Düzce University
Journal of Science & Technology

Research Article

A Note on Function Spaces with Fractional Fourier Transforms in Wiener-type Spaces

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DOI: 10.29130/dubited.1068024

ABSTRACT

The purpose of this paper is to introduce and study a function space $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ to be a linear space of functions $h \in L_w^1(\mathbb{R}^d)$ whose fractional Fourier transforms $F_\alpha h$ belong to the Wiener-type space $W(B,Y)(\mathbb{R}^d)$, where w is a Beurling weight function on $\mathbb{R}^d$. We show that this space becomes a Banach algebra with the sum norm $\|h\|_{1,w} + \|F_\alpha h\|_{W(B,Y)}$ and Θ convolution operation under some conditions. We find an approximate identity in this space and show that this space is an abstract Segal algebra with respect to $L_w^1(\mathbb{R}^d)$ under some conditions.

Keywords: *Fractional Fourier transform, convolution, Wiener-type spaces*

Kesirli Fourier Dönüşümleri Wiener-tipi Uzaylarda olan Fonksiyon Uzayları Üzerine Bir Not

ÖZET

Bu çalışmanın amacı w, $\mathbb{R}^d$ kümesi üzerinde bir Beurling ağırlık fonksiyonu olmak üzere $F_\alpha h$ kesirli Fourier dömüşümü $W(B,Y)(\mathbb{R}^d)$ Wiener-tipi uzayına ait $h \in L_w^1(\mathbb{R}^d)$ fonksiyonlarının bir vektör uzayı olan $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ fonksiyon uzayını tanıtmak ve çalışmaktır. Bu uzayın bazı koşullar altında, $\|h\|_{1,w} + \|F_\alpha h\|_{W(B,Y)}$ toplam normu ve Θ girişim işlemiyle birlikte bir Banach cebiri olduğu gösterildi. Bu uzayda bir yaklaşık birim bulundu ve bu uzayın $L_w^1(\mathbb{R}^d)$ uzayına göre bir soyut Segal cebiri olduğu gösterildi.

Anahtar Kelimeler: *Kesirli Fourier dönüşümü, girişim işlemi, Wiener-tipi uzaylar*

Received: 03/02/2022, Revised: 25/05/2022, Accepted: 20/06/2022

I. INTRODUCTION

In this paper, we study on $\mathbb{R}^d$. $C_c(\mathbb{R}^d)$ denotes the space of all continuous, complex-valued functions on $\mathbb{R}^d$ with compact support, and $C_0(\mathbb{R}^d)$ indicates the space of continuous, complex-valued functions on $\mathbb{R}^d$ which vanish at infinity, [1]. $(L^p(\mathbb{R}^d), \|.\|_p)$ denotes the usual Lebesgue spaces for $1 \leq p < \infty$. Throughout this paper, we will use Beurling weights, i.e. measurable and locally bounded functions w on $\mathbb{R}^d$ which satisfy $w(x) \geq 1$ and $w(x+y) \leq w(x)w(y)$, for all $x, y \in \mathbb{R}^d$. Let $\eta \geq 1$. A weight function w is called weighted function of regular growth if $w\left(\frac{x}{\eta}\right) \leq w(x)$ and there are constants $C > 0$ and $\sigma > 0$ such that $w(\eta x) \leq C\eta^\sigma w(x)$ for all $x \in \mathbb{R}^d$. $L_w^p(\mathbb{R}^d)$ denotes weighted Lebesgue space i.e.

$$L_w^p(\mathbb{R}^d) = \{h | hw \in L^p(\mathbb{R}^d)\},$$

for $1 \leq p < \infty$. $L_w^p(\mathbb{R}^d)$ is a Banach space with the norm $\|h\|_{p,w} = \|hw\|_p$, [2].

