Turkish Journal of Science & Technology
17(1), 99-108, 2022

Research Paper
https://doi.org/10.55525/tjst.1065429

Multiplicative Conformable Fractional Differential Equations

Sertaç GÖKTAŞ[1*]
[1] Department of Mathematics, Faculty of Science and Letters, Mersin University, Mersin, Turkey
[*1]srtcgoktas@gmail.com

(Geliş/Received: 30/01/2022; Kabul/Accepted: 26/02/2022)

Abstract: In this study, multiplicative conformable fractional differential equations are presented. Wronskian concept, linear dependence-independence concepts are defined on multiplicative conformable fractional calculus and some theorems and results are given among them. Finally, some examples are solved by giving some methods for finding general solutions of multiplicative conformable fractional differential equations.

Key words: Conformable fractional derivative, multiplicative conformable fractional derivative, non-Newtonian calculus, variation of parameters.

Çarpımsal Uyumlu Kesirli Diferansiyel Denklemler

Öz: Bu çalışmada, çarpımsal uyumlu kesirli diferansiyel denklemler sunulmuştur. Çarpımsal uyumlu kesirli analiz üzerinde Wronskian kavramı, lineer bağımlılık-bağımsızlık kavramları tanımlanarak bunlar arasında bazı teoremler ve sonuçlar verilmiştir. Son olarak, çarpımsal uyumlu kesirli diferansiyel denklemlerin genel çözümlerinin bulunması üzerine bazı metotlar verilerek bazı örnekler çözülmüştür.

Anahtar kelimeler: Uyumlu kesirli türev, çarpımsal uyumlu kesirli türev, non-Newtonian analiz, parametrelerin değişimi.

1. Introduction

In 1970's, non-Newtonian calculus with infinite sub-branches was firstly presented as an alternative to usual calculus in [1,2]. The sub-branches such as geometric, anageometric, bigeometric, quadratic and harmonic calculus, etc. can be given as examples. The geometric calculus, which is one of these, is also defined as multiplicative calculus by some authors [3-9]. In this calculus, changes of arguments and values of a function are measured by differences and ratios, respectively. On the other hand, they are measured by differences in the classical case.

Many events such as the levels of sound signals, the acidities of chemicals and the magnitudes of earthquakes change exponentially. For this reason, examining these problems in nature using multiplicative calculus offers great convenience and benefits. In the study of these physical properties, it would be more accurate to prefer the multiplicative differential equations. In numerous fields as biology, chaos theory, demography, earthquakes, engineering, economics, business and medicine [5,10-15], this calculus yields better outcomes than the classical case.

Fractional calculus, which is frequently encountered with various applications [16-20] in different fields of engineering and science, is defined as a generalization of classical calculus. We prefer the conformable fractional (CF) calculus in the present study. Because the other fractional derivatives used in the literature fail to satisfy some basic properties. Thus, it can be found basic properties and main results of CF calculus in [21,22]. Some applications of fractional derivatives are given in [23-27].

Multiplicative fractional calculus theory is a combination of both fractional calculus theory and multiplicative calculus theory. We refer to the paper [28] that encourages us and from which the main concepts of the multiplicative fractional calculus are set. Here, it has been defined conformable multiplicative fractional derivative and multiplicative fractional integral and has been studied some of their properties.

In [29-30], the constructs and methods on CF calculus guided us in the preparation of this study.

2. Preliminaries

In this section, some basic definitions and properties of CF calculus, the multiplicative calculus and the multiplicative CF calculus theories will be given.

[*] Corresponding author: srtcgoktas@gmail.com. ORCID Number of authors: [1] 0000-0001-7842-6309

Definition 2.1. [21,22] Consider the function $f:[a,\infty)\to\mathbb{R}$. Then, CF derivative and CF integral of f order $\alpha\in(0,1]$ are defined by:

$$T_\alpha^a f(x) := \lim_{h\to 0}\frac{f(x+h(x-a)^{1-\alpha})-f(x)}{h},$$

$$I_\alpha^a f(x) := \int_a^x f(t)d_\alpha(t,a) = \int_a^x (t-a)^{\alpha-1}f(t)dt, \qquad \text{for } x>0,$$

respectively. Here, the last integral to the right of this equality is the usual Riemann integral. Moreover, when $a=0$, the CF derivative be written T_α and the CF integral be written I_α and $d_\alpha(t,a)=d_\alpha t$. In addition, if f is usual differentiable, then $T_\alpha f(x)=x^{1-\alpha}f'(x)$.

Definition 2.2. [28] Consider the function $f:\mathbb{R}\to\mathbb{R}^+$. Then, the forward multiplicative and the backward multiplicative derivative of f are defined by:

$$\frac{d^*}{dx^*}f(x) = f^*(x) := \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}},$$

$$\frac{d_*}{dx_*}f(x) = f_*(x) := \lim_{h\to 0}\left(\frac{f(x)}{f(x-h)}\right)^{\frac{1}{h}},$$

respectively. It can easily be shown that

$$f^{*(n)}(x) = f_*^{(n)}(x) = \exp\left(\frac{d^n}{dx^n}\ln f(x)\right).$$

Definition 2.3. [28] Consider the function $f:[a,b]\to\mathbb{R}^+$. Then, the forward or the backward multiplicative integrals of $f(x)$ are defined by:

$$\int_a^b f(x)^{dx} = \int_a^b f(x)_{dx} = \exp\left(\int_a^b ln\, f(x)\, dx\right).$$

