Y)\times(X\times Y))$ such that $\mathbb{H}\in\Lambda_X\otimes\Lambda_Y$. By the definition of $\Lambda_X\otimes\Lambda_Y$, we obtain

$$(p_X\times p_X)^{\Rightarrow}(\mathbb{H})\in\Lambda_X$$

and

$$(p_Y\times p_Y)^{\Rightarrow}(\mathbb{H})\in\Lambda_Y.$$

This shows that $p_X:(X\times Y,\Lambda_X\otimes\Lambda_Y)\longrightarrow(X,\Lambda_X)$ and $p_Y:(X\times Y,\Lambda_X\otimes\Lambda_Y)\longrightarrow(Y,\Lambda_Y)$ are uniformly continuous. □

By means of the unique $\top$-semiuniform convergence structure $\Lambda_{\{\infty\}}$ on the single point set $\{\infty\}$, we construct two transformations

$$l:(\{\infty\},\Lambda_{\{\infty\}})\otimes(-)\longrightarrow id_{\top\text{-}\mathbf{SUConv}}$$

and

$$r:(-)\otimes(\{\infty\},\Lambda_{\{\infty\}})\longrightarrow id_{\top\text{-}\mathbf{SUConv}}$$

by for each (X,Λ_X) in $|\top\text{-}\mathbf{SUConv}|$,

$$l_X:(\{\infty\},\Lambda_{\{\infty\}})\otimes(X,\Lambda_X)\longrightarrow(X,\Lambda_X),\ r_X:(X,\Lambda_X)\otimes(\{\infty\},\Lambda_{\{\infty\}})\longrightarrow(X,\Lambda_X).$$

Concretely, for each $x \in X$,

$$l_X((\infty, x)) = x,\ r_X((x, \infty)) = x.$$

Next we will show that transformations l and r are natural isomorphisms.

Proposition 4.8. *The transformations*

$$l : (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (-) \longrightarrow id_{\top\text{-}\mathbf{SUConv}}$$

and

$$r : (-) \otimes (\{\infty\}, \Lambda_{\infty}) \longrightarrow id_{\top\text{-}\mathbf{SUConv}}$$

are natural isomorphisms.

Proof. It is easy to verify that $l : (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (-) \longrightarrow id_{\top\text{-}\mathbf{SUConv}}$ and $r : (-) \otimes (\{\infty\}, \Lambda_{\{\infty\}}) \longrightarrow id_{\top\text{-}\mathbf{SUConv}}$ are natural transformations. Next it remains to show that $l_X : (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (X, \Lambda_X) \longrightarrow (X, \Lambda_X)$ and $r_X : (X, \Lambda_X) \otimes (\{\infty\}, \Lambda_{\{\infty\}}) \longrightarrow (X, \Lambda_X)$ are isomorphisms for each (X, Λ_X). Here we only show the case of $l_X : (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (X, \Lambda_X) \longrightarrow (X, \Lambda_X)$. To this end, we need to verify

(1) l_X is a bijection.

(2) $l_X : (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (X, \Lambda_X) \longrightarrow (X, \Lambda_X)$ is uniformly continuous.

(3) $l_X^{-1} : (X, \Lambda_X) \longrightarrow (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (X, \Lambda_X)$ is uniformly continuous.

For (1), define a map $l_X^{-1} : (X, \Lambda_X) \longrightarrow (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (X, \Lambda_X)$ by $l_X^{-1}(x) = (\infty, x)$ for $x \in X$. It is easy to see that $l_X^{-1} \circ l_X = id_{\{\infty\} \times X}$ and $l_X \circ l_X^{-1} = id_X$. This means l_X is a bijection and l_X^{-1} is the inverse map of l_X.

For (2), by Proposition 4.7, it is easy to see that $l_X : (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (X, \Lambda_X) \longrightarrow (X, \Lambda_X)$ is uniformly continuous.

For (3), take each $\mathbb{F} \in F_L^{\top}(X \times X)$ such that $\mathbb{F} \in \Lambda_X$. Let $p_{\{\infty\}} : \{\infty\} \times X \longrightarrow \{\infty\}$ and $p_X : \{\infty\} \times X \longrightarrow X$ denote the projection maps, respectively. Then it follows that $((p_X \times p_X) \circ (l_X^{-1} \times l_X^{-1}))^{\Rightarrow}(\mathbb{F}) = \mathbb{F}$. By Proposition 3.5, we also have

$$((p_{\{\infty\}} \times p_{\{\infty\}}) \circ (l_X^{-1} \times l_X^{-1}))^{\Rightarrow}(\mathbb{F}) = [\infty]_{\top} \otimes [\infty]_{\top}.$$

This implies

$$((p_X \times p_X) \circ (l_X^{-1} \times l_X^{-1}))^{\Rightarrow}(\mathbb{F}) \in \Lambda_X$$

and

$$((p_{\{\infty\}} \times p_{\{\infty\}}) \circ (l_X^{-1} \times l_X^{-1}))^{\Rightarrow}(\mathbb{F}) \in \Lambda_{\{\infty\}}.$$

By the definition of $\Lambda_{\{\infty\}} \otimes \Lambda_X$, we obtain

$$(l_X^{-1} \times l_X^{-1})^{\Rightarrow}(\mathbb{F}) \in \Lambda_{\{\infty\}} \otimes \Lambda_X,$$

which means $l_X^{-1} : (X, \Lambda_X) \longrightarrow (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (X, \Lambda_X)$ is uniformly continuous. This shows $l : (\{\infty\}, \Lambda_{\{\infty\}}) \otimes (-) \longrightarrow id_{\top\text{-}\mathbf{SUConv}}$ is a natural isomorphism.