Let B be any subset of $\mathbb{R}^d$. χ_B indicates characteristic function of B. The space $L_{loc}^1(\mathbb{R}^d)$ is the set of all measurable functions (equivalence classes) h such that $h\chi_K \in L^1(\mathbb{R}^d)$ for any compact subset K of $\mathbb{R}^d$. This space is topological vector space with the senimorms $h \to \|h\chi_K\|_1$. A BF-space on $\mathbb{R}^d$ is a Banach space that is continuously embedded into $L_{loc}^1(\mathbb{R}^d)$, [3]. A normed space of measurable functions is called F-space, if every convergent sequence has a subsequence converging almost everywhere. If the space is complete, then it is called BF-spaces, [4]. A normed space $(A, \|.\|_A)$ of measurable functions is called solid, if for all $f \in A$ and any measurable function h satisfying $|h(x)| \leq |f(x)|$ almost everywhere, implies $h \in A$ and $\|h\|_A \leq \|f\|_A$, [4]. Let h be any function from $\mathbb{R}^d$ into $\mathbb{C}$. The translation and character (modulation) operators are defined by $T_y h(x) = h(x-y)$ and $M_\omega h(x) = \exp(i\omega x)h(x)$ for all $y, \omega \in \mathbb{R}^d$, respectively, [5]. $(X, \|.\|_X)$ is called (strongly) translation invariant if $T_y h \in X$ (and $\|T_y h\|_X = \|h\|_X$ i.e. strongly) for all $h \in X$ and $y \in \mathbb{R}^d$. The strongly character invariance similar to definition of the strongly translation invariance. A commutative Banach algebra $(B, \|.\|_B)$ that is a subset of commutative Banach algebra $(A, \|.\|_A)$ is called a Banach ideal of A if $hf \in B$ and the inequalities $\|h\|_A \leq \|h\|_B$ and $\|hf\|_B \leq \|h\|_B\|f\|_A$ hold for all $h \in B$, $f \in A$, [6]. A Banach space $(X(\mathbb{R}^d), \|.\|_X)$ of complex-valued measurable functions on $\mathbb{R}^d$ is called homogeneous Banach space if it is strongly translation invariant and the function $y \to T_y h$ from $\mathbb{R}^d$ into $X(\mathbb{R}^d)$ is continuous for $h \in X(\mathbb{R}^d)$, [7]. Let $(X, \|.\|_X)$ be a Banach algebra. $(Y, \|.\|_Y)$ is said to be an abstract Segal algebra with respect to $(X, \|.\|_X)$ if it has the following properties [8]:

1. $(Y, \|.\|_Y)$ is a Banach algebra and is a dense ideal in X.
2. There exists $M_1 > 0$ such that $\|h\|_X \leq M_1\|h\|_Y$ for all $h \in Y$.
3. There exists $M_2 > 0$ such that $\|hf\|_Y \leq M_2\|h\|_X\|f\|_Y$ for all $h, f \in Y$.

In order to introduce the Wiener-type space, let us give some expressions: For any Banach space $(B, \|.\|_B)$ there exists a homogeneous Banach space $(A, \|.\|_A)$, continuously embedded into $(C_b(\mathbb{R}^d), \|.\|_\infty)$, which is a regular Banach algebra under pointwise multiplication operation (i.e. separating points from closed sets), and which is closed under complex conjugation, such that $(B, \|.\|_B)$ is continuously embedded into topological dual of $A_0(\mathbb{R}^d) = \mathrm{A}(\mathbb{R}^d) \cap C_c(\mathbb{R}^d)$ and is a Banach module over A under pointwise multiplication operation (i.e. $\|fg\|_B \leq \|g\|_B\|f\|_A$ for all $f \in A$, $g \in B$). Here $A_0(\mathbb{R}^d)$ that is given above is a topological vector space with respect to usual inductive limit topology. Let $B_{loc}(\mathbb{R}^d)$ be the space of all $h \in A_0'(\mathbb{R}^d)$ such that $\varphi h \in B$ for all $\varphi \in A_0(\mathbb{R}^d)$, where $A_0'(\mathbb{R}^d)$ is the topological dual of $A_0(\mathbb{R}^d)$. The space $B_{loc}(\mathbb{R}^d)$ is a topological vector space with respect to the family of seminorms $h \to \|\varphi h\|_B$. Let O be any open subset of $\mathbb{R}^d$ with compact closure. Let $(Y, \|.\|_Y)$ be a solid translation invariant BF-space on $\mathbb{R}^d$. Then the Wiener-type space $W(B,Y)(\mathbb{R}^d)$ consist of all $g \in B_{loc}(\mathbb{R}^d)$ such that the mapping $G := x \to \|g\|_{B(xO)}$ belongs to the space Y, where $\|g\|_{B(xO)}$ is the restriction norm of g over xO. This space has a norm that defined as $\|g\|_{W(B,Y)} = \|G\|_Y$. The spaces B and Y are called the local and the global component of $W(B,Y)(\mathbb{R}^d)$, respectively, [3]. Let $f \in C_c(\mathbb{R}^d)$ be any non-zero window-function and $h \in B_{loc}(\mathbb{R}^d)$. The control function $K(f,h)$ is defined as $K(f,h)(y) = \|(T_y f)h\|_B$ for $y \in \mathbb{R}^d$. This function is a continuous function from $\mathbb{R}^d$ into $(0,\infty)$. Then we also define the Wiener-type space $W(B,Y)(\mathbb{R}^d)$ as