Definition 2.4. [28] Consider the function $f:[a,b]\to\mathbb{R}^+$. Then, the multiplicative CF derivative and the multiplicative CF integral of f order $\alpha\epsilon(0,1]$ are defined by:

$${}^*T_\alpha^a f(x) := \lim_{h\to 0}\left(\frac{f(x+h(x-a)^{1-\alpha})}{f(x)}\right)^{\frac{1}{h}}.$$

$$({}^*I_\alpha^a f)(x) := \int_a^x f(t)_{d_\alpha^*(t,a)} = \exp\left\{\int_a^x \ln f(t)d_\alpha(t,a)\right\}$$

$$= \int_a^x f(t)_{dt}^{(t-a)^{\alpha-1}} = \exp\left\{\int_a^x (t-a)^{\alpha-1}\ln f(t)dt\right\}, \qquad \text{for } x>0$$

respectively. Hence, the last integral to the right of this equality is the usual Riemann integral.
When $a=0$, the multiplicative CF derivative be written ${}^*T_\alpha$ and the multiplicative CF integral be written ${}^*I_\alpha$, and $d_\alpha^*(t,a)=d_\alpha^*t$.

Now, let $\alpha\epsilon(0,1]$ and $n\in\mathbb{Z}^+$. The sequential multiplicative CF derivatives of order n is defined by

$${}^{*(n)}T_\alpha f(x) = \underbrace{{}^*T_\alpha\, {}^*T_\alpha \ldots {}^*T_\alpha}_{n-times} f(x).$$

Proposition 2.1. [28] Consider the function $f:[0,b]\to\mathbb{R}^+$and $\alpha\epsilon(0,1]$. Then,

i) ${}^*T_\alpha f(x) = \exp\{T_\alpha \ln f(x)\} = \exp\left\{\frac{T_\alpha f(x)}{f(x)}\right\}$,

ii) $({}^*I_\alpha f)(x) = \exp\{I_\alpha \ln f(x)\}$.

Proposition 2.2.[28] Consider the function $f:[0,b]\to\mathbb{R}^+$ and $\alpha\epsilon(0,1]$. Then,

i) $({}^*T_\alpha\, {}^*I_\alpha f)(x) = f(x)$, for f is continuous,

$\boldsymbol{ii})\ ({}^{*}I_{\alpha}\,{}^{*}T_{\alpha}f)(x)=\dfrac{f(x)}{f(a)}.$

Theorem 2.1. [31] Let $f,g:[0,b]\to\mathbb{R}^{+}$ be multiplicative CF differentiable of order $\alpha\epsilon(0,1]$ and h be CF differentiable $\alpha\epsilon(0,1]$ at x. Then,

$\boldsymbol{i})\ \tau(cf)(x)=\tau f(x),$

$\boldsymbol{ii})\ \tau(fg)(x)=\tau f(x)\tau g(x),$

$\boldsymbol{iii})\ \tau\left(\dfrac{f}{g}\right)(x)=\dfrac{\tau f(x)}{\tau g(x)},$

$\boldsymbol{iv})\ \tau(f^{h})(x)=\{\tau f(x)\}^{h(x)}f(x)^{T_{\alpha}h(x)},$

$\boldsymbol{v})\ \tau(f\circ h)(x)=\left\{(\tau f)\big(h(x)\big)\right\}^{T_{\alpha}h(x)h(x)^{\alpha-1}},$

$\boldsymbol{vi})\ \tau(f+g)(x)=[\tau f(x)]^{\frac{f(x)}{f(x)+g(x)}}[\tau g(x)]^{\frac{g(x)}{f(x)+g(x)}},$

where c is a positive constant and $\tau y={}^{*}T_{\alpha}y$.

Theorem 2.2. [31] Let $f,g:[0,b]\to\mathbb{R}^{+}$ be multiplicative CF integrable of order $\alpha\epsilon(0,1]$ at x. Then,

$\boldsymbol{i})\ \displaystyle\int_{0}^{b}[f(x)]^{k}_{d_{\alpha}^{*}x}=\left[\int_{0}^{b}f(x)_{d_{\alpha}^{*}x}\right]^{k},$

$\boldsymbol{ii})\ \displaystyle\int_{0}^{b}[f(x)g(x)]_{d_{\alpha}^{*}x}=\int_{0}^{b}f(x)_{d_{\alpha}^{*}x}\int_{0}^{b}g(x)_{d_{\alpha}^{*}x},$

$\boldsymbol{iii})\ \displaystyle\int_{0}^{b}\left[\frac{f(x)}{g(x)}\right]_{d_{\alpha}^{*}x}=\frac{\int_{0}^{b}f(x)_{d_{\alpha}^{*}x}}{\int_{0}^{b}g(x)_{d_{\alpha}^{*}x}},$

$\boldsymbol{iv})\ \displaystyle\int_{0}^{b}f(x)_{d_{\alpha}^{*}x}=\int_{0}^{c}f(x)_{d_{\alpha}^{*}x}\int_{c}^{b}f(x)_{d_{\alpha}^{*}x},$

$\boldsymbol{v})\ \displaystyle\int_{0}^{b}[\tau f(x)]^{g(x)}_{d_{\alpha}^{*}x}=\frac{f(b)^{g(b)}}{f(0)^{g(0)}}\left\{\int_{0}^{b}f(x)^{T_{\alpha}g(x)}_{d_{\alpha}^{*}x}\right\}^{-1},$

where $k\in\mathbb{R}$ and $c\in[0,b]$ is a positive constant. The last formula is called $\alpha-{}^{*}$integration by parts.