In a similar way, we can show that $r : (-) \otimes (\{\infty\}, \Lambda_{\{\infty\}}) \longrightarrow id_{\top\text{-}\mathbf{SUConv}}$ is also a natural isomorphism, as desired. □

Next we construct another transformation $c : \otimes \longrightarrow \otimes$ defined by for each $(X, \Lambda_X), (Y, \Lambda_Y) \in |\top$-**SUConv**$|$,

$$c_{XY} : (X, \Lambda_X) \otimes (Y, \Lambda_Y) \longrightarrow (Y, \Lambda_Y) \otimes (X, \Lambda_X).$$

Concretely, for each $(x, y) \in X \times Y$,

$$c_{XY}((x, y)) = (y, x).$$

Proposition 4.9. *The transformation*

$$c : \otimes \longrightarrow \otimes$$

is a natural isomorphism.

Proof. It is easy to see that $c : \otimes \longrightarrow \otimes$ is a natural transformation. Then it remains to show that for each $(X, \Lambda_X), (Y, \Lambda_Y) \in |\mathsf{T}\text{-}\mathbf{SUConv}|$,

$$c_{XY} : (X, \Lambda_X) \otimes (Y, \Lambda_Y) \longrightarrow (Y, \Lambda_Y) \otimes (X, \Lambda_X)$$

is an isomorphism. For this, we need to show that

(1) c_{XY} is a bijection.

(2) $c_{XY} : (X, \Lambda_X) \otimes (Y, \Lambda_Y) \longrightarrow (Y, \Lambda_Y) \otimes (X, \Lambda_X)$ is uniformly continuous.

(3) $c_{XY}^{-1} : (Y, \Lambda_Y) \otimes (X, \Lambda_X) \longrightarrow (X, \Lambda_X) \otimes (Y, \Lambda_Y)$ is uniformly continuous.

For (1), it is straightforward.

For (2), take each $\mathbb{H} \in F_L^{\mathsf{T}}((X \times Y) \times (X \times Y))$ such that $\mathbb{H} \in \Lambda_X \otimes \Lambda_Y$. Let

$$p_X : X \times Y \longrightarrow X, \quad p_Y : X \times Y \longrightarrow Y,$$

$$q_X : Y \times X \longrightarrow X, \quad q_Y : Y \times X \longrightarrow Y$$

be the projection maps. Then

$$p_X \times p_X = (q_X \times q_X) \circ (c_{XY} \times c_{XY})$$

and

$$p_Y \times p_Y = (q_Y \times q_Y) \circ (c_{XY} \times c_{XY}).$$

By the definition of $\Lambda_X \otimes \Lambda_Y$, we have

$$((q_X \times q_X) \circ (c_{XY} \times c_{XY}))^{\Rightarrow}(\mathbb{H}) = (p_X \times p_X)^{\Rightarrow}(\mathbb{H}) \in \Lambda_X$$

and

$$((q_Y \times q_Y) \circ (c_{XY} \times c_{XY}))^{\Rightarrow}(\mathbb{H}) = (p_Y \times p_Y)^{\Rightarrow}(\mathbb{H}) \in \Lambda_Y.$$

This implies

$$(c_{XY} \times c_{XY})^{\Rightarrow}(\mathbb{H}) \in \Lambda_Y \otimes \Lambda_X,$$

which means $c_{XY} : (X, \Lambda_X) \otimes (Y, \Lambda_Y) \longrightarrow (Y, \Lambda_Y) \otimes (X, \Lambda_X)$ is uniformly continuous.

For (3), it is similar to (2). □

Theorem 4.10. *The category* T-**SUConv** *of* T*-semiuniform convergence spaces is a symmetric monoidal category with respect to the tensor operation* $\otimes$*.*

Proof. By Propositions 4.6, 4.8 and 4.9, we know for objects $(X, \Lambda_X), (Y, \Lambda_Y), (Z, \Lambda_Z)$ in T-**SUConv**, a_{XYZ}, l_X, r_X and c_{XY} are natural isomorphisms. Further, for each $((x, \infty), y) \in (X \times \{\infty\}) \times Y$, it follows that

$$(id_X \otimes l_Y) \circ a_{X\{\infty\}Y}((x, \infty), y) = id_X \otimes l_Y(x, (\infty, y)) = (x, y) = r_X \otimes id_Y((x, \infty), y),$$

which means the Diagram 2 is commutative. The commutativity of other Diagrams are obvious. Hence, we have shown the corresponding coherence axioms in Definitions 2.4 and 2.5 hold. That is to say, the category T-**SUConv** of T-semiuniform convergence spaces is a symmetric monoidal category. □

At the end of the section, we point out that for each object (X, Λ_X) in T-**SUConv**, we obtain a functor

$$\mathbf{F}_X : \mathsf{T}\text{-}\mathbf{SUConv} \longrightarrow \mathsf{T}\text{-}\mathbf{SUConv}$$

determined by (X, Λ_X) in the sense of

$$\mathbf{F}_X(Y, \Lambda_Y) = (Y, \Lambda_Y) \otimes (X, \Lambda_X)$$

for each object (Y, Λ_Y) in T-**SUConv**, and

$$\mathbf{F}_X(\varphi) = \varphi \times id_X : (Y, \Lambda_Y) \otimes (X, \Lambda_X) \longrightarrow (Z, \Lambda_Z) \otimes (X, \Lambda_X)$$

for each uniformly continuous map $\varphi : (Y, \Lambda_Y) \longrightarrow (Z, \Lambda_Z)$.

5. Monoidal closedness of $\top$-SUConv

In this section, we will show the monoidal closedness of the category of $\top$-semiuniform convergence spaces. Firstly, we introduce the following map.

Suppose that X and Y be two nonempty sets. Define $\eta_{XY} : (X \times X) \times (Y \times Y) \longrightarrow (X \times Y) \times (X \times Y)$ by for each $((x,x'),(y,y')) \in (X \times X) \times (Y \times Y)$,

$$\eta_{XY}((x,x'),(y,y')) = ((x,y),(x',y')).$$

Then we have the following proposition.

