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Research Article

Uniform integrability of sequences of random elements with respect to weak topologies and weak integrals

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Abstract

In probability theory, uniform integrability of families of random variables or random elements plays an important role in the mean convergence. In this paper, we introduce a new version of uniform integrability for sequences in normed spaces in the weak sense. We study the relationship of this new concept with summability theory by considering statistical convergence. We also define a new type of uniform integrability of random elements taking values in topological vector spaces by considering weak integrals. Moreover, we study the connection of summability theory with this new concept as well.

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1. Introduction

The notion of uniform integrability that has a crucial role in establishing probability limit theorems is one of the main concepts of the probability theory. It is well known that mean convergence of a sequence of pth order integrable random variables implies convergence in probability of this sequence for $p > 0$. However, convergence in probability is equivalent to mean convergence under the uniform integrability condition. Various types of uniform integrability can be found in [1, 4, 15–18, 20].

Throughout the paper, we assume that the triple $(H, \Sigma, \mathcal{P})$ denotes a probability space. Let (X_k) be a sequence of random variables and let I denote the indicator function. Then (X_k) said to be uniformly integrable if

$$\lim_{c\to\infty} \sup_{k\in\mathbb{N}} \int_H |X_k|\, I_{\{|X_k|>c\}} d\mathcal{P} = 0$$

where $\mathbb{N}$ denotes the set of all positive integers [4].

The notion of compact uniform integrability which is a generalization of the notion of uniform integrability [6, 18, 21] in a separable Banach space has been generalized to the notion of A-compactly uniformly integrable by Ordóñez Cabrera [18]. Let $(\mathcal{Y}, \|.\|)$ be a

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separable Banach space, let (X_k) be a sequence of random elements from H to $\mathcal{Y}$ and $A = (a_{nk})$ be a summability matrix. Then (X_k) is said to be A-compactly uniformly integrable if for any $\varepsilon > 0$ there exists a compact subset K of $\mathcal{Y}$ such that

$$\sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} |a_{nk}| \int_H \|X_k\| \, I_{\{X_k \notin K\}} d\mathcal{P} < \varepsilon.$$

The concept of A-compact uniform integrability was generalized via Bochner integral by Unver and Uluçay [20]. Let $(\mathcal{Y}, \|.\|)$ be a separable Banach space. Then a sequence (X_k) of Bochner integrable random elements is said to be A-compactly uniformly Bochner integrable if for any $\varepsilon > 0$ there exists a compact subset K of $\mathcal{Y}$ such that

$$\sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} |a_{nk}| \left\| \int_H X_k I_{\{X_k \notin K\}} d\mathcal{P} \right\| < \varepsilon$$

where $A = (a_{nk})$ is a summability matrix and the integral is the Bochner integral. On the other hand, Godet-Thobie and Satco [11] introduced a version of uniform integrability in terms of weak (Pettis) integral: Let (X_k) be a sequence of weakly integrable random elements in a separable Banach space $(\mathcal{Y}, \|.\|)$. Then (X_k) is said to be weakly uniformly integrable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $k \in \mathbb{N}$

$$\left\| \int_H X_k I_F d\mathcal{P} \right\| < \varepsilon$$

whenever $\mathcal{P}(F) \leq \delta$ where the integral is the weak integral. It was also given in [11] that (X_k) is weakly uniformly integrable if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{k\in\mathbb{N}, h\in\mathcal{B}'} \int_H |hX_k I_F| \, d\mathcal{P} < \varepsilon$$

whenever $P(F) \leq \delta$ where $\mathcal{B}'$ is the closed unit ball of the continuous dual $\mathcal{Y}'$ of $\mathcal{Y}$. Moreover, the weak convergence of probability measures which is a different type of weak convergence plays an important role in the probability theory [2] and Bochner's theorem deals with the Fourier transforms of probability measures [3]. Recently, introducing a version of uniform integrability, A-uniform integrability, for scalar sequences, Khan and Orhan [12] have brought the notion of uniform integrability to summability theory and have characterized the notion of strong convergence by proving that a sequence is A-strongly convergent if and only if it is A-statistically convergent to same number and A-uniformly integrable.

A real sequence (x_k) is said to be A-uniformly integrable [12] if

$$\lim_{c\to\infty} \sup_{n\in\mathbb{N}} \sum_{k:|x_k|>c} |x_k| \, |a_{nk}| = 0$$

and (x_k) is said to be A-strongly convergent to a number L if (see e.g., [9])

$$\lim_{n\to\infty} \sum_{k\in\mathbb{N}} |x_k - L| \, |a_{nk}| = 0$$

where $A = (a_{nk})$ is a summability matrix.