3. Multiplicative Conformable Fractional Differential Equations

It is aimed to apply conformable fractional differential equations to multiplicative calculus with a method similar to the application of classical differential equations to multiplicative calculus such as in [10].

Definition 3.1. The multiplicative differential equation

$$(\tau^{n}y)^{a_{0}(x)}(\tau^{n-1}y)^{a_{1}(x)}\cdots(\tau y)^{a_{n-1}(x)}y^{a_{n}(x)}=b(x)\tag{3.1}$$

is defined as *multiplicative CF differential equation of n order*, where $b(x)$ is a positive valued function. Here, $\tau y={}^{*}T_{\alpha}y,\quad \tau^{2}y={}^{*(2)}T_{\alpha}y={}^{*}T_{\alpha}\,{}^{*}T_{\alpha}y,\quad \ldots\quad \tau^{n}y={}^{*(n)}T_{\alpha}y=\underbrace{{}^{*}T_{\alpha}\,{}^{*}T_{\alpha}\ldots{}^{*}T_{\alpha}}_{n-times}y.$

If the exponents $a_{0}(x)\neq 0,\ \ a_{k}(x),\ k=\overline{1,n}$ are constants, Eq.(3.1) is called *linear multiplicative CF differential equation with constant exponents*; otherwise *linear multiplicative CF differential equation with variable exponents*. Moreover, if $b(x)=1$, Eq.(3.1) is called *homogeneous multiplicative CF differential equation,* that is

$$(\tau^{n}y)^{a_{0}(x)}(\tau^{n-1}y)^{a_{1}(x)}\cdots(\tau y)^{a_{n-1}(x)}y^{a_{n}(x)}=1,\tag{3.2}$$

otherwise *nonhomogeneous multiplicative CF differential equation*.

Theorem 3.1. Let $y_h(x)$ be the general solution of Eq.(3.2) and $y_p(x)$ be any particular solution of Eq.(3.1). Then, $y_h(x)y_p(x)$ is a general solution of Eq.(3.1).

Proof. Let $y(x)$ be a solution of Eq.(3.1). Since $y_p(x)$ be any solution of Eq.(3.1), it must be shown that $\frac{y(x)}{y_p(x)}$ is also the solution of Eq.(3.2) to complete the proof. Indeed, considering the properties on theorem 2.1 of the multiplicative CF derivative, we obtain

$$\left(\tau^n\left(\frac{y}{y_p}\right)\right)^{a_0(x)}\left(\tau^{n-1}\left(\frac{y}{y_p}\right)\right)^{a_1(x)}\cdots\left(\tau\left(\frac{y}{y_p}\right)\right)^{a_{n-1}(x)} y^{a_n(x)} =$$

$$= \frac{(\tau^n y)^{a_0(x)}(\tau^{n-1}y)^{a_1(x)}\cdots(\tau y)^{a_{n-1}(x)}y^{a_n(x)}}{\left(\tau^n y_p\right)^{a_0(x)}\left(\tau^{n-1}y_p\right)^{a_1(x)}\cdots\left(\tau y_p\right)^{a_{n-1}(x)}{y_p}^{a_n(x)}} = \frac{b(x)}{b(x)} = 1.$$

Hereby, $y_h(x)$ being the general solution of Eq.(3.2) causes $\frac{y(x)}{y_p(x)}$ to be the general solution of Eq.(3.2) too. Consequently, we reach $y(x) = y_h(x)y_p(x)$. □

Theorem 3.2. Let the functions $y_1(x),\ y_2(x), \dots, y_m(x)$ be any solutions of Eq.(3.2) on an interval I. Then, the function $y(x) = y_1^{c_1}(x)y_2^{c_2}(x) \dots y_m^{c_m}(x)$ is also a solution of Eq.(3.2) for any real constants $c_k,\ k = \overline{1,m}$.

Proof. The theorem is easily proved if the properties on theorem 2.1 of the multiplicative CF derivative, we obtain are taken into account.□

Definition 3.2. Consider the positive functions $y_1(x),\ y_2(x), \dots, y_n(x)$ on an interval I. If $c_k,\ k = \overline{1,n}$ are scalars, then the *multiplicative linear combination* of the functions $y_1(x),\ y_2(x), \dots, y_n(x)$ is
$y_1^{c_1}(x)\, y_2^{c_2}(x) \dots y_n^{c_n}(x)$.

Definition 3.3. Consider the positive functions $y_1(x),\ y_2(x), \dots, y_n(x)$ on an interval I. If a sequence of the functions $y_1(x),\ y_2(x), \dots, y_n(x)$ is said to be *multiplicative linearly independent* if it is not *multiplicative linearly dependent*, that is, if the equation

$$y_1^{c_1}(x)\, y_2^{c_2}(x) \dots y_n^{c_n}(x) = 1, \quad \forall x \in I \tag{3.3}$$

can only be satisfied by $c_k = 0,\ k = \overline{1,n}$.