$$W(B,Y)(\mathbb{R}^d) = \{h \in B_{loc}(\mathbb{R}^d) \,|\, K(f,h) \in Y\}.$$

This space is endowed with the norm $\|h\|_{W(B,Y)} = \|K(f,h)\|_Y$, [9]. Some families of Wiener-type spaces are studied in [10–12].

Let $h \in L^1(\mathbb{R})$. The Fourier transform $\hat{h}$ (or Fh) of the function h is defined as

$$\hat{h}(\omega) = Fh(\omega) = \left(\sqrt{2\pi}\right)^{-1} \int_{-\infty}^{+\infty} h(x)\exp(-i\omega x)\, dx.$$

The fractional Fourier transform is a generalization of the Fourier transform with a paramater α. Let δ be Dirac delta function (i.e. $\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$ and $\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$). The fractional Fourier transform with angle α of $h \in L^1(\mathbb{R})$ is given by

$$F_\alpha h(x) = \int_{-\infty}^{+\infty} K_\alpha(x,y)h(y)dy$$

such that

$$K_\alpha(x,y) = \begin{cases} \sqrt{\dfrac{1 - i\cot\alpha}{2\pi}} \exp\left(\dfrac{i}{2}(x^2 + y^2)\cot\alpha - ixy\operatorname{cosec}\alpha\right), & \alpha \neq m\pi, m \in \mathbb{Z} \\ \delta(y - x), & \alpha = 2m\pi, m \in \mathbb{Z} \\ \delta(y + x), & \alpha = (2m+1)\pi, m \in \mathbb{Z}. \end{cases}$$

If we take $\alpha = \frac{\pi}{2}$, then the fractional Fourier transform coincides the Fourier transform, [13–17]. The definition of the fractional Fourier transform on $\mathbb{R}^d$ is given below [18]: Let us take $\alpha = (\alpha_1, \cdots, \alpha_d)$ such that each α_j is related to j-th coordinates of the variables of the function $K_\alpha(x,y)$, where $x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d) \in \mathbb{R}^d$. Then the fractional Fourier transform of $h \in L^1(\mathbb{R}^d)$ is

$$F_\alpha h(x) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K_\alpha(x,y)h(y)dy$$

such that

$$K_\alpha(x,y) = K_{(\alpha_1,\cdots,\alpha_d)}(x_1, \cdots, x_d; y_1, \cdots, y_d) = K_{\alpha_1}(x_1, y_1)K_{\alpha_2}(x_2, y_2) \cdots K_{\alpha_d}(x_d, y_d).$$

Throughout this paper, we get $\alpha_j \neq m\pi$, $m \in \mathbb{Z}$ for all $j = 1,2,\cdots,d$. Therefore, the fractional Fourier transform of $h \in L^1(\mathbb{R}^d)$ is taken

$$F_\alpha h(x) = \prod_{j=1}^{d} \left| \sqrt{\frac{1-i\cot\alpha_j}{2\pi}} \right| \int_{\mathbb{R}^d} h(y)\exp\left(\sum_{j=1}^{d} \frac{i}{2}\left(x_j^2 + y_j^2\right)\cot\alpha_j - i x_j y_j \mathrm{cosec}\alpha_j\right) dy. \quad (1)$$

The fractional Fourier transform $F_\alpha h$ of $h \in L^1(\mathbb{R}^d)$ belongs to $C_0(\mathbb{R}^d)$, [19]. Hence the operator F_α is an integral operator with kernel function $K_\alpha(x,y)$. Then the operator F_α is a linear operator from $L^1(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$. Let $z = (-y_1\cot\alpha_1, \cdots, -y_d\cot\alpha_d)$ for all $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$. The Θ convolution operation is defined as

$$\begin{aligned}(h\Theta f)(x) &= \int_{\mathbb{R}^d} h(y)f(x-y)\exp\left(\sum_{j=1}^{d} i y_j (y_j - x_j)\cot\alpha_j\right) dy \\ &= \int_{\mathbb{R}^d} h(y) T_y M_z f(x) dy\end{aligned}$$

for all $h, f \in L^1(\mathbb{R}^d)$, [20,21].