A real sequence (x_k) is said to be A-statistically convergent [8, 10] to a real number L if for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \sum_{k:|x_k-L|\geq\varepsilon} a_{nk} = 0$$

where $A = (a_{nk})$ is a non-negative regular summability matrix.

Indeed, the notions of statistical and A-statistical convergences have been given in general Hausdorff topological spaces. Let $(\mathcal{Y}, \tau)$ be a Hausdorff topological space, let $A = (a_{nk})$ be a non-negative regular summability matrix and let (x_k) be a sequence in $\mathcal{Y}$.

Then (x_k) is said to be A-statistically convergent to $x \in \mathcal{Y}$ (see, e.g. [7,14]) if for any open set U such that $x \in U$ we get

$$\lim_{n\to\infty} \sum_{k:x_k\notin U} a_{nk} = 0.$$

Considering the open sets of a norm topology we obtain the following definition of A-statistical convergence in a normed space $(\mathcal{Y}, \|.\|)$. Let $A = (a_{nk})$ be a non-negative regular summability matrix and let (x_k) be a sequence in $\mathcal{Y}$. Then (x_k) is said to be A-statistically convergent to $x \in \mathcal{Y}$ if for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \sum_{k:\|x_k-x\|\geq\varepsilon} a_{nk} = 0.$$

This definition can be found in [13] for Cesàro matrix.

Furthermore, Connor et. al. [5] introduced the concept of weak statistical convergence in a normed space. A sequence (x_k) is said to be weakly statistically convergent to $x \in X$ in a normed space $(\mathcal{Y}, \|.\|)$ provided that, for any $g \in \mathcal{Y}'$, the sequence $(g(x_k - x))$ is statistically convergent to zero. This concept can be given with general summability matrix: Let $A = (a_{nk})$ be non-negative regular summability matrix. Then (x_k) is said to be weakly $A-$statistically convergent to $x \in \mathcal{Y}$, if for any $\varepsilon > 0$ and $g \in \mathcal{Y}'$ we have

$$\lim_{n\to\infty} \sum_{k:|g(x_k-x)|\geq\varepsilon} a_{nk} = 0.$$

Note that weak A-statistical convergence is linear.

In this paper, we introduce the concept of weak uniform integrability and weak strong convergence of a sequence in a normed space with respect to a summability matrix in the functional analysis sense. We investigate the relationship between these concepts and statistical convergence in the weak topology. We also define versions of these three concepts for sequences of random elements taking values in a topological vector space by using weak integrals. We investigate the connection between these concepts and some previous ones in the literature.

2. Weak uniform integrability

2.1. Weak uniform integrability of sequences of vectors

In this subsection we define the concepts of weak strong convergence and weak uniform integrability of sequences of vectors with respect to a non-negative regular summability matrix and investigate the relationship of these concepts with weak statistical convergence. Throughout this subsection we assume that $(\mathcal{Y}, \|.\|)$ is a normed space and it is equipped with the Borel sigma algebra of the norm topology, $A = (a_{nk})$ is a non-negative regular summability matrix and (x_k) is a sequence in $\mathcal{Y}$ unless we indicate converse.

Next proposition proves that weak A-statistical convergence is more general than A-statistical convergence.

Proposition 2.1. *If (x_k) is $A-$statistically convergent to $x \in \mathcal{Y}$ then it is weakly $A-$statistically convergent to x.*

Proof. Let $\varepsilon > 0$ and $g \neq 0 \in \mathcal{Y}'$. Using the fact that $|g(x_k - x)| \leq \|g\| \, \|(x_k - x)\|$ we have

$$\{k : |g(x_k - x)| > \varepsilon\} \subset \left\{k : \|(x_k - x)\| > \frac{\varepsilon}{\|g\|}\right\}.$$

Since A is non-negative we can write

$$0 \leq \sum_{k:|g(x_k-x)|>\varepsilon} a_{nk} < \sum_{k:\|(x_k-x)\|>\frac{\varepsilon}{\|g\|}} a_{nk}.$$

Hence, from the $A-$statistical convergence of (x_k) to x, we obtain that

$$\lim_{n\to\infty} \sum_{k:|g(x_k-x)|>\varepsilon} a_{nk} = 0.$$

□

Now we prove that weak A-statistical convergence is actually A-statistical convergence of the weak topology.