Definition 3.4. Consider the positive functions $y_1(x),\ y_2(x), \dots, y_n(x)$ which are $(n-1)$ −times multiplicative CF differentiable on an interval I. The determinant

$${}^*W_\alpha(y_1, y_2, \dots, y_n)(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \tau y_1 & \tau y_2 & \cdots & \tau y_n \\ \vdots & \vdots & \ddots & \cdots \\ \tau^{n-1}y_1 & \tau^{n-1}y_2 & \cdots & \tau^{n-1}y_n \end{vmatrix}^*$$

is called *multiplicative CF Wronskian* ($\alpha -{}^*$*Wronskianı*) of the functions $y_1(x),\ y_2(x), \dots, y_n(x)$, where $|\cdot|^*$ is multiplicative determinant and the abbreviation ${}^*W_\alpha(x)$ will be used instead of ${}^*W_\alpha(y_1, y_2, \dots, y_n)(x)$ [32]. For example, when $n = 2$,

$${}^*W_\alpha(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ \tau y_1 & \tau y_2 \end{vmatrix}^* = \frac{y_1^{\ln \tau y_2}}{y_2^{\ln \tau y_1}} \tag{3.4}$$

Theorem 3.3.($\alpha -{}^*$Abel Formula) Consider the continuous functions $a_k(x),\ k = \overline{0,n}$ and $a_0(x) \neq 0$ on an interval I. If the positive functions $y_1(x),\ y_2(x), \dots, y_n(x)$ are multiplicative linearly independent solutions of Eq.(3.2) on an interval I, then the formula

$$^{*}W_{\alpha}(x)=\{^{*}W_{\alpha}(x_0)\}^{^{*}I_{\alpha}^{x_0}\left(e^{-\frac{a_1(x)}{a_0(x)}}\right)}$$

holds for $\forall x\in I$, where $^{*}I_{\alpha}^{x_0}(\cdot)$ will symbolize the multiplicative CF integral from definition 2.4..

Proof. For the sake of clarity, the proof in the case $n=2$ will be given instead of the proof of the general case. Let y_1 and y_2 be the continuous functions on an interval I and be multiplicative linearly independent solutions of Eq.(3.2) for $\forall x\in I$. Then, we can write the following

$$(\tau^2 y_1)=(\tau y_1)^{-\frac{a_1(x)}{a_0(x)}}y_1^{-\frac{a_2(x)}{a_0(x)}} \quad \text{and} \quad (\tau^2 y_2)=(\tau y_2)^{-\frac{a_1(x)}{a_0(x)}}y_2^{-\frac{a_2(x)}{a_0(x)}}. \tag{3.5}$$

On the other hand, taking the multiplicative CF derivative of both sides of (3.4) with respect to x when $n=2$, we obtain

$$\tau\left(^{*}W_{\alpha}(y_1,y_2)\right)=\tau\left(\frac{\{\tau y_2\}^{\ln y_1}}{\{\tau y_1\}^{\ln y_2}}\right)=\frac{\{\tau^2 y_2\}^{\ln y_1}}{\{\tau^2 y_1\}^{\ln y_2}}. \tag{3.6}$$

Substituting Eqs. (3.5) into (3.6) yields to

$$\tau\left(^{*}W_{\alpha}(y_1,y_2)\right)\{^{*}W_{\alpha}(y_1,y_2)\}^{\frac{a_1(x)}{a_0(x)}}=1. \tag{3.7}$$

If the algebraic operations on the multiplicative calculus are taken into account in Eq.(3.7), this equation is written in the following form:

$$\tau\left(\ln{}^{*}W_{\alpha}(y_1,y_2)\right)=e^{-\frac{a_1(x)}{a_0(x)}}.$$

Taking the multiplicative CF integral of both sides of the last equality with respect to x, the proof is completed.□

Corollary 3.1. Consider the continuous functions $a_k(x)$, $k=\overline{0,n}$ and $a_0(x)\neq 0$ on an interval I. If the positive function $y_1,y_2,\dots,y_n$ are the solutions of Eq.(3.2) on an interval I, then for $\forall x\in I$ either $^{*}W_{\alpha}(y_1,y_2,\dots,y_n)(x)=1$ or $^{*}W_{\alpha}(y_1,y_2,\dots,y_n)(x)\neq 1$ is provided.

Proof. From the $\alpha-{}^{*}$Abel formula, for $\forall x\in I$, $^{*}I_{\alpha}^{x_0}\left(e^{-\frac{a_1(x)}{a_0(x)}}\right)\neq 0$ dir. If $^{*}W_{\alpha}(x_0)=1$ is provided at a point $x_0\in I$, then $^{*}W_{\alpha}(x)=1$ is provided for $\forall x\in I$; if $^{*}W_{\alpha}(x_0)\neq 1$ is provided at a point $x_0\in I$, then $^{*}W_{\alpha}(x)\neq 1$ is provided for $\forall x\in I$. This completes the proof. □

Theorem 3.4. Let the positive function $y_1,y_2,\dots,y_n$ be the solutions of Eq.(3.2) on an interval I. The functions $y_1,y_2,\dots,y_n$ are multiplicative linearly dependent solutions of Eq.(3.2) on an interval I if and only if $^{*}W_{\alpha}(y_1,y_2,\dots,y_n)(x)=1$.