Let G be a locally compact Abelian group and $\hat{G}$ is dual group of G. The space $A_p(\mathrm{G})$ to be the space of $g \in L^1(\mathrm{G})$ such that $\hat{g} \in L^p(\hat{G})$ for $1 \leq p < \infty$. This space and its properties investigate in [22–25]. The weighted type of this spaces are studied in [26,27]. For the some other spaces that define by Fourier transform, we refer [28–31]. Also there are some spaces which define by other time-frequency operators, [32,33].

II. MAIN RESULTS

Definition 2.1. Let w be a weight function on $\mathbb{R}^d$. Let B and Y be a solid translation invariant BF-space on $\mathbb{R}^d$, and local and the global component of $W(B,Y)(\mathbb{R}^d)$, respectively. The set $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ consist of all functions $h \in L_w^1(\mathbb{R}^d)$ such that the fractional Fourier transforms $F_\alpha h \in W(B,Y)(\mathbb{R}^d)$. Since the space $L_w^1(\mathbb{R}^d)$ is a linear space, then $0 \in L_w^1(\mathbb{R}^d)$. By using (1), we get $F_\alpha 0 = 0$. From the linearity of space $W(B,Y)(\mathbb{R}^d)$ clearly $0 \in W(B,Y)(\mathbb{R}^d)$. This means that the zero function belongs to $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ and so the set $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is non-empty. By using the linearity of the spaces $L_w^1(\mathbb{R}^d)$ and $W(B,Y)(\mathbb{R}^d)$, and the linearity property of the operator F_α, it is easy to see that $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is a linear space. Let us define a function on this linear space as

$$\|h\|_{A_{\alpha,w}^{B,Y}} = \|h\|_{1,w} + \|F_\alpha h\|_{W(B,Y)}$$

for all $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. Since $\left(L_w^1(\mathbb{R}^d), \|.\|_{1,w}\right)$ and $\left(W(B,Y)(\mathbb{R}^d), \|.\|_{W(B,Y)}\right)$ are normed spaces, then $\|h\|_{1,w} \geq 0$ and $\|F_\alpha h\|_{W(B,Y)} \geq 0$. Then we have

$$\|h\|_{A_{\alpha,w}^{B,Y}} = \|h\|_{1,w} + \|F_\alpha h\|_{W(B,Y)} \geq 0$$

for all $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. By using the norms $\|.\|_{1,w}$ and $\|.\|_{W(B,Y)}$, and the linearity property of the operator F_α, we obtain

$$\begin{aligned}\|\lambda h\|_{A_{\alpha,w}^{B,Y}} &= \|\lambda h\|_{1,w} + \|F_\alpha \lambda h\|_{W(B,Y)} \\ &= |\lambda|\|h\|_{1,w} + |\lambda|\|F_\alpha h\|_{W(B,Y)} = |\lambda|\|h\|_{A_{\alpha,w}^{B,Y}}\end{aligned}$$

and

$$\begin{aligned}\|h+g\|_{A_{\alpha,w}^{B,Y}} &= \|h+g\|_{1,w} + \|F_\alpha(h+g)\|_{W(B,Y)} \\ &= \|h+g\|_{1,w} + \|F_\alpha h + F_\alpha g\|_{W(B,Y)} \\ &\le \|h\|_{1,w} + \|g\|_{1,w} + \|F_\alpha h\|_{W(B,Y)} + \|F_\alpha g\|_{W(B,Y)} \\ &= \|h\|_{A_{\alpha,w}^{B,Y}} + \|g\|_{A_{\alpha,w}^{B,Y}}\end{aligned}$$