Theorem 2.2. *Let $\mathcal{Y}$ be a topological vector space. Then (x_k) is weakly A-statistically convergent to $x \in \mathcal{Y}$ if and only if (x_k) is A-statistically convergent to x in the weak topology.*

Proof. If (x_k) is weakly A-statistically convergent to x, then we have for any $\varepsilon > 0$ and $g \in \mathcal{Y}'$ that

$$\lim_{n\to\infty} \sum_{k:|g(x_k)-g(x)|\geq\varepsilon} a_{nk} = 0. \tag{2.1}$$

Let U be a weakly open set such that $x \in U$. Then, there exist some $r \in \mathbb{N}$ and $g_1, g_2, ..., g_r \in \mathcal{Y}'$ such that

$$U = U(x, g_1, g_2, ..., g_r, \varepsilon)$$

where

$$U(x, g_1, g_2, ..., g_r, \varepsilon) := \{y \in \mathcal{Y} : |g_i(y) - g_i(x)| < \varepsilon, i = 1, ..., r\}.$$

If $x_k \notin U$, then there exists $i_0 \in \{1, 2, ..., r\}$ such that $|g_{i_0}(x_k) - g_{i_0}(x)| \geq \varepsilon$. On the other hand, by (2.1) we obtain

$$\lim_{n\to\infty} \sum_{k:\left|g_{i_0}(x_k)-g_{i_0}(x)\right|\geq\varepsilon} a_{nk} = 0. \tag{2.2}$$

Therefore we have

$$0 \leq \sum_{k:x_k\notin U} a_{nk} \leq \sum_{k:\left|g_{i_0}(x_k)-g_{i_0}(x)\right|\geq\varepsilon} a_{nk}$$

which yield with (2.2) that

$$\lim_{n\to\infty} \sum_{x_k\notin U} a_{nk} = 0.$$

Hence, (x_k) is A-statistically convergent to x in the weak topology.
Conversely, assume that for any weakly open set U such that $x \in U$

$$\lim_{n\to\infty} \sum_{x_k\notin U} a_{nk} = 0.$$

Let $g \in \mathcal{Y}'$ and $\varepsilon > 0$. Consider the set

$$U_g = \{y \in \mathcal{Y} : |g(y) - g(x)| < \varepsilon\}$$

which is open in the weak topology such that $x \in U_g$. Since

$$\sum_{x_k\notin U_g} a_{nk} = \sum_{k:|g(x_k)-g(x)|\geq\varepsilon} a_{nk}$$

we have from the hypothesis that

$$\lim_{n\to\infty} \sum_{k:|g(x_k)-g(x)|\geq\varepsilon} a_{nk} = 0.$$

□

Now we define the concept of weak A-strong convergence of a sequence.

Definition 2.3. (x_k) is called weakly A-strongly convergent to $x \in \mathcal{Y}$, if for any $g \in \mathcal{Y}'$

$$\lim_{n\to\infty} \sum_{k\in\mathbb{N}} a_{nk} \left| g\left(x_k - x\right)\right| = 0.$$

Next theorem gives a connection between the concepts of weak A-strong convergence and weak A-statistical convergence with the help of weak A-uniform integrability. We omit the proof since it can be proved like the ordinary case.

Theorem 2.4. ***i)*** *If the sequence of (x_k) is weakly A-strongly convergent to $x \in \mathcal{Y}$, then it is weakly A-statistically convergent to x.*
ii) *If the sequence of (x_k) is bounded and weakly A-statistically convergent to $x \in \mathcal{Y}$, then it is weakly A-strongly convergent to x.*

Now we define a new version of uniform integrability in the weak sense which is called weak A-uniform integrability.

Definition 2.5. Let $A = (a_{nk})$ be a summability matrix. $\{x_k\}$ is said to be weakly A-uniformly integrable if for any $g \in \mathcal{Y}'$

$$\lim_{c\to\infty} \sup_{n\in\mathbb{N}} \sum_{k:|g(x_k)|\geq c} |a_{nk}| \left| g\left(x_k\right)\right| = 0.$$

The following theorem is a characterization of the concept of weak A-uniform integrability which is an analogue of the characterization of A-uniform integrability that was given in [12]. We omit the proof since it can be proved like the ordinary case.