Proof. For the sake of clarity, the proof in the case $n=2$ will be given instead of the proof of the general case. Suppose that the functions y_1 and y_2 are multiplicative linearly dependent solutions of Eq.(3.2) on an interval I. Then, there is a constant C such that $y_2(x)=y_1^C(x)$. From the properties of multiplicative CF derivative, we get $\tau y_2(x)=\{\tau y_1(x)\}^C$. If the constant C is eliminated in this last equalities, the equality

$$\{\tau y_2(x)\}^{\ln y_1(x)}\{\tau y_1(x)\}^{-\ln y_2(x)}=1 \tag{3.8}$$

is obtained, that is, $^{*}W_{\alpha}(y_1,y_2)(x)=1$.
On the other hand, Let $^{*}W_{\alpha}(y_1,y_2)(x)=1$, that is, let the equality (3.8) be provided. If $y_1(x)=1$ on an interval I, then the function y_2 becomes multiplicative linearly dependent on y_1. Now, let's assume $y_1(x)\neq 1$ on a subinterval of the interval I. Then, taking the power $\{\ln y_1(x)\}^{-2}$ of both sides of the equality (3.8) gives

$\tau\left(\ln\{y_2(x)\}^{\{\ln y_1(x)\}^{-1}}\right) = 1$. From here and the properties of multiplicative CF derivative, we get $\ln\{y_2(x)\}^{\{\ln y_1(x)\}^{-1}} = C$ or $y_2(x) = y_1^C(x)$. This completes the proof. □

Theorem 3.5. Let the functions $y_1, y_2, \dots, y_n$ be multiplicative linearly independent solutions of Eq.(3.2) on an interval I. Then, a $y(x)$ solution of Eq.(3.2) in the same interval, that is, the general solution, is in the form below:

$$y(x) = y_1^{c_1}(x)\, y_2^{c_2}(x) \dots y_n^{c_n}(x). \tag{3.9}$$

Proof. Let the functions $y(x)$ be a solution of Eq.(3.2) on an interval I. Since the functions $y_1^{c_1}(x)\, y_2^{c_2}(x) \dots y_n^{c_n}(x)$ ve $y(x)$ are solutions of Eq.(3.2) on an interval I, at some x_0 of the interval, the arbitrary constants $c_k,\ k = \overline{1,n}$ must to be found such that the following system is provided:

$$\begin{gathered} y_1^{c_1}(x_0)\, y_2^{c_2}(x_0) \dots y_n^{c_n}(x_0) = y(x_0), \\ \tau y_1^{c_1}(x_0)\, \tau y_2^{c_2}(x_0) \dots \tau y_n^{c_n}(x_0) = \tau y(x_0), \\ \vdots \\ \tau^{n-1} y_1^{c_1}(x_0)\, \tau^{n-1} y_2^{c_2}(x_0) \dots \tau^{n-1} y_n^{c_n}(x_0) = \tau^{n-1} y(x_0). \end{gathered}$$

In order for this system to be solvable according to the arbitrary constants, the coefficients matrix of this system must be ${}^*W_\alpha(y_1, y_2, \dots, y_n)(x_0) \neq 1$. Consequently, from theorem 3.3, it is obtained that ${}^*W_\alpha(y_1, y_2, \dots, y_n)(x) \neq 1$ for $\forall x \in I$. This completes the proof. □

4. Some Methods for General Solution of Multiplicative CF Differential Equations

In this section, for the sake of clarity, some methods for finding the general solution of the homogeneous multiplicative CF differential equation (3.2) for $n = 2$ will be given. When $n = 2$, let's rearrange Eq.(3.2) as follows:

$$(\tau^2 y)(\tau y)^{p(x)} y^{q(x)} = 1, \tag{4.1}$$

where $p(x) = \frac{a_1(x)}{a_0(x)}$, $q(x) = \frac{a_2(x)}{a_0(x)}$ and $a_0(x) \neq 0$

4.1. The Solution of Type $y_2(x) = \{y_1(x)\}^{\ln u(x)}$

For brevity's sake, let's use these abbreviations $y_1 = y_1(x)$, $y_2 = y_2(x)$ and $u = u(x)$. Suppose that the function $y_1 \neq 1$ which known and $y_2 = y_1^{\ln u}$ functions are solutions of Eq.(4.1), where $u(x)$ is an unknown positive function. Thus, the following system is obtained:

$$\tau y_2 = \{\tau y_1\}^{\ln u} y_1^{T_\alpha(\ln u)}$$

$$\tau^2 y_2 = \{\tau^2 y_1\}^{\ln u} \{\tau y_1\}^{2T_\alpha(\ln u)} y_1^{{}^{(2)}T_\alpha(\ln u)}.$$

After these equalities are written in Eq.(4.1), considering that $y_1(x)$ is a solution to Eq.(4.1), we get

$$\left\{\{\tau y_1\}^2 y_1^{p(x)}\right\}^{T_\alpha(\ln u)} y_1^{{}^{(2)}T_\alpha(\ln u)} = 1$$

or

$$\frac{{}^{(2)}T_\alpha(\ln u)}{T_\alpha(\ln u)} = -2\frac{T_\alpha(\ln y_1)}{\ln y_1} - p(x).$$

CF integrating both sides of last equality, we get

$$T_\alpha(\ln u) = \ln^{-2} y_1\, e^{-I_\alpha p(x)} \qquad \Rightarrow \qquad u = e^{I_\alpha\left(\ln^{-2} y_1 e^{-I_\alpha p(x)}\right)}, \tag{4.2}$$

or

$$\tau u = e^{\ln^{-2} y_1\ {}^*I_\alpha e^{-p(x)}} \qquad \Rightarrow \qquad u = {}^*I_\alpha\left(e^{\ln^{-2} y_1\, {}^*I_\alpha e^{-p(x)}}\right), \tag{4.3}$$

where $I_\alpha \cdot$ is CF integral from definition 2.1 and ${}^*I_\alpha \cdot$ is multiplicative CF integral from definition 2.4. Consequently, the general solution of Eq.(4.1) is as follows

$$y(x) = y_1^{c_1}(x)\{y_1(x)\}^{c_2 \ln u(x)}, \tag{4.4}$$

where $u(x)$ is defined by (4.2) or (4.3).

