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Numerical Treatment of Uniformly Convergent Method for Convection Diffusion Problem

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Abstract − In this paper, we will study the convergence properties of the method designed for the convection-diffusion problem. We will prove that the analytical and numerical methods give the same result. Merging the ideas in previous research, we introduce a numerical algorithm on a uniform mesh that requires no exact solution to the local convection-diffusion problem. We display how to obtain the numerical solution of the local Boundary Value Problem (BVP) in a suitable way to ensure that the resulting numerical algorithm recaptures the same convergence properties when using the exact solution of the local BVP. We prove that the proposed algorithm nodally converges to the exact solution.

Keywords − *Trapezoidal rule, convection-diffusion problem, boundary value problem, singular points, Green's function, Lagrange interpolation*

Mathematics Subject Classification (2020) − 34B27, 65L10

1. Introduction

It is well-known that the piecewise- uniform fitted meshes studied by Shishkin [1] and the corresponding numerical algorithms were developed and shown to be ε −uniform in various studies including the book by Shishkin [2]. The numerical results using a fitted mesh method were firstly presented in [3]. We refer the readers to Bakhvalov [4], Gartland [5] and Vulanovic [6] for other approaches to adapting the mesh, involving complicated redistribution of the mesh points [7, 8]. We note that none has the simplicity of the piecewise uniform fitted meshes.

Motivating by ~~this~~ these considerations, we remark that both fitted operators and fitted meshes need to be studied. Since the methods using fitted meshes are usually easier to implement than the methods using fitted operators in practice, they recommended to be applied whenever possible. We also note that the methods using fitted meshes are easier to generalize to the problems in more than one dimension and to the nonlinear problems.

In this paper, the following convection−diffusion problem with a concentrated source is considered and we prove that ε-uniformly convergent methods can be designed for the problem (1). In other words, in this article, to investigate the numerical solution of equation (1) and to obtain a suitable method, we will focus on the following boundary value problem [9]

$$Lu = -\varepsilon u'' + bu' + c\,u = f(x), \quad u(0) = 0, \quad u(1) = 0 \tag{1}$$

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It is worth mentioning that the modelling of real-world problems including physical, chemical, and biological phenomena contain interactions of convection and diffusion processes, which can be described by the convection-diffusion- problem [10].

We remark that, we have studied the following

$$-\varepsilon g_i'(x_{i-1})\, u_{i-1} + u_i + \varepsilon g_i'(x_{i+1})\, u_{i+1} = (f - cu) \int_{x_{i-1}}^{x_{i+1}} g_i \, dx$$

and we obtained the analytical solution

$$-\frac{e^{\rho_i}}{e^{\rho_i}+1} U_{i-1} + U_i - \frac{1}{e^{\rho_i}+1} U_{i+1} = (f_i - c_i\, U_i)\, \frac{h}{b} \left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right) \tag{2}$$

see [10] and [11] for details. In this article, we will use the equation (1), and after applying various numerical treatments, we will get the same solution given by the equation (2) which was studied before in [12]. In this study, we have,

$$g_i'(x_{i-1}) \cong D^+ G_0 = \frac{G_1 - G_0}{h_1^*} \text{ and } g_i'(x_{i+1}) \cong D^- G_M = \frac{G_M - G_{M-1}}{h_2^*}$$

$$-\varepsilon\, D^+ G_0 U_{i-1} + U_i + \varepsilon\, D^- G_M U_{i+1} = (f_i - c\, U_i) \int_{x_{i-1}}^{x_{i+1}} G^i \, dx$$

$$T_1(\varepsilon, b_i, c_i, h, M) = \varepsilon D^+ G_0$$

$$T_2(\varepsilon, b_i, c_i, h, M) = -\varepsilon D^- G_M$$

$$T_3(\varepsilon, b_i, c_i, M) = \int_{x_{i-1}}^{x_{i+1}} G^i \, dx$$

At the end of this paper, we will show that

$$\lim_{M\to\infty} T_1(\varepsilon, b_i, c_i, h, M) = \frac{e^{\rho_i}}{e^{\rho_i}+1}$$

$$\lim_{M\to\infty} T_2(\varepsilon, b_i, c_i, h, M) = -\frac{1}{\varepsilon}\left(\frac{1}{e^{\rho_i}+1}\right)$$

$$\lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i \, dx = \lim_{M\to\infty} T_3(\varepsilon, b_i, c_i, M) = \left(\frac{h}{b_i}\right)\left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right)$$