Theorem 2.6. *Let $A = (a_{nk})$ be a non-negative summability matrix whose each row is adding up to 1. Then (x_k) is weakly A-strongly convergent to zero vector θ if and only if (x_k) is weakly A-statistically convergent to θ and weakly A-uniformly integrable.*

2.2. Weak uniformly integrability of sequences of random elements

In this subsection we define A-uniform integrability, A-statistical convergence and A-strong convergence of sequences of random elements taking values in a topological vector space by using the weak integral. Then we investigate the relationship of these concepts and some existing concepts in the literature. Throughout this subsection we assume that $(\mathcal{Y}, \|.\|)$ is a topological vector space and it is equipped with the Borel sigma algebra of the topology, $A = (a_{nk})$ is a non-negative regular summability matrix and (X_k) is a sequence of weakly integrable random elements taking values in $\mathcal{Y}$ unless we indicate converse. First of all we recall the weak integral of a random element.

Definition 2.7. [19] A random element X taking values in $\mathcal{Y}$ is called weakly integrable if
i) gX is Lebesgue integrable for any $g \in \mathcal{Y}'$,
ii) There exists a vector $z \in \mathcal{Y}$ such that for any $g \in \mathcal{Y}'$

$$g(z) = \int_H gX d\mathcal{P}.$$

In this case, the vector z is called the weak integral of X and denoted by

$$\int_H X d\mathcal{P}.$$

Definition 2.8. Let $A = (a_{nk})$ be a summability matrix. Then (X_k) is said to be weakly A-boundedly uniformly integrable if for any $\varepsilon > 0$ and $g \in \mathcal{Y}'$ there exists a subset $K_g := K_g(\varepsilon)$ of $\mathcal{Y}$ such that $g\left(K_g\right)$ is bounded and

$$\sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} |a_{nk}| \left| g\left(\int_H X_k I_{\{X_k \notin K_g\}} d\mathcal{P}\right)\right| < \varepsilon.$$

Following two propositions show that the concept of weak A-bounded uniform integrability generalizes the concepts of weak uniform integrability of [11] and A-compact uniform Bochner integrability of [20], respectively.

Proposition 2.9. *Let $(\mathcal{Y},\|.\|)$ be a separable Banach space. If (X_k) is weakly uniformly integrable and $\sup\limits_{k\in\mathbb{N},h\in\mathcal{B}'}\int_H |hX_k|\,d\mathcal{P}<\infty$, then it is weakly A-boundedly uniformly integrable where A is the identity matrix.*

Proof. Let $\varepsilon>0$ and let $g\neq 0\in\mathcal{Y}'$. If we define $f:=\dfrac{g}{\|g\|}$, then $f\in\mathcal{B}'$. Now for an arbitrary $C>0$ let us define

$$D_k(C):=\{s\in H:|fX_k|>C\}.$$

Then from the Chebyshev's inequality we get

$$P(|fX_k|>C)\leq\frac{\int_H|fX_k|\,d\mathcal{P}}{C}\leq\frac{1}{C}\sup_{k\in\mathbb{N},h\in\mathcal{B}'}\int_H|hX_k|\,d\mathcal{P}=\frac{M}{C}$$

where $\sup\limits_{k\in\mathbb{N},h\in\mathcal{B}'}\int_H(|hX_k|)\,d\mathcal{P}=M<\infty$. So if $C>\dfrac{M}{\delta}$, then $P(D_k(C))<\delta$ which imply with the hypothesis that

$$\sup_{k\in\mathbb{N},h\in\mathcal{B}'}\int_H|hX_k|\,I_{D_k(C)}d\mathcal{P}<\frac{\varepsilon}{\|g\|}.$$

Since $f\in\mathcal{B}'$ we get

$$\sup_{k\in\mathbb{N}}\int_H|fX_k|\,I_{D_k(C)}d\mathcal{P}<\frac{\varepsilon}{\|g\|}.$$

Therefore we have

$$\sup_{k\in\mathbb{N}}\int_H\left|gX_kI_{D_k(C)}\right|d\mathcal{P}<\varepsilon. \tag{2.3}$$

On the other hand if we define $K_g:=g^{-1}[-C,C]$, then we obtain

$$\left|g\left(\int_H X_kI_{\{X_k\notin K_g\}}d\mathcal{P}\right)\right|=\left|\int_H\left(gX_kI_{\{X_k\notin K_g\}}\right)d\mathcal{P}\right|\leq\int_H\left|gX_kI_{D_k(C)}\right|d\mathcal{P}. \tag{2.4}$$

So from (2.3) and (2.4) we have

$$\sup_{k\in\mathbb{N}}\left|g\left(\int_H X_kI_{\{X_k\notin K_g\}}d\mathcal{P}\right)\right|<\varepsilon.$$