for all $h, g \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ and $\lambda \in \mathbb{C}$. Let $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. If $\|h\|_{A_{\alpha,w}^{B,Y}} = 0$, then we get $h = 0$ by using the norms $\|.\|_{1,w}$ and $\|.\|_{W(B,Y)}$. If $h = 0$, then $F_\alpha h = 0$ by (1), and so $\|h\|_{A_{\alpha,w}^{B,Y}} = 0$. Since the above mentioned properties are satisfied, the function $\|.\|_{A_{\alpha,w}^{B,Y}}$ is a norm on $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. If we take $B = L^p(\mathbb{R}^d)$ and $Y = L_w^p(\mathbb{R}^d)$, then $W\left(L^p(\mathbb{R}^d), L_w^p(\mathbb{R}^d)\right)(\mathbb{R}^d) = L_w^p(\mathbb{R}^d)$, [3]. Therefore the space $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ coincides to the space $A_{\alpha,p}^{w,w}(\mathbb{R}^d)$ which is given in [21].

Theorem 2.2. The space $\left(A_{\alpha,w}^{B,Y}(\mathbb{R}^d), \|.\|_{A_{\alpha,w}^{B,Y}}\right)$ is a Banach space.

Proof. Let $(h_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. Hence, $(h_n)_{n\in\mathbb{N}}$ and $(F_\alpha h_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $L_w^1(\mathbb{R}^d)$ and $W(B,Y)(\mathbb{R}^d)$, respectively. It is well known that the spaces $L_w^1(\mathbb{R}^d)$ and $W(B,Y)(\mathbb{R}^d)$ are Banach spaces. Thus there exist $h \in L_w^1(\mathbb{R}^d)$ and $f \in W(B,Y)(\mathbb{R}^d)$ such that $\|h_n - h\|_{1,w} \to 0$ and $\|F_\alpha h_n - f\|_{W(B,Y)} \to 0$. Since $(B, \|.\|_B)$ and $(Y, \|.\|_Y)$ are solid translation invariant BF-spaces, then the space $W(B,Y)(\mathbb{R}^d)$ is also a solid translation invariant BF-space [34,35]. Besides, since the space $W(B,Y)(\mathbb{R}^d)$ is a BF-space, then the sequence $(F_\alpha h_n)_{n\in\mathbb{N}}$ that satisfies $\|F_\alpha h_n - f\|_{W(B,Y)} \to 0$ has a subsequence $(F_\alpha h_{n_k})_{n_k\in\mathbb{N}}$ that converges to the function f almost everywhere [4]. Therefore by using the inequality

$$\begin{aligned}|F_\alpha h(u) - f(u)| &= \left|F_\alpha h(u) - F_\alpha h_{n_k}(u) + F_\alpha h_{n_k}(u) - f(u)\right| \\ &\le \prod_{j=1}^{d} \left|\sqrt{\frac{1 - i\cot\alpha_j}{2\pi}}\right| \int_{\mathbb{R}^d} |(h_{n_k} - h)(t)| dt + |F_\alpha h_{n_k}(u) - f(u)| \\ &\le \prod_{j=1}^{d} \left|\sqrt{\frac{1 - i\cot\alpha_j}{2\pi}}\right| \|h_{n_k} - h\|_{1,w} + |F_\alpha h_{n_k}(u) - f(u)|,\end{aligned}$$

we may write $F_\alpha h = f$ almost everywhere. Thus $\|h_n - h\|_{A_{\alpha,w}^{B,Y}} \to 0$ and $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. This means $\left(A_{\alpha,w}^{B,Y}(\mathbb{R}^d), \|.\|_{A_{\alpha,w}^{B,Y}}\right)$ is a Banach space.

Theorem 2.3. The space $\left(A_{\alpha,w}^{B,Y}(\mathbb{R}^d), \|.\|_{A_{\alpha,w}^{B,Y}}\right)$ is a Banach algebra with Θ convolution operation.

Proof. The space $\left(A_{\alpha,w}^{B,Y}(\mathbb{R}^d), \|.\|_{A_{\alpha,w}^{B,Y}}\right)$ is a Banach space by Theorem 2.2. Let $g, h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. Then $g, h \in L_w^1(\mathbb{R}^d)$ and $F_\alpha g, F_\alpha h \in W(B,Y)(\mathbb{R}^d)$ by the definition of the space $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. Since the space $L_w^1(\mathbb{R}^d)$ is a Banach algebra with Θ convolution operation (see [21]), we have