Now, consider

$$-\varepsilon\, D^+ D^- G_j - b D^+ G_j = \Delta X_{i,j}, \qquad j = 1, 2, 3, \dots, M-1$$

$$-\varepsilon\left(\frac{G_{j+1} - G_j}{h_{j+1}} - \frac{G_j - G_{j-1}}{h_j}\right)\frac{1}{h_j} - b\left(\frac{G_{j+1} - G_j}{h_{j+1}}\right) = \Delta X_{i,j}$$

where

$$\Delta X_{i,j} = \begin{cases} \dfrac{1}{h_{j+1}}, & x_i \in (x_j, x_{j+1}) \\ 0, & \text{otherwise} \end{cases}$$

If $j=0$ or $j=M$, then $G_0=0$ or $G_M=0$.

$$h_j=\begin{cases} h_1\ ,1\le j\le \frac{M}{4}-1\\ h_2,\ \frac{M}{4}\le j\le \frac{2M}{4}-1\\ h_1,\ \frac{2M}{4}\le j\le \frac{3M}{4}-1\\ h_2,\ \frac{3M}{4}\le j\le \frac{4M}{4}-1\end{cases}$$

$$-G_{j+1}\left(1+\frac{b\,h_1}{\varepsilon}\right)+G_j\left(2+\frac{b\,h_1}{\varepsilon}\right)+G_{j-1}(-1)=0$$

In the previous equation, if we take $G_{j+1}=r^2$, $G_j=r$, $G_{j-1}=r^0=1$ and $\lambda_1=1+\frac{b\,h_1}{\varepsilon}$ then we will get

$$-r^2\left(1+\frac{b\,h_1}{\varepsilon}\right)+r\left(2+\frac{b\,h_1}{\varepsilon}\right)-1=0$$

$$-r^2\lambda_1+r(1+\lambda_1)-1=0\ \Rightarrow(1-r\lambda_1)(r-1)=0$$

Then, the roots of the quadratic equation are given by: $r_1=1$ and $r_2=\frac{1}{\lambda_1}$. Similarly, we get

$$-G_{j+1}\left(1+\frac{b\,h_2}{\varepsilon}\right)+G_j\left(2+\frac{b\,h_2}{\varepsilon}\right)+G_{j-1}(-1)=0$$

In the previous equation, if we take $G_{j+1}=r^2$, $G_j=r$, $G_{j-1}=r^0=1$ and $\lambda_2=1+\frac{b\,h_2}{\varepsilon}$, then we will get

$$-r^2\left(1+\frac{b\,h_2}{\varepsilon}\right)+r\left(2+\frac{b\,h_2}{\varepsilon}\right)-1=0$$

$$-r^2\lambda_2+r(1+\lambda_2)-1=0\ \Rightarrow(1-r\lambda_2)(r-1)=0$$

Then, the roots of the quadratic equation are given by: $r_1=1$ and $r_2=\frac{1}{\lambda_2}$.

2. Derivation of Trapezoidal Rule

We can derive the trapezoidal rule by using polynomial interpolants of $f(x)$ function. The usage of a Lagrange interpolant for each sub-interval $[x_{i-1},x_i]$, $i=1,2,3,\dots,n$ leads to the trapezoidal rule in [13], that is,

$$\int_{x_{i-1}}^{x_i} f(x)\,dx\approx\int_{x_{i-1}}^{x_i} P(x)\,dx$$

where

$$P(x)=\frac{x-x_i}{x_{i-1}-x_i}f(x_{i-1})+\frac{x-x_{i-1}}{x_i-x_{i-1}}f(x_i)$$

$$\begin{aligned}\int_{x_{i-1}}^{x_i} f(x)\,dx\approx\int_{x_{i-1}}^{x_i} P(x)dx &=\int_{x_{i-1}}^{x_i}\left(\frac{(x-x_i)}{x_{i-1}-x_i}f(x_{i-1})+\frac{(x-x_{i-1})}{x_i-x_{i-1}}f(x_i)\right)dx\\ &=\frac{f(x_{i-1})}{x_{i-1}-x_i}\int_{x_{i-1}}^{x_i}(x-x_i)dx+\frac{f(x_i)}{x_i-x_{i-1}}\int_{x_{i-1}}^{x_i}(x-x_{i-1})dx\\ &=\frac{f(x_{i-1})}{-(x_i-x_{i-1})}\,\frac{(x-x_i)^2}{2}\,\Bigg|_{x=x_{i-1}}^{x_i}+\frac{f(x_i)}{x_i-x_{i-1}}\,\frac{(x-x_{i-1})^2}{2}\,\Bigg|_{x=x_{i-1}}^{x_i}\end{aligned}$$