Furthermore, it is obvious that $g(K_g)\subset[-C,C]$ is bounded. Hence $\{X_k\}$ is weakly A-boundedly uniformly integrable. □

Proposition 2.10. *Let $(\mathcal{Y},\|.\|)$ be a separable Banach space and $A=(a_{nk})$ be a summability matrix. If (X_k) is A-compactly uniformly Bochner integrable, then it is weakly A-boundedly uniformly integrable.*

Proof. Let $g\neq 0\in\mathcal{Y}'$. Since (X_k) is A-compactly uniformly Bochner integrable for any $\varepsilon>0$ there exists a compact subset K of $\mathcal{Y}$ such that

$$\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\left\|\int_H X_kI_{\{X_k\notin K\}}d\mathcal{P}\right\|<\frac{\varepsilon}{\|g\|}.$$

Since g is a continuous and the sequence $\{X_k\}$ is weakly integrable whenever it is Bochner integrable, it is easy to check that

$$\left|g\left(\int_H X_kI_{\{X_k\notin K\}}d\mathcal{P}\right)\right|\leq\|g\|\left\|\int_H X_kI_{\{X_k\notin K\}}d\mathcal{P}\right\|.$$

Thus we have

$$\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\left|g\left(\int_H X_k I_{\{X_k\notin K\}}d\mathcal{P}\right)\right|\leq \|g\|\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\left\|\int_H X_k I_{\{X_k\notin K\}}d\mathcal{P}\right\|<\varepsilon.$$

On the other hand compactness of K and continuity of g imply that $g(K)$ is bounded. Hence (X_k) is weakly A-boundedly uniformly integrable. □

Now we introduce the concepts of weak A-statistical and weak A-strong convergences of sequences of random elements.

Definition 2.11. (X_k) is said to be weakly A-statistically convergent to a weakly integrable random element X if the sequence of vectors $(\int_H X_k)$ is A-statistically convergent to the vector $\int_H X d\mathcal{P}$ in the weak topology, i.e., for any $\varepsilon>0$ any $g\in\mathcal{Y}'$

$$\lim_{n\to\infty}\sum_{\left|g\left(\int_H (X_k-X)d\mathcal{P}\right)\right|>\varepsilon} a_{nk}=0.$$

Definition 2.12. Let $A=(a_{nk})$ be a summability matrix. Then (X_k) is said to be weakly A-strongly convergent to a weakly integrable random element X if for any $g\in\mathcal{Y}'$

$$\lim_{n\to\infty}\sum_{k\in\mathbb{N}}|a_{nk}|\left|g\left(\int_H (X_k-X)d\mathcal{P}\right)\right|=0.$$

Definition 2.13. Let $A=(a_{nk})$ be a summability matrix. Then (X_k) is said to be Lebesgue weakly A-strongly convergent to a weakly integrable random element X if for any $g\in\mathcal{Y}'$

$$\lim_{n\to\infty}\sum_{k\in\mathbb{N}}|a_{nk}|\int_H |g(X_k-X)|\,d\mathcal{P}=0.$$

Since for any weakly integrable random element X

$$\left|g\left(\int_H X d\mathcal{P}\right)\right|=\left|\int_H gX d\mathcal{P}\right|\leq\int_H |gX|\,d\mathcal{P}$$

we get a sequence of random elements is weakly A-strongly convergent whenever it is Lebesgue weakly A-strongly convergent.

Following examples give the relationship between the new concepts for sequences of random elements and for scalar sequences.

Example 2.14. Let $(\mathcal{Y},\|.\|)$ be a normed space, let (x_k) be a sequence and $x\in\mathcal{Y}$. Now for each $k\in\mathbb{N}$ let us define the random element $X_k:(J,\sigma(J),\lambda)\to(\mathcal{Y},\|.\|)$ as

$$X_k(s)=x_k$$

where $J=[0,1]$, $\sigma(J)$ is the Borel sigma field of J and λ is the Lebesgue measure. Now define the random element $X:(J,\sigma(J),\lambda)\to(\mathcal{Y},\|.\|)$ by

$$X(s)=x.$$

Then it is easy to see that

$$\left|g\left(\int_J (X_k-X)d\lambda\right)\right|=\int_J |g(X_k-X)|\,d\lambda=|g(x_k-x)|$$

for $g\in\mathcal{Y}'$. Thus weak A-statistical convergence and weak (or equivalently Lebesgue weak) A-strong convergence is a generalization of weak A-statistical convergence and weak A-strong convergence of scalar sequences to sequences of random elements, respectively.