Proposition 5.1. *Let* $\mathbb{F} \in F_L^{\top}(X \times X), \mathbb{G} \in F_L^{\top}(Y \times Y), \mathbb{H} \in F_L^{\top}((X \times Y) \times (X \times Y)), H_1, H_2 \in L^{(X \times Y) \times (X \times Y)}$*. Then*

(1) $\mathbb{F} = ((p_X \times p_X) \circ \eta_{XY})^{\Rightarrow}(\mathbb{F} \otimes \mathbb{G})$ *and* $\mathbb{G} = ((p_Y \times p_Y) \circ \eta_{XY})^{\Rightarrow}(\mathbb{F} \otimes \mathbb{G})$*;*
(2) $H_1 * H_2 \leq \eta_{XY}^{\rightarrow}\big((p_X \times p_X)^{\rightarrow}(H_1) \otimes (p_Y \times p_Y)^{\rightarrow}(H_2)\big)$*;*
(3) $\eta_{XY}^{\Rightarrow}\big((p_X \times p_X)^{\Rightarrow}(\mathbb{H}) \otimes (p_Y \times p_Y)^{\Rightarrow}(\mathbb{H})\big) \subseteq \mathbb{H}$*;*
(4) $\eta_{XY}^{\Rightarrow}(\mathbb{F}^{-1} \otimes \mathbb{G}) = (\eta_{XY}^{\Rightarrow}(\mathbb{F} \otimes \mathbb{G}^{-1}))^{-1}$.

where $p_X : X \times Y \longrightarrow X$ *and* $p_Y : X \times Y \longrightarrow Y$ *are the projection maps.*

Proof. (1) Take each $A \in \mathbb{F}$, $B \in \mathbb{G}$ and $(x,x') \in X \times X$. Then

$$\begin{aligned}
&((p_X \times p_X) \circ \eta_{XY})^{\rightarrow}(A \otimes B)((x,x')) \\
=& \bigvee_{(p_X \times p_X) \circ \eta_{XY}((x,x'),(y,y'))=(x,x')} A \otimes B((x,x'),(y,y')) \\
=& \bigvee_{(y,y') \in Y \times Y} A((x,x')) * B((y,y')) \\
=& A((x,x')) * \bigvee_{(y,y') \in Y \times Y} B((y,y')) \\
=& A((x,x')) * \top \\
=& A((x,x')).
\end{aligned}$$

This shows $((p_X \times p_X) \circ \eta_{XY})^{\rightarrow}(A \otimes B) = A$. Then for each $D \in L^{X \times X}$, it follows that

$$\begin{aligned}
D \in \mathbb{F} &\Longleftrightarrow \bigvee_{A \in \mathbb{F}} \mathcal{S}_{X \times X}(A, D) = \top \\
&\Longleftrightarrow \bigvee_{A \in \mathbb{F}, B \in \mathbb{G}} \mathcal{S}_{X \times X}\Big(((p_X \times p_X) \circ \eta_{XY})^{\rightarrow}(A \otimes B), D\Big) \\
&\Longleftrightarrow D \in ((p_X \times p_X) \circ \eta_{XY})^{\Rightarrow}(\mathbb{F} \otimes \mathbb{G}), \quad \text{(by Propositions 3.4 and 3.8)}
\end{aligned}$$

which implies $\mathbb{F} = ((p_X \times p_X) \circ \eta_{XY})^{\Rightarrow}(\mathbb{F} \otimes \mathbb{G})$. In a similar way, we can get $\mathbb{G} = ((p_Y \times p_Y) \circ \eta_{XY})^{\Rightarrow}(\mathbb{F} \otimes \mathbb{G})$.

(2) Take each $((x,y),(x',y')) \in (X \times Y) \times (X \times Y)$. Then

$$\begin{aligned}
&\eta_{XY}^{\rightarrow}\big((p_X \times p_X)^{\rightarrow}(H_1) \otimes (p_Y \times p_Y)^{\rightarrow}(H_2)\big)((x,y),(x',y')) \\
=&(p_X \times p_X)^{\rightarrow}(H_1)(x,x') * (p_Y \times p_Y)^{\rightarrow}(H_2)(y,y') \\
=& \bigvee_{p_X \times p_X((x,z),(x',z'))=(x,x')} H_1((x,z),(x',z')) * \bigvee_{p_Y \times p_Y((w,y),(w',y'))=(y,y')} H_2((w,y),(w',y')) \\
\geq& H_1((x,y),(x',y')) * H_2((x,y),(x',y')) \\
=& H_1 * H_2((x,y),(x',y')),
\end{aligned}$$

which means $H_1 * H_2 \leq \eta_{XY}^{\rightarrow}\big((p_X \times p_X)^{\rightarrow}(H_1) \otimes (p_Y \times p_Y)^{\rightarrow}(H_2)\big)$.

(3) Take each $D \in \eta_{XY}^{\Rightarrow}\big((p_X \times p_X)^{\Rightarrow}(\mathbb{H}) \otimes (p_Y \times p_Y)^{\Rightarrow}(\mathbb{H})\big)$. By Propositions 3.4 and 3.8, we have

$$\begin{aligned}\top &= \bigvee_{A,B\in\mathbb{H}} \mathcal{S}_{(X\times Y)\times(X\times Y)}\Big(\eta_{XY}^{\rightarrow}\big((p_X \times p_X)^{\rightarrow}(A) \otimes (p_Y \times p_Y)^{\rightarrow}(B)\big), D\Big)\\ &\le \bigvee_{A,B\in\mathbb{H}} \mathcal{S}_{(X\times Y)\times(X\times Y)}(A * B, D) \qquad \text{(by (2))}\\ &\le \bigvee_{C\in\mathbb{H}} \mathcal{S}_{(X\times Y)\times(X\times Y)}(C, D),\end{aligned}$$

which means $D \in \mathbb{H}$. This shows $\eta_{XY}^{\Rightarrow}\big((p_X \times p_X)^{\Rightarrow}(\mathbb{H}) \otimes (p_Y \times p_Y)^{\Rightarrow}(\mathbb{H})\big) \subseteq \mathbb{H}$.