Example 4.1. Consider the following multiplicative CF equation for which $y_1(x) = e^x$ has a known solution:

$$(\tau^2 y)(\tau y)^{\frac{1}{2\sqrt{x}}} y^{-\frac{1}{x}} = 1, \tag{4.5}$$

where $\tau y = {}^*T_{\frac{1}{2}} y$ and $\tau^2 y = {}^*T_{\frac{1}{2}} {}^*T_{\frac{1}{2}} y$.

Then considering the formula (4.2), we have $u(x) = e^{-\frac{x^{-2}}{2}}$. Consequently, from (4.4), the general solution of Eq.(4.5) is as follows:

$$y(x) = e^{c_1 x + c_2 x^{-1}}.$$

4.2. The Solution When the Coefficients are Constant

In Eq.(4.1), let $p(x)$ and $q(x)$ be constants. So let's look for the solutions of Eq.(4.1) in type $y(x) = e^{e^{k\left(\frac{1}{\alpha}x^\alpha\right)}}$. If the first and second multiplicative CF derivatives of this function are taken, it is obtained as follows, respectively:

$$\tau y(x) = e^{k e^{k\left(\frac{1}{\alpha}x^\alpha\right)}} \text{ and } \tau^2 y(x) = e^{k^2 e^{k\left(\frac{1}{\alpha}x^\alpha\right)}}.$$

If these derivatives are substituted in (4.1), we have

$$e^{(k^2+pk+q)e^{k\left(\frac{1}{\alpha}x^\alpha\right)}} = 1,$$

or

$$k^2 + pk + q = 0. \tag{4.6}$$

The following three cases exist for the roots k_1 and k_2 of Eq.(4.6).

i) If $k_1 \neq k_2 \in \mathbb{R}$, the general solution of Eq.(4.1) is as follows

$$y(x) = e^{c_1 e^{k_1\left(\frac{1}{\alpha}x^\alpha\right)} + c_2 e^{k_2\left(\frac{1}{\alpha}x^\alpha\right)}} \quad \text{or} \quad y(x) = a^{e^{k_1\left(\frac{1}{\alpha}x^\alpha\right)}} b^{e^{k_2\left(\frac{1}{\alpha}x^\alpha\right)}} \quad (a = e^{c_1}, \ \ b = e^{c_2}). \tag{4.7}$$

ii) If $k_1 = k_2 \in \mathbb{R}$, the general solution of Eq.(4.1) is as follows

$$y(x) = e^{\left(c_1 + c_2\left(\frac{1}{\alpha}x^\alpha\right)\right) e^{k_1\left(\frac{1}{\alpha}x^\alpha\right)}} \quad \text{or} \quad y(x) = a^{e^{k_1\left(\frac{1}{\alpha}x^\alpha\right)}} b^{\left(\frac{1}{\alpha}x^\alpha\right) e^{k_1\left(\frac{1}{\alpha}x^\alpha\right)}} \quad (a = e^{c_1}, \ \ b = e^{c_2}). \tag{4.8}$$

iii) If $k_1 = \sigma - i\tau, \ k_2 = \sigma + i\tau$ the general solution of Eq.(4.1) is as follows

$$y(x) = e^{e^{\sigma\left(\frac{1}{\alpha}x^\alpha\right)}\left(c_1 \cos\tau\left(\frac{1}{\alpha}x^\alpha\right) + c_2 \sin\tau\left(\frac{1}{\alpha}x^\alpha\right)\right)}. \tag{4.9}$$

Example 4.2. Consider the following multiplicative CF equation

$$(\tau^2 y)(\tau y) y^{-3} = 1. \tag{4.10}$$

Then, from (4.7), the general solution of Eq.(4.10) is as follows:

$$y(x) = e^{c_1 e^{\left(\frac{1}{\alpha}x^\alpha\right)} + c_2 e^{-2\left(\frac{1}{\alpha}x^\alpha\right)}} \quad \text{or} \quad y(x) = a^{e^{\left(\frac{1}{\alpha}x^\alpha\right)}} b^{e^{-2\left(\frac{1}{\alpha}x^\alpha\right)}} \quad (a = e^{c_1}, \ \ b = e^{c_2}).$$

Example 4.3. Consider the following multiplicative CF equation

$$(\tau^2 y)(\tau y)^{-2} y^2 = 1. \tag{4.11}$$

Then, from (4.9), the general solution of Eq.(4.11) is as follows:

$$y(x) = e^{e^{\left(\frac{1}{\alpha}x^\alpha\right)}\left(c_1 \cos 2\left(\frac{1}{\alpha}x^\alpha\right) + c_2 \sin 2\left(\frac{1}{\alpha}x^\alpha\right)\right)}.$$