Example 2.15. Let $A=(a_{nk})$ be a summability matrix and let $x=(x_k)$ be a scalar sequence that is weakly A-uniformly integrable. Take $g\in\mathcal{Y}'$. Then for any $\varepsilon>0$ there exists $L=L(\varepsilon,g)>0$ such that

$$\sup_{n\in\mathbb{N}}\sum_{k:|g(x_k)|>c}|g(x_k)|\,|a_{nk}|<\varepsilon$$

whenever $c\geq L$. Now consider the set $K_g=\{x\in\mathcal{Y}:|g(x)|\leq L\}\subset\mathcal{Y}$ and the sequence of random elements (X_k) in Example 2.14. Then we have

$$\begin{aligned}\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\left|g\left(\int_J X_k I_{\{X_k\notin K_g\}}d\lambda\right)\right| &= \sup_{n\in\mathbb{N}}\sum_{k:|g(x_k)|>L}|a_{nk}|\int_J|g(x_k)|\,d\lambda\\ &= \sup_{n\in\mathbb{N}}\sum_{k:|g(x_k)|>L}|a_{nk}|\,|g(x_k)|\,\lambda(J)\\ &= \sup_{n\in\mathbb{N}}\sum_{k:|g(x_k)|>L}|g(x_k)|\,|a_{nk}|\\ &<\varepsilon.\end{aligned}$$

Since $g(K_g)$ is bounded, weak A-bounded uniform integrability is a generalization of weak A-uniform integrability of scalar sequences to sequences of random elements.

Now we deal with the relationship between the concepts of weak A-strong and weak A-statistical convergences of sequences of random elements and give some necessary and sufficient conditions for sequences of random elements to be weakly A-boundedly uniformly integrable in topological vector spaces.

Theorem 2.16. *Assume that $A=(a_{nk})$ is a non-negative regular summability matrix whose each row adds up to 1 and $X(s)=\theta$ for all $s\in H$. Then the following statements hold:*
i) *If (X_k) is weakly A-statistically convergent to X and weakly A-boundedly uniformly integrable, then it is weakly A-strongly convergent to X.*
ii) *If (X_k) is weakly A-strongly convergent to X, then it is weakly A-statistically convergent to X.*

Proof. **i)** Suppose that (X_k) is weakly A-statistically convergent to X and it is weakly A-boundedly uniformly integrable. Then for any $\varepsilon>0$ and $g\in\mathcal{Y}'$ we have

$$\lim_{n\to\infty}\sum_{k:\left|g\left(\int_H X_k d\mathcal{P}\right)\right|>\varepsilon/2}a_{nk}=0 \tag{2.5}$$

and there exists $K_g\subset\mathcal{Y}$ such that $g(K_g)$ is bounded and

$$\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\left|g\left(\int_H X_k I_{\{X_k\notin K_g\}}d\mathcal{P}\right)\right|<\varepsilon/2. \tag{2.6}$$

On the other hand we have for any $\varepsilon > 0$ and $g \in \mathcal{Y}'$ that

$$\begin{aligned}\sum_{k\in\mathbb{N}} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right| &= \sum_{k:\left|g\left(\int_H X_k d\mathcal{P}\right)\right|>\varepsilon/2} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right| \\ &+ \sum_{k:\left|g\left(\int_H X_k d\mathcal{P}\right)\right|\le\varepsilon/2} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right| \\ &\le \sum_{k\in\mathbb{N}} a_{nk}\left|g\left(\int_H X_k I_{\{X_k\notin K_g\}} d\mathcal{P}\right)\right| \\ &+ \sum_{k:\left|g\left(\int_H X_k d\mathcal{P}\right)\right|>\varepsilon/2} a_{nk}\left|g\left(\int_H X_k I_{\{X_k\in K_g\}} d\mathcal{P}\right)\right| + \varepsilon/2. \qquad (2.7)\end{aligned}$$

Since $g(K_g)$ is bounded there exists $M > 0$ such that for all $x \in K_g$, $|g(x)| \le M$. Thus for any $k \in \mathbb{N}$ we have from the definition of weak integral that

$$\begin{aligned}\left|g\left(\int_H X_k I_{\{X_k\in K_g\}} d\mathcal{P}\right)\right| &= \left|\int_H g\left(X_k I_{\{X_k\in K_g\}}\right) d\mathcal{P}\right| \\ &\le \int_H \left|g\left(X_k I_{\{X_k\in K_g\}}\right)\right| d\mathcal{P} \\ &\le M\mathcal{P}(H) \\ &= M. \qquad (2.8)\end{aligned}$$