(4) Take each $D \in L^{(X\times Y)\times(X\times Y)}$. Then

$$\begin{aligned}& D \in \eta_{XY}^{\Rightarrow}(\mathbb{F}^{-1} \otimes \mathbb{G})\\ \Longleftrightarrow & \bigvee_{A\in\mathbb{F}^{-1},B\in\mathbb{G}} \mathcal{S}_{(X\times Y)\times(X\times Y)}(\eta_{XY}^{\rightarrow}(A \otimes B), D) = \top \quad \text{(by Propositons 3.4 and 3.8)}\\ \Longleftrightarrow & \bigvee_{A\in\mathbb{F}^{-1},B\in\mathbb{G}} \mathcal{S}_{(X\times Y)\times(X\times Y)}\big((\eta_{XY}^{\rightarrow}(A^{-1} \otimes B^{-1}))^{-1}, D\big) = \top\\ \Longleftrightarrow & \bigvee_{A\in\mathbb{F}^{-1},B\in\mathbb{G}} \mathcal{S}_{(X\times Y)\times(X\times Y)}(\eta_{XY}^{\rightarrow}(A^{-1} \otimes B^{-1}), D^{-1}) = \top\\ \Longleftrightarrow & \bigvee_{C\in\mathbb{F},E\in\mathbb{G}^{-1}} \mathcal{S}_{(X\times Y)\times(X\times Y)}(\eta_{XY}^{\rightarrow}(C \otimes E), D^{-1}) = \top\\ \Longleftrightarrow & D^{-1} \in \eta_{XY}^{\Rightarrow}(\mathbb{F} \otimes \mathbb{G}^{-1}) \qquad \text{(by Propositons 3.4 and 3.8)}\\ \Longleftrightarrow & D \in \big(\eta_{XY}^{\Rightarrow}(\mathbb{F} \otimes \mathbb{G}^{-1})\big)^{-1}.\end{aligned}$$

This shows $\eta_{XY}^{\Rightarrow}(\mathbb{F}^{-1} \otimes \mathbb{G}) = (\eta_{XY}^{\Rightarrow}(\mathbb{F} \otimes \mathbb{G}^{-1}))^{-1}$. □

For each $(X, \Lambda_X), (Y, \Lambda_Y) \in |\top$-**SUConv**$|$, we denote the set of morphisms from (X, Λ_X) to (Y, Λ_Y) by $[X, Y]$, i.e.,

$$[X, Y] := \{\varphi : (X, \Lambda_X) \longrightarrow (Y, \Lambda_Y) \mid \varphi \text{ is uniformly continuous}\}.$$

Then we define a set $\Lambda_{[X,Y]} \subseteq F_L^{\top}([X, Y] \times [X, Y])$ by

$$\Lambda_{[X,Y]} = \left\{\mathbb{H} \in F_L^{\top}([X, Y] \times [X, Y]) \,\middle|\, \begin{array}{l}\forall \mathbb{F} \in F_L^{\top}(X \times X), \mathbb{F} \in \Lambda_X \text{ implies}\\ ((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H} \otimes \mathbb{F}) \in \Lambda_Y\end{array}\right\},$$

where $ev_{XY} : [X, Y] \times X \longrightarrow Y$ is defined by

$$ev_{XY}((\varphi, x)) = \varphi(x)$$

for each $(\varphi, x) \in [X, Y] \times X$, and $\eta_{[X,Y]X} : ([X, Y] \times [X, Y]) \times (X \times X) \longrightarrow ([X, Y] \times X) \times ([X, Y] \times X)$ is defined by

$$\eta_{[X,Y]X}\big(((\varphi, \psi), (x, x'))\big) = \big((\varphi, x), (\psi, x')\big)$$

for each $((\varphi, \psi), (x, x')) \in ([X, Y] \times [X, Y]) \times (X \times X)$.

In order to show that the map $\Lambda_{[X,Y]}$ is a $\top$-semiuniform convergence structure on $[X, Y]$, the following lemmas are necessary.

Lemma 5.2. *Let $\varphi : X \longrightarrow Y$ be a map and $A \in L^{X\times X}$. Then*

$$(\varphi \times \varphi)^{\rightarrow}(A) = ((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\rightarrow}(\top_{\{(\varphi,\varphi)\}} \otimes A),$$

where $\top_{\{(\varphi,\varphi)\}} \in L^{[X,Y]\times[X,Y]}$ is defined by $\top_{\{(\varphi,\varphi)\}}(\psi, \chi) = \top$ if $(\psi, \chi) = (\varphi, \varphi)$ and $\top_{\{(\varphi,\varphi)\}}(\psi, \chi) = \bot$ otherwise.

Proof. Take each $(y,y') \in Y \times Y$. Then

$$\begin{aligned}
&((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\rightarrow}(\top_{\{(\varphi,\varphi)\}} \otimes A)(y,y') \\
=& \bigvee_{ev_{XY} \times ev_{XY}((\psi,x),(\chi,x'))=(y,y')} \eta^{\rightarrow}_{[X,Y]X}(\top_{\{(\varphi,\varphi)\}} \otimes A)((\psi,x),(\chi,x')) \\
=& \bigvee_{\psi(x)=y,\chi(x')=y'} \top_{\{(\varphi,\varphi)\}}(\psi,\chi) * A(x,x') \\
=& \bigvee_{\varphi(x)=y,\varphi(x')=y'} A(x,x') \\
=&(\varphi \times \varphi)^{\rightarrow}(A)(y,y'),
\end{aligned}$$

as desired. □

Lemma 5.3. *Let* $\varphi : X \longrightarrow Y$ *be a map and* $\mathbb{F} \in F^{\top}_L(X \times X)$*. Then*

$$(\varphi \times \varphi)^{\Rightarrow}(\mathbb{F}) \subseteq ((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}(([\varphi]_\top \otimes [\varphi]_\top) \otimes \mathbb{F}).$$