4.3. Variation of Parameters

Suppose that $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two multiplicative linearly independent solutions for Eq.(4.1). The aim is to find the particular solution $y_p = y_p(x)$ for

$$(\tau^2 y)(\tau y)^{p(x)} y^{q(x)} = b(x). \tag{4.12}$$

Let

$$y_p = y_1^{c_1} y_2^{c_2}, \tag{4.13}$$

where the functions $c_1 = c_1(x)$ and $c_2 = c_2(x)$ are two unknown functions.
By taking the multiplicative CF derivative of Eq.(4.13) with respect to x, we get

$$\tau y_p = (\tau y_1)^{c_1} y_1^{T_\alpha c_1} (\tau y_2)^{c_2} y_2^{T_\alpha c_2}. \tag{4.14}$$

If the equality

$$y_1^{T_\alpha c_1} y_2^{T_\alpha c_2} = 1 \tag{4.15}$$

is assumed in Eq.(4.14), we get

$$\tau y_p = (\tau y_1)^{c_1} (\tau y_2)^{c_2}. \tag{4.16}$$

Later, taking the multiplicative CF derivative of Eq.(4.16) with respect to x, we obtain

$$\tau^2 y_p = (\tau^2 y_1)^{c_1} (\tau y_1)^{T_\alpha c_1} (\tau^2 y_2)^{c_2} (\tau y_2)^{T_\alpha c_2}. \tag{4.17}$$

Considering that the functions y_1 and y_2 are the solution of Eq.(4.1), substituting Eqs.(4.13), (4.14), and (4.17) into (4.12) yields to

$$(\tau y_1)^{T_\alpha c_1} (\tau y_2)^{T_\alpha c_2} = b(x). \tag{4.18}$$

Now, solving the system of the Eqs.(4.15) and (4.18), we get

$$T_\alpha c_1 = \frac{-\ln b(x) \ln y_2(x)}{\ln\left({}^*W_\alpha(y_1, y_2)(x)\right)} \quad \text{and} \quad T_\alpha c_2 = \frac{\ln b(x) \ln y_1(x)}{\ln\left({}^*W_\alpha(y_1, y_2)(x)\right)},$$

or

$$c_1 = I_\alpha\left(\frac{-\ln b(x) \ln y_2(x)}{\ln\left({}^*W_\alpha(y_1, y_2)(x)\right)}\right) \quad \text{and} \quad c_2 = I_\alpha\left(\frac{\ln b(x) \ln y_1(x)}{\ln\left({}^*W_\alpha(y_1, y_2)(x)\right)}\right).$$

Consequently, the particular solution is as follows:

$$y_p = y_1^{I_\alpha\left(\frac{-\ln b(x) \ln y_2(x)}{\ln\left({}^*W_\alpha(y_1, y_2)(x)\right)}\right)} y_2^{I_\alpha\left(\frac{\ln b(x) \ln y_1(x)}{\ln\left({}^*W_\alpha(y_1, y_2)(x)\right)}\right)}. \tag{4.19}$$

Corollary 4.1. Let the functions $y_1, y_2, \dots, y_n$ be multiplicative linearly independent solutions of Eq.(3.2) on an interval I. If the above method is applied for these functions, it can be easily shown that

$$c_n = I_\alpha\left(\frac{\ln\left({}_n^*W_\alpha(y_1, y_2, \dots, y_n)(x)\right)}{\ln\left({}^*W_\alpha(y_1, y_2, \dots, y_n)(x)\right)}\right),$$

where ${}_n^*W_\alpha(y_1, y_2, \dots, y_n)(x)$ is the determinant obtained by replacing the *n*th column on ${}^*W_\alpha(y_1, y_2, \dots, y_n)(x)$ by the column $[0\ 0\ \cdots 0\ \ln b(x)]_{1\times n}$. Consequently, the particular solution is as follows:

$$y_p = \prod_{k=1}^{n} y_k^{I_\alpha\left(\frac{\ln\left({}_k^*W_\alpha(y_1, y_2, \dots, y_n)(x)\right)}{\ln\left({}^*W_\alpha(y_1, y_2, \dots, y_n)(x)\right)}\right)}.$$

Example 4.4. Consider the following multiplicative CF equation

$$(\tau^2 y)(\tau y)^6 y^9 = e^{e^{-6\sqrt{x}}}, \tag{4.20}$$

where $\tau y = {}^*T_{\frac{1}{2}} y$ and $\tau^2 y = {}^*T_{\frac{1}{2}} {}^*T_{\frac{1}{2}} y$.
Then, from (4.8), the homogeneous solution of Eq.(4.20) is as follows:

$$y_h(x) = e^{\left(c_1 + c_2(2\sqrt{x})\right)e^{-6\sqrt{x}}}.$$

Namely, $y_1(x) = e^{e^{-6\sqrt{x}}}$ and $y_2(x) = e^{2\sqrt{x}e^{-6\sqrt{x}}}$. Therefore,

$c_1 = I_\alpha(-2\sqrt{x}) = -2x$ and $c_2 = I_\alpha(1) = 2\sqrt{x}$

holds and from (4.19),

$$y_p = \left(e^{e^{-6\sqrt{x}}}\right)^{-2x}\left(e^{2\sqrt{x}e^{-6\sqrt{x}}}\right)^{2\sqrt{x}} = e^{2xe^{-6\sqrt{x}}}. \tag{4.21}$$

is a particular solution of Eq.(4.21). Consequently, from Theorem 3.1 with (4.21) , the general solution of Eq.(4.20) is as follows:

$$y(x) = y_h(x)y_p(x) = e^{\left(c_1+c_2(2\sqrt{x})\right)e^{-6\sqrt{x}}}e^{2xe^{-6\sqrt{x}}} = e^{\left(c_1+c_2(2\sqrt{x})\right)e^{-6\sqrt{x}}+2xe^{-6\sqrt{x}}}.$$

5. Conclusion

In multiplicative conformable fractional calculus which brings together multiplicative calculus and conformable fractional calculus, the concept of differential equations is presented. The basic definitions, properties, and results of this concept are given. Some solution methods of this equation are explained. A few examples have been solved for clarity.