$$= -\frac{f(x_{i-1})}{(x_i - x_{i-1})}\left[\frac{(x_i - x_i)^2}{2} - \frac{(x_{i-1} - x_i)^2}{2}\right] + \frac{f(x_i)}{(x_i - x_{i-1})}\left[\frac{(x_i - x_{i-1})^2}{2} - \frac{(x_{i-1} - x_{i-1})^2}{2}\right]$$

$$= -\frac{f(x_{i-1})}{(x_i - x_{i-1})}\cdot\left[0 - \frac{(x_{i-1} - x_i)^2}{2}\right] + \frac{f(x_i)}{(x_i - x_{i-1})}\left[\frac{(x_i - x_{i-1})^2}{2} - 0\right]$$

$$= \frac{f(x_{i-1})}{(x_i - x_{i-1})}\left[\frac{(x_{i-1} - x_i)^2}{2}\right] + \frac{f(x_i)}{(x_i - x_{i-1})}\left[\frac{(x_i - x_{i-1})^2}{2}\right]$$

$$\int_{x_{i-1}}^{x_i} P(x)\,dx = \frac{(x_i - x_{i-1})}{2}\left[\frac{f(x_{i-1})}{2} + \frac{f(x_i)}{2}\right]$$

For the composite trapezoidal rule, we have,

$$\int_a^b f(x)dx = \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\,dx \quad = \sum_{i=1}^{n}(x_i - x_{i-1})\left[\frac{f(x_{i-1})}{2} + \frac{f(x_i)}{2}\right]$$

$$\int_a^b P(x)\,dx = \frac{h}{2}\sum_{i=1}^{n}[\,f(x_{i-1}) + f(x_i)\,] = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n} f(x_i) + (x_n)\right]$$

We note that, this is known as the composite trapezoidal rule in [13].

Lemma 2.1: If

$$T_1(\varepsilon, b_i, c_i, h, M) = \varepsilon D^+ G_0$$

then

$$\lim_{M\to\infty} T_1(\varepsilon, b_i, c_i, h, M) = \frac{e^{\rho_i}}{e^{\rho_i} + 1}$$

Proof. Consider the uniform case, that is $\tau = \frac{h}{2}$. Then, the mesh parameters can be written as $h_1^* = h_2^* = \frac{2h}{M}$ and $\lambda_1 = \lambda_2 = 1 + \frac{2bh}{\varepsilon M}$.

$$\lim_{M\to\infty} T_1(\varepsilon, b_i, c_i, h, M) = \lim_{M\to\infty}\frac{G_1 - G_0}{h_1^*} = \frac{e^{\rho_i}}{e^{\rho_i} + 1}$$

Lemma 2.2: If

$$T_2(\varepsilon, b_i, c_i, h, M) = -\varepsilon D^- G_M$$

then

$$\lim_{M\to\infty} T_2(\varepsilon, b_i, c_i, h, M) = -\frac{1}{\varepsilon}\left(\frac{1}{e^{\rho_i} + 1}\right)$$

Proof. We follow the same steps in the proof of Lemma 2.1. For the uniform case when $\tau = \frac{h}{2}$, we use the difference solution G^i and the fact that $h_1^* = h_2^* = \frac{2h}{M}$;

$$\lim_{M\to\infty} T_2(\varepsilon, b_i, c_i, h, M) = \lim_{M\to\infty}\frac{G_M - G_{M-1}}{h_2^*} = -\frac{1}{\varepsilon}\left(\frac{1}{e^{\rho_i} + 1}\right)$$

Lemma 2.3: If

$$T_3(\varepsilon, b_i, c_i, M) = \int_{x_{i-1}}^{x_{i+1}} G^i\,dx$$

then

$$\lim_{M\to\infty} T_3(\varepsilon, b_i, c_i, M) = \left(\frac{h}{b_i}\right)\left(\frac{e^{\rho_i}-1}{e^{\rho_i}+1}\right)$$