$$\|g\Theta h\|_{1,w} \le \|g\|_{1,w}\|h\|_{1,w}. \tag{2}$$

Also, we shall write

$$|F_\alpha(g\Theta h)(u)| = \prod_{j=1}^{d}\left|\sqrt{\frac{2\pi}{1-i\cot\alpha_j}}\right| \exp\left(\sum_{j=1}^{d} -\frac{i}{2}u_j^2\cot\alpha_j\right)|F_\alpha g(u)||F_\alpha h(u)|$$
$$\leq |F_\alpha h(u)| \int_{\mathbb{R}^d} |g(t)|dt \leq |F_\alpha h(u)| \|g\|_{1,w} \tag{3}$$

by Theorem 7 in [21]. It is known that the fractional Fourier transform of a function belongs to $C_0(\mathbb{R}^d)$, [19] and so it is continuous on $\mathbb{R}^d$. Thus $F_\alpha(g\Theta h)$ is a measurable function on $\mathbb{R}^d$. Since $(B, \|.\|_B)$ and $(Y, \|.\|_Y)$ are solid translation invariant BF-spaces, then the space $W(B,Y)(\mathbb{R}^d)$ is also a solid translation invariant BF-space [34,35]. By using the solidity of the space $W(B,Y)(\mathbb{R}^d)$ and inequality (3), we obtain $F_\alpha(g\Theta h) \in W(B,Y)(\mathbb{R}^d)$ and

$$\|F_\alpha(g\Theta h)\|_{W(B,Y)} \leq \|F_\alpha h\|g\|_{1,w}\|_{W(B,Y)} = \|g\|_{1,w}\|F_\alpha h\|_{W(B,Y)}. \tag{4}$$

Combining (2) and (4), we get

$$\begin{aligned}\|g\Theta h\|_{A_{\alpha,w}^{B,Y}} &= \|g\Theta h\|_{1,w} + \|F_\alpha(g\Theta h)\|_{W(B,Y)} \\ &\leq \|g\|_{1,w}\|h\|_{1,w} + \|g\|_{1,w}\|F_\alpha h\|_{W(B,Y)} \leq \|g\|_{A_{\alpha,w}^{B,Y}}\|h\|_{A_{\alpha,w}^{B,Y}}.\end{aligned} \tag{5}$$

Theorem 2.4. The space $\left(A_{\alpha,w}^{B,Y}(\mathbb{R}^d), \|.\|_{A_{\alpha,w}^{B,Y}}\right)$ is a Banach ideal on $L_w^1(\mathbb{R}^d)$ with Θ convolution operation.

Proof. Let $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ and $g \in L_w^1(\mathbb{R}^d)$. By the definition of the space $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$, clearly $h \in L_w^1(\mathbb{R}^d)$. Then we have the inequality (2). By using the inequality (3) and solidity of the space $W(B,Y)(\mathbb{R}^d)$, we get $F_\alpha(g\Theta h) \in W(B,Y)(\mathbb{R}^d)$ and the inequality (4). Hence, by combining (2) and (4), we obtain

$$\|g\Theta h\|_{A_{\alpha,w}^{B,Y}} \leq \|g\|_{1,\mathrm{w}}\|h\|_{A_{\alpha,w}^{B,Y}}.$$

Besides, by the definition of the norm $\|.\|_{A_{\alpha,w}^{B,Y}}$, we have $\|h\|_{1,\mathrm{w}} \leq \|h\|_{A_{\alpha,w}^{B,Y}}$. Thus, the space $\left(A_{\alpha,w}^{B,Y}(\mathbb{R}^d), \|.\|_{A_{\alpha,w}^{B,Y}}\right)$ is a Banach ideal on $L_w^1(\mathbb{R}^d)$.

Proposition 2.5. Let w be a weight function of regular growth on $\mathbb{R}^d$. If $C_c(\mathbb{R}^d) \subset W(B,Y)(\mathbb{R}^d)$, then $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is dense in $L_w^1(\mathbb{R}^d)$.

Proof. Let us take a set $F_{0,w}^\alpha(\mathbb{R}^d) = \{g \in L_w^1(\mathbb{R}^d) | F_\alpha g \in C_c(\mathbb{R}^d)\}$. Then it is known that the set $F_{0,w}^\alpha(\mathbb{R}^d)$ is dense in $L_w^1(\mathbb{R}^d)$ by Corollary 2.14 in [36]. Since $C_c(\mathbb{R}^d) \subset W(B,Y)(\mathbb{R}^d)$, then we get

$$F_{0,w}^\alpha(\mathbb{R}^d) \subset A_{\alpha,w}^{B,Y}(\mathbb{R}^d) \subset L_w^1(\mathbb{R}^d).$$

By using this inclusion and the density of $F_{0,w}^\alpha(\mathbb{R}^d)$ in $L_w^1(\mathbb{R}^d)$, it is easy to see that $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is dense in $L_w^1(\mathbb{R}^d)$.