Proof. Take each $D \in (\varphi \times \varphi)^{\Rightarrow}(\mathbb{F})$. Then

$$\begin{aligned}
\top =& \bigvee_{A\in\mathbb{F}} \mathcal{S}_{Y\times Y}((\varphi \times \varphi)^{\rightarrow}(A), D) \\
=& \bigvee_{A\in\mathbb{F}} \mathcal{S}_{Y\times Y}\Big(((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\rightarrow}(\top_{\{(\varphi,\varphi)\}} \otimes A), D\Big) \quad \text{(by Lemma 5.2)} \\
\le& \bigvee_{E\in[\varphi]_\top\otimes[\varphi]_\top, A\in\mathbb{F}} \mathcal{S}_{Y\times Y}\Big(((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\rightarrow}(E \otimes A), D\Big).
\end{aligned}$$

This implies $D \in ((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}(([\varphi]_\top \otimes [\varphi]_\top) \otimes \mathbb{F})$, as desired. □

Proposition 5.4. *Let* $(X,\Lambda_X),(Y,\Lambda_Y) \in |\top$-**SUConv**$|$. *Then* $\Lambda_{[X,Y]}$ *defined above is a* $\top$*-semiuniform convergence structure on* $[X,Y]$.

Proof. It suffices to show that $\Lambda_{[X,Y]}$ satisfies (TSUC1)–(TSUC3). Since (TSUC2) is obvious, it remains to show (TSUC1) and (TSUC3).

(TSUC1) Take each $\varphi \in [X,Y]$ and $\mathbb{F} \in \Lambda_X$. By the uniform continuity of φ, we have $(\varphi \times \varphi)^{\Rightarrow}(\mathbb{F}) \in \Lambda_Y$. Then it follows from Lemma 5.3 that

$$\Big((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X}\Big)^{\Rightarrow}\Big(([\varphi]_\top \otimes [\varphi]_\top) \otimes \mathbb{F}\Big) \in \Lambda_Y.$$

This shows $[\varphi]_\top \otimes [\varphi]_\top \in \Lambda_{[X,Y]}$.

(TSUC3) Suppose that $\mathbb{H} \in \Lambda_{[X,Y]}$. Then take each $\mathbb{F} \in F^{\top}_L(X \times X)$ such that $\mathbb{F} \in \Lambda_X$. It follows that $\mathbb{F}^{-1} \in \Lambda_X$. By the definition of $\Lambda_{[X,Y]}$, we have

$$((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H} \otimes \mathbb{F}^{-1}) \in \Lambda_Y.$$

This implies

$$\Big(((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H} \otimes \mathbb{F}^{-1})\Big)^{-1} \in \Lambda_Y.$$

Next we show

$$\Big(((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H} \otimes \mathbb{F}^{-1})\Big)^{-1} \subseteq (ev_{XY} \times ev_{XY})^{\Rightarrow}\Big((\eta^{\Rightarrow}_{[X,Y]X}(\mathbb{H} \otimes \mathbb{F}^{-1}))^{-1}\Big).$$

Take each $D \in \Big(((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H} \otimes \mathbb{F}^{-1})\Big)^{-1}$, i.e., $D^{-1} \in ((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H} \otimes \mathbb{F}^{-1})$. By Propositions 3.4 and 3.8, we have

$$\bigvee_{B\in\mathbb{H}, A\in\mathbb{F}} \mathcal{S}_{Y\times Y}\Big(((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\rightarrow}(B \otimes A^{-1}), D^{-1}\Big) = \top,$$

which is equivalent to

$$\bigvee_{B\in\mathbb{H}, A\in\mathbb{F}} \mathcal{S}_{Y\times Y}\Big(\big(((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\rightarrow}(B \otimes A^{-1})\big)^{-1}, D\Big) = \top.$$

Further, it follows from

$$\Big(\big((ev_{XY}\times ev_{XY})\circ\eta_{[X,Y]X}\big)^{\rightarrow}(B\otimes A^{-1})\Big)^{-1}=(ev_{XY}\times ev_{XY})^{\rightarrow}\Big((\eta^{\rightarrow}_{[X,Y]X}(B\otimes A^{-1}))^{-1}\Big)$$

that

$$\bigvee_{B\in\mathbb{H},A\in\mathbb{F}}\mathcal{S}_{Y\times Y}\Big((ev_{XY}\times ev_{XY})^{\rightarrow}\big((\eta^{\rightarrow}_{[X,Y]X}(B\otimes A^{-1}))^{-1}\big),D\Big)=\top.$$

This implies $D\in(ev_{XY}\times ev_{XY})^{\Rightarrow}\big((\eta^{\Rightarrow}_{[X,Y]X}(\mathbb{H}\otimes\mathbb{F}^{-1}))^{-1}\big)$. By Proposition 5.1 (4), we have

$$(ev_{XY}\times ev_{XY})^{\Rightarrow}\big((\eta^{\Rightarrow}_{[X,Y]X}(\mathbb{H}\otimes\mathbb{F}^{-1}))^{-1}\big)=\big((ev_{XY}\times ev_{XY})\circ\eta_{[X,Y]X}\big)^{\Rightarrow}(\mathbb{H}^{-1}\otimes\mathbb{F}).$$

Thus we get $((ev_{XY}\times ev_{XY})\circ\eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H}^{-1}\otimes\mathbb{F})\in\Lambda_Y$, which means $\mathbb{H}^{-1}\in\Lambda_{[X,Y]}$, as desired. □