PROOF. In order to calculate the following integral

$$\int_{x_{i-1}}^{x_{i+1}} G^i \, dx$$

the trapezoidal rule is used for the exact solution of

$$G_j^i = \begin{cases} a_1 r_1^j + a_2 r_2^j & 0 \le j \le \dfrac{M}{4} \\ a_3 r_3^j + a_4 r_4^j, & \dfrac{M}{4} \le j \le \dfrac{2M}{4} \\ a_5 r_1^j + a_6 r_2^j, & \dfrac{2M}{4} \le j \le \dfrac{3M}{4} \\ a_7 r_3^j + a_8 r_4^j, & \dfrac{3M}{4} \le j \le \dfrac{4M}{4} \end{cases}$$

$$G_j^i = \begin{cases} a_1 + a_2 \lambda_1^{-j}, & 0 \le j \le \dfrac{M}{4} \\ a_3 + a_4 \lambda_2^{-j}, & \dfrac{M}{4} \le j \le \dfrac{2M}{4} \\ a_5 + a_6 \lambda_1^{-j}, & \dfrac{2M}{4} \le j \le \dfrac{3M}{4} \\ a_7 + a_8 \lambda_2^{-j}, & \dfrac{3M}{4} \le j \le \dfrac{4M}{4} \end{cases}$$

Using the properties of Green's function in [14–18], we get,

$$a_1 + a_2\lambda_1^{-j} = a_3 + a_4\lambda_2^{-j}$$
$$a_3 + a_4\lambda_2^{-j} = a_5 + a_6\lambda_1^{-j}$$
$$a_5 + a_6\lambda_1^{-j} = a_7 + a_8\lambda_2^{-j}$$

For $G_0 = G_M = 0$, we have,

$$a_1 + a_2\lambda_1^{-j} = 0$$
$$a_7 + a_8\lambda_2^{-j} = 0$$

For j = M/4, we have, $a_1\lambda_1 + a_2(k_1^{-1}(1-\lambda_2+\lambda_1)) - a_3\lambda_1 + a_4 k_3 k_2^{-1} = 0$.

For $\frac{h_2}{\varepsilon}$, we have, $a_3\lambda_2 + a_4(k_2^{-2}(1-\lambda_2+\lambda_1)) - a_5\lambda_2 - a_6 k_1^{-2} k_3^{-1} = \frac{h_2}{\varepsilon}$.

For j=3M/4, we have, ⇒ $a_5\lambda_1 + a_6(k_1^{-3}(1-\lambda_2+\lambda_1)) - a_7\lambda_1 - a_8 k_3 k_2^{-3} = 0$.

In order to get the difference solution exactly, we need to determine the eight unknown coefficients. Two equations can be obtained by using the boundary conditions: $G_0 = G_M = 0$; the difference equations related to the nodes $x_{M/4}$, $x_{2M/4}$ and $x_{3M/4}$ give us other three equations; and finally, the continuity conditions can be applied to obtain the other three equations. Next, the corresponding numerical algorithm can be obtained by using the fitted finite difference operator in order to get a system of finite difference equations on a standard mesh. We remark that the mesh is often a uniform mesh in practice. Finally, the obtained system can be solved in a practical way to get the numerical solutions. We refer the readers to [16] for other approaches in constructing fitted finite difference operators.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & k_1^{-1} & -1 & k_2^{-1} & 0 & 0 & 0 & 0 \\ \lambda_1 & \xi_1 & -\lambda_1 & -k_3 k_2^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & k_2^{-2} & -1 & -k_1^{-2} & 0 & 0 \\ 0 & 0 & \lambda_2 & \xi_2 & -\lambda_2 & -k_1^{-2}k_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & k_1^{-3} & -1 & -k_2^{-3} \\ 0 & 0 & 0 & 0 & \lambda_1 & \xi_3 & -\lambda_1 & -k_3 k_2^{-3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & k_2^{-4} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{h_2}{\varepsilon} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $k_1 = \lambda_1^{\frac{M}{4}}$, $k_2 = \lambda_2^{\frac{M}{4}}$, $k_3 = \lambda_1 \lambda_2^{-1}$, $\xi_1 = k_1^{-1}(1 - \lambda_2 + \lambda_1)$, $\xi_2 = k_2^{-2}(1 - \lambda_1 + \lambda_2)$, $\xi_3 = k_1^{-3}(1 - \lambda_2 + \lambda_1)$, and $\eta = \varepsilon(\lambda_1 - 1)(1 + \lambda_1^{\frac{M}{4}} \lambda_2^{\frac{M}{4}})$.