Proposition 2.6. Let w be a weight function of regular growth on $\mathbb{R}^d$. If $C_c(\mathbb{R}^d) \subset W(B,Y)(\mathbb{R}^d)$, then $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is an abstract Segal algebra with respect to $L_w^1(\mathbb{R}^d)$.

Proof. The space $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is a Banach algebra and also is a Banach ideal on $L_w^1(\mathbb{R}^d)$, in addition the inequality $\|g\Theta h\|_{A_{\alpha,w}^{B,Y}} \leq \|g\|_{1,w}\|h\|_{A_{\alpha,w}^{B,Y}}$ holds for all $g, h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ by Theorem 2.3 and Theorem

2.4. Furthermore, from the structure of the norm $\|.\|_{A_{\alpha,w}^{B,Y}}$, we may write an inequality $\|h\|_{1,w} \leq \|h\|_{A_{\alpha,w}^{B,Y}}$ for all $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. Finally, it is shown that $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is dense in $L_w^1(\mathbb{R}^d)$ by Proposition 2.5. Thus under the given conditions, $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is an abstract Segal algebra with respect to $L_w^1(\mathbb{R}^d)$.

Theorem 2.7. Let B be a strongly character invariant space on $\mathbb{R}^d$. Suppose that translation and character operators are continuous in B and also $C_c(\mathbb{R}^d)$ is dense in Y. Let $z = (-y_1\cot\alpha_1, \ldots, -y_d\cot\alpha_d)$ for all $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$.

1. $T_y M_z h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ and

$$\|T_y M_z h\|_{A_{\alpha,w}^{B,Y}} \leq w(y)\|h\|_{A_{\alpha,w}^{B,Y}}$$

for all $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$.

2. Assume that $C_c(\mathbb{R}^d) \cap A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is dense in $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. Then the mapping $y \to T_y M_z h$ from $\mathbb{R}^d$ into $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is continuous.

Proof. 1. Let $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$. Then the definition of $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ implies $h \in L_w^1(\mathbb{R}^d)$ and $F_\alpha h \in W(B,Y)(\mathbb{R}^d)$. It is well known that the space $L_w^1(\mathbb{R}^d)$ is translation and character invariant space and the inequality $\|T_y h\|_{1,w} \leq w(y)\|h\|_{1,w}$ holds for all $y \in \mathbb{R}^d$. Therefore we shall write

$$\|T_y M_z h\|_{1,w} \leq w(y)\|h\|_{1,w}. \tag{6}$$

Let us take $v = (-y_1\operatorname{cosec}\alpha_1, \cdots, -y_d\operatorname{cosec}\alpha_d)$ for all $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$. Thus we have

$$F_\alpha(T_y M_z h)(u) = \exp\left(\sum_{j=1}^{d} \frac{i}{2} y_j^2 \cot\alpha_j\right) M_v F_\alpha h(u) \tag{7}$$

by the equality (2.55) in [36]. Since B is strongly character invariant, then $W(B,Y)(\mathbb{R}^d)$ is also strongly character invariant by Corollary 1.4 in [35]. Hence we obtain

$$\exp\left(\sum_{j=1}^{d} \frac{i}{2} y_j^2 \cot\alpha_j\right) M_v F_\alpha h \in W(B,Y)(\mathbb{R}^d)$$

and

$$\begin{aligned}\|F_\alpha(T_y M_z \mathrm{h})\|_{W(B,Y)} &= \left|\exp\left(\sum_{j=1}^{d} \frac{i}{2} y_j^2 \cot\alpha_j\right)\right| \|M_v F_\alpha h\|_{W(B,Y)} \\ &= \|M_v F_\alpha h\|_{W(B,Y)} = \|F_\alpha h\|_{W(B,Y)}.\end{aligned} \tag{8}$$