For a fixed object $(X,\Lambda_X)\in|\top\text{-}\mathbf{SUConv}|$, we can define an object map $\mathbf{G}_X:\top\text{-}\mathbf{SUConv}\longrightarrow\top\text{-}\mathbf{SUConv}$, which is ensured by the above proposition. Concretely,

$$\forall(Y,\Lambda_Y)\in|\top\text{-}\mathbf{SUConv}|,\ \mathbf{G}_X(Y,\Lambda_Y)=([X,Y],\Lambda_{[X,Y]}).$$

In order to show $\mathbf{G}_X$ is a functor, we define a morphism map $\mathbf{G}_X$ from $\top$-**SUConv** to itself by for each $\Phi\in[Y,Z]$,

$$\mathbf{G}_X(\Phi):([X,Y],\Lambda_{[X,Y]})\longrightarrow([X,Z],\Lambda_{[X,Z]}),$$

and for each $\varphi\in[X,Y]$,

$$\mathbf{G}_X(\Phi)(\varphi)=\Phi\circ\varphi:(X,\Lambda_X)\longrightarrow(Z,\Lambda_Z).$$

It is easy to see that $\mathbf{G}_X(\Phi)$ is well-defined. In order to show that $\mathbf{G}_X$ is a functor from $\top$-**SUConv** to itself, we need to show the following proposition.

Proposition 5.5. *Let $\Phi:(Y,\Lambda_Y)\longrightarrow(Z,\Lambda_Z)$ be a uniformly continuous map. Then $\mathbf{G}_X(\Phi)$ defined above is uniformly continuous.*

Proof. Take each $\mathbb{H}\in F_L^{\top}([X,Y]\times[X,Y])$ such that $\mathbb{H}\in\Lambda_{[X,Y]}$. Then for each $\mathbb{F}\in\Lambda_X$, it follows that

$$((ev_{XY}\times ev_{XY})\circ\eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H}\otimes\mathbb{F})\in\Lambda_Y.$$

By the uniform continuity of $\Phi:(Y,\Lambda_Y)\longrightarrow(Z,\Lambda_Z)$, we obtain

$$((\Phi\times\Phi)\circ(ev_{XY}\times ev_{XY})\circ\eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H}\otimes\mathbb{F})\in\Lambda_Z.$$

Since

$$\begin{aligned}&((\Phi\times\Phi)\circ(ev_{XY}\times ev_{XY})\circ\eta_{[X,Y]X})\\=&(ev_{XZ}\times ev_{XZ})\circ\eta_{[X,Z]X}\circ\big((\mathbf{G}_X(\Phi)\times\mathbf{G}_X(\Phi))\times(id_X\times id_X)\big),\end{aligned}$$

we have

$$\begin{aligned}&((\Phi\times\Phi)\circ(ev_{XY}\times ev_{XY})\circ\eta_{[X,Y]X})^{\Rightarrow}(\mathbb{H}\otimes\mathbb{F})\\=&\Big((ev_{XZ}\times ev_{XZ})\circ\eta_{[X,Z]X}\circ\big((\mathbf{G}_X(\Phi)\times\mathbf{G}_X(\Phi))\times(id_X\times id_X)\big)\Big)^{\Rightarrow}(\mathbb{H}\otimes\mathbb{F})\\=&((ev_{XZ}\times ev_{XZ})\circ\eta_{[X,Z]X})^{\Rightarrow}\big(((\mathbf{G}_X(\Phi)\times\mathbf{G}_X(\Phi))^{\Rightarrow}(\mathbb{H}))\otimes\mathbb{F}\big).\end{aligned}$$

This implies

$$((ev_{XZ}\times ev_{XZ})\circ\eta_{[X,Z]X})^{\Rightarrow}\big(((\mathbf{G}_X(\Phi)\times\mathbf{G}_X(\Phi))^{\Rightarrow}(\mathbb{H}))\otimes\mathbb{F}\big)\in\Lambda_Z.$$

By the definition of $\Lambda_{[X,Z]}$, we get

$$(\mathbf{G}_X(\Phi)\times\mathbf{G}_X(\Phi))^{\Rightarrow}(\mathbb{H})\in\Lambda_{[X,Z]},$$

as desired. □

By Propositions 5.4 and 5.5, we know that for a fixed object (X, Λ_X) in $\top$-**SUConv**, $\mathbf{G}_X$ defined above is a functor from $\top$-**SUConv** to itself.

Recall another functor $\mathbf{F}_X$ defined at the end of the above section, we realize that they are both functors from $\top$-**SUConv** to itself. In the sequel, we will aim to show that $\mathbf{F}_X$ is a left adjoint of $\mathbf{G}_X$. To this end, we need to find a natural bijective map

$$\omega := \omega_{ZY} : [\mathbf{F}_X(Z, \Lambda_Z), (Y, \Lambda_Y)] \longrightarrow [(Z, \Lambda_Z), \mathbf{G}_X(Y, \Lambda_Y)]$$

for each $(X, \Lambda_X), (Y, \Lambda_Y)$ and (Z, Λ_Z) in $|\top$-**SUConv**$|$. For each map $\varphi : Z \times X \longrightarrow Y$, we define the map $\omega_{ZY}(\varphi) : Z \longrightarrow Y^X$ by $\omega_{ZY}(\varphi)(z)(x) = \varphi(z, x)$ for each $z \in Z$ and $x \in X$, where Y^X is the set of maps from X to Y. In order to show ω_{ZY} is the desired map, the following propositions are necessary.

Proposition 5.6. *If* $\varphi : (Z \times X, \Lambda_Z \otimes \Lambda_X) \longrightarrow (Y, \Lambda_Y)$ *is uniformly continuous, then so is* $\omega_{ZY}(\varphi)(z) : (X, \Lambda_X) \longrightarrow (Y, \Lambda_Y)$ *for each* $z \in Z$.