Using the symbolic programming MATHEMATICA, one can solve $AX = B$ linear system and obtain the following results:

$$a_1 = \frac{h_2}{\eta} k_1\ k_2\ k_3$$

$$a_2 = -\frac{h_2}{\eta} k_1\ k_2\ k_3$$

$$a_3 = \frac{h_2}{\eta}(-k_2\ k_3\ + k_1\ k_2\ k_3\ + k_2\ )$$

$$a_4 = -\frac{h_2}{\eta} k_2^2$$

$$a_5 = -\frac{h_2}{\eta}(\lambda_2 - \lambda_2 k_2 + \lambda_1 k_2)\ \lambda_2^{-1}$$

$$a_6 = \frac{h_2}{\eta} k_1^3 k_2\ k_3$$

$$a_7 = \frac{h_2}{\eta}$$

$$a_8 = \frac{h_2}{\eta} k_2^4$$

$$\int_{x_{i-1}}^{x_{i+1}} G^i\, dx = \int_{x_{i-1}}^{x_{i-1}+\tau} G^i\, dx + \int_{x_{i-1}+\tau}^{x_i} G^i\, dx + \int_{x_i}^{x_i+\tau} G^i\, dx + \int_{x_i+\tau}^{x_{i+1}} G^i\, dx$$

where $\tau = \frac{h}{2}$.

$$\lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i\, dx = \lim_{M\to\infty}(I_1 + I_2 + I_3 + I_4)$$

Unless otherwise indicated, we will apply the trapezoidal rule for numerical integration until the end of this work.

$$I_1 = \int_{x_{i-1}}^{x_{i-1}+\tau} G^i\, dx = h_1 \left[\frac{G_0}{2} + G_1 + G_2 + \cdots + G_{\frac{M}{4}-1} + \frac{G_{\frac{M}{4}}}{2}\right]$$

$$I_1 = \int_{x_{i-1}}^{x_{i-1}+\tau} G^i \, dx = h_1 \left[\frac{0}{2} + G_1 + G_2 + \cdots + G_{\frac{M}{4}-1} + \frac{G_{\frac{M}{4}}}{2} \right]$$

$$I_1 = h_1 \left[\sum_{j=1}^{\frac{M}{4}-1} G^j \right] + \frac{h_1}{2} G_{\frac{M}{4}}$$

$$I_1 = h_1 \sum_{j=1}^{\frac{M}{4}-1} \left(a_1 r_1^j + a_2 r_2^j \right) + \frac{h_1}{2} \left(a_1 r_1^{M/4} + a_2 r_2^{M/4} \right)$$

$$I_1 = h_1 \sum_{j=1}^{\frac{M}{4}-1} \left(a_1 r_1^j \right) + h_1 \sum_{j=1}^{\frac{M}{4}-1} \left(a_2 r_2^j \right) + \frac{h_1}{2} \left(a_1 r_1^{M/4} + a_2 r_2^{M/4} \right)$$

$$I_2 = \int_{x_{i-1}+\tau}^{x_i} G^i \, dx = h_2 \left[\frac{G_{\frac{M}{4}}}{2} + G_{\frac{M}{4}+1} + G_{\frac{M}{4}+2} + \cdots + G_{\frac{2M}{4}-1} + \frac{G_{\frac{2M}{4}}}{2} \right]$$

$$I_2 = \int_{x_{i-1}+\tau}^{x_i} G^i \, dx = h_2 \left[\frac{G_{\frac{M}{4}}}{2} + \sum_{j=\frac{M}{4}+1}^{\frac{2M}{4}-1} G^j + \frac{G_{\frac{2M}{4}}}{2} \right]$$

$$I_2 = \frac{h_2}{2} \left(a_3 r_3^{M/4} + a_4 r_4^{M/4} \right) + h_2 \sum_{j=\frac{M}{4}+1}^{\frac{2M}{4}-1} \left(a_3 r_3^j + a_4 r_4^j \right) + \frac{h_2}{2} \left(a_3 r_3^{M/2} + a_4 r_4^{M/2} \right)$$