Now from (2.7) and (2.8) we can write for any $\varepsilon > 0$ and $g \in \mathcal{Y}'$ that

$$\begin{aligned}\limsup_{n\to\infty}\sum_{k\in\mathbb{N}} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right| &\le \sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} a_{nk}\left|g\left(\int_H X_k I_{\{X_k\notin K_g\}} d\mathcal{P}\right)\right| \\ &+ M\limsup_{n\to\infty}\sum_{k:\left|g\int_H X_k d\mathcal{P}\right|>\varepsilon/2} a_{nk} + \varepsilon/2. \qquad (2.9)\end{aligned}$$

Thus by (2.5),(2.6) and (2.9) we get for any $\varepsilon > 0$ and $g \in \mathcal{Y}'$ that

$$\limsup_{n\to\infty}\sum_{k\in\mathbb{N}} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right| < \varepsilon$$

which implies that (X_k) is weakly A-strongly convergent to X.
ii) Assume that (X_k) is weakly A-strongly convergent to X. Then we have for any $\varepsilon > 0$ and $g \in \mathcal{Y}'$ that

$$\begin{aligned}\sum_{k\in\mathbb{N}} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right| &= \sum_{k:\left|g\left(\int_H X_k\right)\right|>\varepsilon} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right| \\ &+ \sum_{k:\left|g\left(\int_H X_k\right)\right|\le\varepsilon} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right| \\ &\ge \sum_{k:\left|g\left(\int_H X_k d\mathcal{P}\right)\right|>\varepsilon} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right| \\ &> \varepsilon \sum_{k:\left|g\left(\int_H X_k d\mathcal{P}\right)\right|>\varepsilon} a_{nk}.\end{aligned}$$

Thus we obtain

$$0 \le \sum_{k:\left|g\left(\int_H X_k d\mathcal{P}\right)\right|>\varepsilon} a_{nk} \le \frac{1}{\varepsilon}\sum_{k\in\mathbb{N}} a_{nk}\left|g\left(\int_H X_k d\mathcal{P}\right)\right|$$

which implies (X_k) is weakly A-statistically convergent to X. □

Next theorem gives a sufficient condition for weak A-bounded uniform integrability of a sequence of random elements.

Theorem 2.17. *Let $A = (a_{nk})$ and X be as in Theorem 2.16 and assume that for all $g \in \mathcal{Y}'$ and for each $k \in \mathbb{N}$, $gX_k(H)$ is bounded. If (X_k) is Lebesgue weakly A-strongly convergent to X, then it is weakly A-statistically convergent to X and weakly A-boundedly uniformly integrable.*

Proof. Assume that (X_k) is Lebesgue weakly A-strongly convergent to X. Then since Lebesgue weak A-strong convergence of (X_k) implies weak A-strong convergence, weak A-statistical convergence of (X_k) is obvious from (ii) of Theorem 2.16. Now let $g \in \mathcal{Y}'$ and $\varepsilon > 0$. Since (X_k) is Lebesgue weakly A-strongly convergent to X there exists $N = N_g(\varepsilon) > 0$ such that for all $n > N$

$$\sum_{k\in\mathbb{N}} a_{nk} \int_H |gX_k|\, d\mathcal{P} < \frac{\varepsilon}{2}. \tag{2.10}$$

Since

$$\sum_{k\in\mathbb{N}} a_{nk} \int_H |gX_k|\, d\mathcal{P} < \infty$$

for $n = 1, 2, ..., N$, there exists $M = M_g > 0$ such that for $n = 1, 2, ..., N$

$$\sum_{k>M} a_{nk} \int_H |gX_k|\, d\mathcal{P} < \frac{\varepsilon}{2}. \tag{2.11}$$