Proof. Take each $\mathbb{F} \in F_L^\top(X \times X)$ such that $\mathbb{F} \in \Lambda_X$. By Proposition 5.1 (1), we have

$$\mathbb{F} \subseteq ((p_X \times p_X) \circ \eta_{ZX})^{\Rightarrow}(([z]_\top \otimes [z]_\top) \otimes \mathbb{F})$$

and

$$[z]_\top \otimes [z]_\top \subseteq ((p_Z \times p_Z) \circ \eta_{ZX})^{\Rightarrow}(([z]_\top \otimes [z]_\top) \otimes \mathbb{F}),$$

where $p_X : Z \times X \longrightarrow X$ and $p_Z : Z \times X \longrightarrow Z$ are projection maps. This implies

$$((p_X \times p_X) \circ \eta_{ZX})^{\Rightarrow}(([z]_\top \otimes [z]_\top) \otimes \mathbb{F}) \in \Lambda_X$$

and

$$((p_Z \times p_Z) \circ \eta_{ZX})^{\Rightarrow}(([z]_\top \otimes [z]_\top) \otimes \mathbb{F}) \in \Lambda_Z.$$

By the definition of $\Lambda_Z \otimes \Lambda_X$, we get

$$\eta_{ZX}^{\Rightarrow}(([z]_\top \otimes [z]_\top) \otimes \mathbb{F}) \in \Lambda_Z \otimes \Lambda_X.$$

Since $\varphi : (Z \times X, \Lambda_Z \otimes \Lambda_X) \longrightarrow (Y, \Lambda_Y)$ is uniformly continuous, it follows that

$$((\varphi \times \varphi) \circ \eta_{ZX})^{\Rightarrow}(([z]_\top \otimes [z]_\top) \otimes \mathbb{F}) \in \Lambda_Y.$$

Take each $D \in L^{Y \times Y}$. Then

$$\begin{aligned}
& D \in ((\varphi \times \varphi) \circ \eta_{ZX})^{\Rightarrow}(([z]_\top \otimes [z]_\top) \otimes \mathbb{F}) \\
\Longrightarrow & \bigvee_{A \in \mathbb{F}} \mathcal{S}_{Y \times Y}\Big((\omega_{ZY}(\varphi)(z) \times \omega_{ZY}(\varphi)(z))^{\rightarrow}(A), D\Big) \\
& = \bigvee_{A \in \mathbb{F}} \mathcal{S}_{Y \times Y}\Big(((\varphi \times \varphi) \circ \eta_{ZX})^{\rightarrow}\big((\top_{\{z\}} \otimes \top_{\{z\}}) \otimes A\big), D\Big) = \top \\
& \qquad (\{\top_{\{z\}} \otimes \top_{\{z\}}\} \text{ is a } \top\text{-filter base of } [z]_\top \otimes [z]_\top) \\
\Longrightarrow & D \in \big(\omega_{ZY}(\varphi)(z) \times \omega_{ZY}(\varphi)(z)\big)^{\Rightarrow}(\mathbb{F}),
\end{aligned}$$

which means

$$((\varphi \times \varphi) \circ \eta_{ZX})^{\Rightarrow}(([z]_\top \otimes [z]_\top) \otimes \mathbb{F}) \subseteq (\omega_{ZY}(\varphi)(z) \times \omega_{ZY}(\varphi)(z))^{\Rightarrow}(\mathbb{F}).$$

This implies

$$(\omega_{ZY}(\varphi)(z) \times \omega_{ZY}(\varphi)(z))^{\Rightarrow}(\mathbb{F}) \in \Lambda_Y,$$

as desired. □

Proposition 5.7. *If* $\varphi : (Z \times X, \Lambda_Z \otimes \Lambda_X) \longrightarrow (Y, \Lambda_Y)$ *is uniformly continuous, then so is* $\omega_{ZY}(\varphi) : (Z, \Lambda_Z) \longrightarrow ([X, Y], \Lambda_{[X,Y]})$.

Proof. Take each $\mathbb{K} \in F_L^\top(Z \times Z)$ such that $\mathbb{K} \in \Lambda_Z$. Then for each $\mathbb{F} \in \Lambda_X$, it follows from Proposition 5.1 (1) that

$$((p_Z \times p_Z) \circ \eta_{ZX})^{\Rightarrow}(\mathbb{K} \otimes \mathbb{F}) = \mathbb{K} \in \Lambda_Z$$

and

$$((p_X \times p_X) \circ \eta_{ZX})^{\Rightarrow}(\mathbb{K} \otimes \mathbb{F}) = \mathbb{F} \in \Lambda_X,$$

where $p_Z : Z \times X \longrightarrow Z$ and $p_X : Z \times X \longrightarrow X$ are projection maps. This implies

$$\eta_{ZX}^{\Rightarrow}(\mathbb{K} \otimes \mathbb{F}) \in \Lambda_Z \otimes \Lambda_X.$$

By the uniform continuity of $\varphi : (Z \times X, \Lambda_Z \otimes \Lambda_X) \longrightarrow (Y, \Lambda_Y)$, we have

$$((\varphi \times \varphi) \circ \eta_{ZX})^{\Rightarrow}(\mathbb{K} \otimes \mathbb{F}) \in \Lambda_Y.$$

Since

$$(\varphi \times \varphi) \circ \eta_{ZX} = (ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X} \circ \big((\omega_{ZY}(\varphi) \times \omega_{ZY}(\varphi)) \times (id_X \times id_X)\big),$$

it follows from Proposition 3.10 that

$$\begin{aligned}
&((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}\big(((\omega_{ZY}(\varphi) \times \omega_{ZY}(\varphi))^{\Rightarrow}(\mathbb{K})) \otimes \mathbb{F}\big) \\
=&\big((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X} \circ ((\omega_{ZY}(\varphi) \times \omega_{ZY}(\varphi)) \times (id_X \times id_X))\big)^{\Rightarrow}(\mathbb{K} \otimes \mathbb{F}) \\
=&((\varphi \times \varphi) \circ \eta_{ZX})^{\Rightarrow}(\mathbb{K} \otimes \mathbb{F}) \in \Lambda_Y.
\end{aligned}$$