$$I_3 = \int_{x_i}^{x_i+\tau} G^i \, dx = h_1 \left[\frac{G_{\frac{M}{2}}}{2} + G_{\frac{M}{2}+1} + G_{\frac{M}{2}+2} + \cdots + G_{\frac{3M}{4}-1} + \frac{G_{\frac{3M}{4}}}{2} \right]$$

$$I_3 = \int_{x_i}^{x_i+\tau} G^i \, dx = h_1 \left[\frac{G_{\frac{M}{2}}}{2} + \sum_{j=\frac{M}{2}+1}^{\frac{3M}{4}-1} G^j + \frac{G_{\frac{3M}{4}}}{2} \right]$$

$$I_3 = \frac{h_1}{2} \left(a_5 r_1^{M/2} + a_6 r_2^{M/2} \right) + h_1 \sum_{j=\frac{M}{2}+1}^{\frac{3M}{4}-1} \left(a_5 r_1^j + a_6 r_2^j \right) + \frac{h_1}{2} \left(a_5 r_1^{3M/4} + a_6 r_2^{3M/4} \right)$$

$$I_4 = \int_{x_i+\tau}^{x_{i+1}} G^i \, dx = h_2 \left[\frac{G_{\frac{3M}{4}}}{2} + G_{\frac{3M}{4}+1} + G_{\frac{3M}{4}+2} + \cdots + G_{M-1} + \frac{G_M}{2} \right]$$

$$I_4 = \int_{x_i+\tau}^{x_{i+1}} G^i \, dx = h_2 \left[\frac{G_{\frac{3M}{4}}}{2} + G_{\frac{3M}{4}+1} + G_{\frac{3M}{4}+2} + \cdots + G_{M-1} + \frac{0}{2} \right]$$

$$I_4 = \int_{x_i+\tau}^{x_{i+1}} G^i \, dx = h_2 \left[\frac{G_{\frac{3M}{4}}}{2} + \sum_{j=\frac{3M}{4}+1}^{M-1} G^j \right]$$

$$I_4 = \frac{h_2}{2}\left(a_7 r_3^{3M/4} + a_8 r_4^{3M/4}\right) + h_2 \sum_{j=\frac{3M}{4}+1}^{M-1} \left(a_7 r_3^j + a_8 r_4^j\right)$$

$$\lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i \, dx = \lim_{M\to\infty} (I_1 + I_2 + I_3 + I_4)$$

Since the integral T_3 ~~integral~~ can be written as the sum of the integrals I_1, I_2 I_3 and I_4, we have

$$\lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i \, dx = \left(\frac{h}{b_i}\right) \tanh\left(\frac{b_i h}{2\varepsilon}\right) = \left(\frac{h}{b_i}\right) \frac{e^{\frac{b_i h}{\varepsilon}} - 1}{e^{\frac{b_i h}{\varepsilon}} + 1}$$

$$T_3 = \lim_{M\to\infty} \int_{x_{i-1}}^{x_{i+1}} G^i \, dx = \lim_{M\to\infty} T_3(\varepsilon, b_i, c_i, M) = \left(\frac{h}{b_i}\right)\left(\frac{e^{\rho_i} - 1}{e^{\rho_i} + 1}\right)$$

Finally, we proved that the numerical and analytical results converge exactly (see [11]), that is,

$$-\frac{e^{\rho_i}}{e^{\rho_i} + 1} U_{i-1} + U_i - \frac{1}{e^{\rho_i} + 1} U_{i+1} = (f_i - c_i\, U_i)\, \frac{h}{b} \left(\frac{e^{\rho_i} - 1}{e^{\rho_i} + 1}\right)$$

3. Conclusion

In this paper, we studied different finite difference methods for the convection-diffusion problem. We presented numerical behaviour of the convection-diffusion problem. We applied a uniformly convergent numerical algorithm, called Il'in-Allen-Southwell scheme, with better accuracy throughout the domain for various values of ε. At the end of the study, we showed how to construct such a method. Finally, we have constructed a uniformly convergent numerical method for the convection-diffusion problem.

Author Contributions

The author read and approved the last version of the manuscript.

Conflict of Interest

The author declares no conflict of interest.

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