References

[1] Grossman M. An introduction to non-Newtonian calculus. Int J Math Educ Sci Technol 1979; 10(4): 525–528.
[2] Grossman M, Katz R. Non-Newtonian calculus, Pigeon Cove, MA: Lee Press, 1972.
[3] Bashirov AE, Kurpınar EM, Özyapıcı A. Multiplicative calculus and its applications. J Math Anal Appl 2008; 337(1): 36-48.
[4] Stanley D. A multiplicative calculus. Primus 1999; IX (4): 310-326.
[5] Bashirov AE, Misirli E, Tandoğdu Y, Özyapıcı, A. On modeling with multiplicative differential equations. Appl Math J Chinese Univ Ser A 2011; 26(4): 425-438.
[6] Bashirov AE, Riza M. On complex multiplicative differentiation. TWMS J of Apl & Eng Math 2011; 1(1): 75-85.
[7] Gurefe Y, Kadak U, Misirli E, Kurdi A. A new look at the classical sequence spaces by using multiplicative calculus. UPB Sci Bul Ser A. 2016; 78(2): 9-20.
[8] Kadak U, Gurefe Y. A generalization on weighted means and convex functions with respect to the Non-Newtonian calculus. Int J Anal 2016; 5416751: 1-10.
[9] Uzer A. Multiplicative type complex calculus as an alternative to the classical calculus. Comput Math Appl 2010; 60(10): 2725-2737.
[10] Yalcin N, Celik E. Solution of multiplicative homogeneous linear differential equations with constant exponentials. New Trend Math Sci 2018; 6(2): 58–67.
[11] Yalcin N, Dedeturk M. Solutions of multiplicative ordinary differential equations via the multiplicative differential transform method. AIMS Mathematics 2021; 6(4): 3393-3409.
[12] Yilmaz E. Multiplicative Bessel equation and its spectral properties, Ric. Mat. 2021; 1-17. https://doi.org/10.1007/s11587-021-00674-1.
[13] Bashirov AE, Bashirova G. Dynamics of literary texts and diffusion. OJCMT 2011; 1(3): 60-82.
[14] Filip D, Piatecki C, A non-newtonian examination of the theory of exogenous economic growth. Math. AEterna 2014; 4(2): 101–117.
[15] Florack L, Assen Hv. Multiplicative calculus in biomedical image analysis. J Math Imaging Vis 2012; 42(1): 64–75.
[16] Baleanu D, Guvenc ZB, Machado JAT. New Trends in Nanotechnology and Fractional Calculus Applications. New York (NY): Springer; 2010.
[17] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations (Vol. 204) Elsevier, 2006.
[18] Miller KS, An Introduction to Fractional Calculus and Fractional Differential Equations. New York (NY): J Wiley and Sons; 1993.
[19] Oldham KB, Spanier J. The Fractional Calculus. New York (NY): Academic Press; 1974.
[20] Podlubny I. Fractional Differential Equations, New York: Academic Press; 1999.
[21] Abdeljawad T. On conformable fractional calculus. J Comput Appl Math 2015; 279: 57-66.
[22] Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. J Comput Appl Math 2014; 264: 65–70.
[23] Atangana A, Baleanu D, Alsaedi A. New properties of conformable derivative. Open Math 2015; 13(1): 889-898.
[24] Benkhettou N, da Cruz AMB, Torres DFM, A fractional calculus on arbitrary time scales: fractional differentiation and fractional integration. Signal Processing 2015, 107: 230–237.
[25] Miller KS. An Introduction to Fractional Calculus and Fractional Differential Equations. NewYork (NY): J Wiley and Sons; 1993.
[26] Gulsen T, Yilmaz E, Goktas S. Conformable fractional Dirac system on time scales. J Inequal Appl 2017; 2017(1): 1-10.

[27] Gulsen T, Yilmaz E, Kemaloglu H. Conformable fractional Sturm-Liouville equation and some existence results on time scales. Turk J Math 2018; 42(3): 1348- 1360.
[28] Abdeljawad T, Grossman M. On geometric fractional calculus. J. Semigroup Theory Appl. 2016; 2016(2): 1-14.
[29] Ilie M, Biazar J, Ayati Z. General solution of second order fractional differential equations. Int J Appl Math Research 2018; 7(2): 56-61.
[30] Al Horani M, Hammad MA, Khalil R. Variation of parameters for local fractional nonhomogenous linear differential equations. J. Math. Computer Sci 2016;16: 147-153.
[31] Goktas S, Kemaloglu H, Yilmaz E. Multiplicative conformable fractional Dirac system. Turk J Math 2022; doi:10.3906/mat-2112-55.
[32] Gurefe Y. Multiplicative differential equations and its applications. Ph.D. Thesis, 2013; Ege University, pp.91.