This implies

$$(\omega_{ZY}(\varphi) \times \omega_{ZY}(\varphi))^{\Rightarrow}(\mathbb{K}) \in \Lambda_{[X,Y]},$$

This shows that $\omega_{ZY}(\varphi) : (Z, \Lambda_Z) \longrightarrow ([X,Y], \Lambda_{[X,Y]})$ is uniformly continuous, as desired. □

Proposition 5.8. *Let (X, Λ_X) be an object in* $\top$-**SUConv**. *Then for each object (Y, Λ_Y), the map*

$$ev_{XY} : ([X,Y] \times X, \Lambda_{[X,Y]} \otimes \Lambda_X) \longrightarrow (Y, \Lambda_Y)$$

is uniformly continuous.

Proof. Take each $\mathbb{H} \in F_L^\top(([X,Y] \times X) \times ([X,Y] \times X))$ such that $\mathbb{H} \in \Lambda_{[X,Y]} \otimes \Lambda_X$. By the definition of $\Lambda_{[X,Y]} \otimes \Lambda_X$, we have

$$(p_{[X,Y]} \times p_{[X,Y]})^{\Rightarrow}(\mathbb{H}) \in \Lambda_{[X,Y]}$$

and

$$(p_X \times p_X)^{\Rightarrow}(\mathbb{H}) \in \Lambda_X,$$

where $p_{[X,Y]} : [X,Y] \times X \longrightarrow [X,Y]$ and $p_X : [X,Y] \times X \longrightarrow X$ are the projection maps. Further it follows from the definition of $\Lambda_{[X,Y]}$ that

$$((ev_{XY} \times ev_{XY}) \circ \eta_{[X,Y]X})^{\Rightarrow}\big((p_{[X,Y]} \times p_{[X,Y]})^{\Rightarrow}(\mathbb{H}) \otimes (p_X \times p_X)^{\Rightarrow}(\mathbb{H})\big) \in \Lambda_Y.$$

By Proposition 5.1 (3), we obtain

$$(ev_{XY} \times ev_{XY})^{\Rightarrow}(\mathbb{H}) \in \Lambda_Y,$$

as desired. □

Theorem 5.9. *The category* $\top$-**SUConv** *of* $\top$-*semiuniform convergence spaces is monoidal closed.*

Proof. By Theorem 4.10, we have shown that $\top$-**SUConv** is a symmetric monoidal category. Then by Definition 2.7, it suffices to show that the functor $\mathbf{F}_X$ has a right adjoint $\mathbf{G}_X$ for each object (X, Λ_X) in $\top$-**SUConv**. By Definition 2.6, we only need to show that for each $(Z, \Lambda_Z), (Y, \Lambda_Y)$ in $\top$-**SUConv**, there exists a natural bijection from $[\mathbf{F}_X(Z, \Lambda_Z), (Y, \Lambda_Y)]$ to $[(Z, \Lambda_Z), \mathbf{G}_X(Y, \Lambda_Y)]$. By Propositions 5.6 and 5.7, we know that the map

$$\omega_{ZY} : [\mathbf{F}_X(Z, \Lambda_Z), (Y, \Lambda_Y)] \longrightarrow [(Z, \Lambda_Z), \mathbf{G}_X(Y, \Lambda_Y)],$$

defined above is well-defined. Actually, it is easy to see that the map is natural and injective. Now it remains to show that ω_{ZY} is surjective. Take each $\psi \in [(Z, \Lambda_Z), \mathbf{G}_X(Y, \Lambda_Y)]$. That is to say, $\psi : (Z, \Lambda_Z) \longrightarrow ([X, Y], \Lambda_{[X,Y]})$ is uniformly continuous. By Proposition 4.5, we know that $\psi \times id_X : (Z \times X, \Lambda_Z \otimes \Lambda_X) \longrightarrow ([X, Y] \times X, \Lambda_{[X,Y]} \otimes \Lambda_X)$ is uniformly continuous. Since compositions of uniformly continuous maps are still uniformly continuous maps, it follows from Proposition 5.8 that the composite map

$$ev_{XY} \circ (\psi \times id_X) : (Z \times X, \Lambda_Z \otimes \Lambda_X) \longrightarrow (Y, \Lambda_Y),$$

is also uniformly continuous. This means $ev_{XY} \circ (\psi \times id_X) \in [\mathbf{F}_X(Z, \Lambda_Z), (Y, \Lambda_Y)]$. Since

$$\omega_{ZY}(ev_{XY} \circ (\psi \times id_X)) = \psi,$$

we obtain ω_{ZY} is surjective, as desired. □

6. Conclusions

Based on a complete residuated lattice, we introduced a new type of $\top$-filters and then used it to define $\top$-semiuniform convergence spaces. In a categorical approach, we showed that the resulting category is a monoidal closed and topological category. Following the contents, we will consider the following problems in the future:

(1) In the classical case, semiuniform limit spaces, uniform limit spaces and principal uniform limit spaces are also important types of semiuniform convergence spaces. By this motivation, we will consider $\top$-semiuniform limit spaces, $\top$-uniform limit spaces and principal $\top$-uniform limit spaces as "$\top$-filter" counterparts of semiuniform limit spaces, uniform limit spaces and principal uniform limit spaces and discuss their relationships from a categorical aspect.

(2) Following (1), we will investigate the monoidal closedness of the categories of $\top$-semiuniform limit spaces, $\top$-uniform limit spaces and principal $\top$-uniform limit spaces. We can adopt a straightforward approach to show how these categories are monoid closed. Also, we can investigate their monoidal closedness by the internal categorical relationships between them.

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