Now if we consider the subset $K_g = \bigcup\limits_{k\leq M} X_k(H)$ of $\mathcal{Y}$, it is obvious that for $k = 1, 2, ..., M$

$$\{s : X_k(s) \notin K_g\} = \varnothing.$$

Thus we get for $n = 1, 2, ..., N$ that

$$\begin{aligned}
\sum_{k\in\mathbb{N}} a_{nk} \left| g\left(\int_H X_k I_{\{X_k\notin K_g\}} d\mathcal{P} \right) \right| &= \sum_{k>M} a_{nk} \left| g\left(\int_H X_k I_{\{X_k\notin K_g\}} d\mathcal{P} \right) \right| \\
&= \sum_{k>M} a_{nk} \left| \int_H g\left(X_k I_{\{X_k\notin K_g\}} \right) d\mathcal{P} \right| \\
&\leq \sum_{k>M} a_{nk} \int_H |gX_k|\, I_{\{X_k\notin K_g\}} d\mathcal{P} \\
&\leq \sum_{k>M} a_{nk} \int_H |g(X_k)|\, d\mathcal{P}.
\end{aligned} \tag{2.12}$$

Also from (2.10) we get for all $g \in \mathcal{Y}'$

$$\begin{aligned}
\sup_{n>N} \sum_{k\in\mathbb{N}} a_{nk} \left| g\left(\int_H X_k I_{\{X_k\notin K_g\}} d\mathcal{P} \right) \right| &\leq \sup_{n>N} \sum_{k\in\mathbb{N}} a_{nk} \int_H |gX_k|\, I_{\{X_k\notin K_g\}} d\mathcal{P} \\
&\leq \sup_{n>N} \sum_{k\in\mathbb{N}} a_{nk} \int_H |gX_k|\, d\mathcal{P} \\
&< \frac{\varepsilon}{2}.
\end{aligned} \tag{2.13}$$

By (2.11), (2.12) and (2.13) we obtain for all $g \in \mathcal{Y}'$

$$\sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} a_{nk} \left| g\left(\int_H X_k I_{\{X_k\notin K_g\}} d\mathcal{P} \right) \right| < \varepsilon.$$

Since

$$g\left(K_g\right)=g\left(\bigcup_{k\leq M}X_k\left(H\right)\right)=\bigcup_{k\leq M}gX_k\left(H\right)$$

is bounded we have (X_k) is weakly A-boundedly uniformly integrable. $\square$

From Examples 2.14 and 2.15 and Theorems 2.16 and 2.17 the following corollary is obtained immediately. State that it is actually Theorem 2.4.

Corollary 2.18. *Let (x_k) be a sequence in a normed space and $A=(a_{nk})$ be a non-negative regular summability matrix. Then (x_k) is weakly A-strongly convergent to some x if and only if it is weakly A-statistically convergent to x and weakly A-uniformly integrable.*

Following proposition gives a sufficient condition for weak A-bounded uniform integrability of sequences of random elements taking values in finite dimensional topological vector spaces.

Proposition 2.19. *Let $\mathcal{Y}$ be a finite dimensional topological vector space and $A=(a_{nk})$ be a summability matrix. If*
i) *for any $g\in\mathcal{Y}'$ $\sup\limits_{n\in\mathbb{N}}\sum\limits_{k\in\mathbb{N}}|a_{nk}|\int_H|gX_k|\,d\mathcal{P}<\infty$,*
ii) *for any $\varepsilon>0$ and $g\in\mathcal{Y}'$ there exists $\delta>0$ such that*

$$\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\left|g\left(\int_H X_kI_{G_k}d\mathcal{P}\right)\right|<\varepsilon$$

whenever $G_k\in\Sigma$ and

$$\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\,\mathcal{P}(G_k)<\delta, \tag{2.14}$$

then (X_k) is weakly A-boundedly uniformly integrable.

Proof. By (i) there exists $T>0$ such that

$$\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\int_H|gX_k|\,d\mathcal{P}<T.$$

Let $\varepsilon>0$ and $g\in\mathcal{Y}'$. Then there exists $\delta>0$ such that (ii) holds. From Chebychev's inequality, for a fixed $a>T/\delta$ we have

$$\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\,\mathcal{P}\left(|gX_k|>a\right)\leq\frac{\delta}{T}\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\int_H|gX_k|\,d\mathcal{P}<\delta. \tag{2.15}$$

Now consider the subset $K_g=\{x\in\mathcal{Y}:|g\left(x\right)|\leq a\}$. Then from (2.15) for each k the measurable subset $G_k=X_k^{-1}(K_g^c)$ realises condition (2.14). So we have from (ii) that

$$\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\left|g\left(\int_H X_kI_{\{X_k\notin K_g\}}d\mathcal{P}\right)\right|=\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{nk}|\left|g\left(\int_H X_kI_{G_k}d\mathcal{P}\right)\right|<\varepsilon.$$

Since $g\left(K_g\right)$ is bounded (X_k) is weakly A-boundedly uniformly integrable. $\square$

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