Consequently, combininig (6) and (8), we get

$$\|T_y M_z h\|_{A_{\alpha,w}^{B,Y}} \leq w(y)\|h\|_{A_{\alpha,w}^{B,Y}}.$$

2. We will show continuity at 0. Assume that $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ and $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that $\lim_{n\to\infty} y_n = 0$. Let $z = (-y_1\cot\alpha_1, \cdots, -y_d\cot\alpha_d)$ for all $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$. It is known that the mapping $y \to T_y M_z \mathrm{h}$ is continuous from $\mathbb{R}^d$ into $L_w^1(\mathbb{R}^d)$ by Theorem 2.1 in [36]. Now, let us take the sequences

$(z_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ where j sequences of coordinates $z_{nj} = -y_{nj}\cot\alpha_j$ and $v_{nj} = -y_{nj}\operatorname{cosec}\alpha_j$. By the continuity of $y \to T_y M_z$h, we shall write

$$\|T_{y_n} M_{z_n} h - h\|_{1,w} \to 0 \tag{9}$$

as n approaches infinity. From the equality (6), we get

$$\begin{aligned}\left\|F_\alpha\left(T_{y_n} M_{z_n} h - h\right)\right\|_{W(B,Y)} &= \left\|F_\alpha\left(T_{y_n} M_{z_n} h\right) - F_\alpha h\right\|_{W(B,Y)} \\ &\leq \left\|\exp\left(\sum_{j=1}^{d} \frac{i}{2} y_{nj}^2 \cot\alpha_j\right) M_{v_n} F_\alpha h \right. \\ &\quad \left. - \exp\left(\sum_{j=1}^{d} \frac{i}{2} y_{nj}^2 \cot\alpha_j\right) F_\alpha h\right\|_{W(B,Y)} \\ &\quad + \left\|\exp\left(\sum_{j=1}^{d} \frac{i}{2} y_{nj}^2 \cot\alpha_j\right) F_\alpha h - F_\alpha h\right\|_{W(B,Y)} \\ &= \left\|M_{v_n} F_\alpha h - F_\alpha h\right\|_{W(B,Y)} \\ &\quad + \left|\exp\left(\sum_{j=1}^{d} \frac{i}{2} y_{nj}^2 \cot\alpha_j\right) - 1\right| \|F_\alpha h\|_{W(B,Y)}.\end{aligned} \tag{10}$$

Let us take $v = (-y_1 \csc\alpha_1, \cdots, -y_d \csc\alpha_d)$ for all $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$. Obviously, the mapping $y \to v$ from $\mathbb{R}^d$ into $\mathbb{R}^d$ is continuous. Using that is given in the hypothesis, we say that the mapping $y \to M_y$h from $\mathbb{R}^d$ into $W(B,Y)(\mathbb{R}^d)$ is continuous (see Lemma 1.5 in [35]). Therefore the composition mapping $y \to M_v$h from $\mathbb{R}^d$ into $W(B,Y)(\mathbb{R}^d)$ is continuous. In the other words, we can write

$$\left\|M_{v_n} F_\alpha h - F_\alpha h\right\|_{W(B,Y)} \to 0 \tag{11}$$

as n approaches infinity. Let us define $p_n = \exp\left(\sum_{j=1}^{d} \frac{i}{2} y_{nj}^2 \cot\alpha_j\right) - 1$ for all $n \in \mathbb{N}$. By using convergence of the sequence $(y_n)_{n\in\mathbb{N}}$ to zero, we get $|p_n| \to 0$ as n approaches infinity. By combining (9), (10) and (11) we obtain

$$\begin{aligned}\left\|T_{y_n} M_{z_n} h - h\right\|_{A_{\alpha,w}^{B,Y}} &= \left\|T_{y_n} M_{z_n} h - h\right\|_{1,w} + \left\|F_\alpha\left(T_{y_n} M_{z_n} h - h\right)\right\|_{W(B,Y)} \\ &\leq \left\|T_{y_n} M_{z_n} h - h\right\|_{1,w} + \left\|M_{v_n} F_\alpha h - F_\alpha h\right\|_{W(B,Y)} \\ &\quad + |p_n| \|F_\alpha h\|_{W(B,Y)} \to 0\end{aligned}$$

as n approaches infinity. This means that the function $y \to T_y M_z h$ is continuous at 0. Let us take any fixed point $y^* = (y_1^*, \cdots, y_d^*) \in \mathbb{R}^d$. Hence we get

$$T_{y-y^*} M_{z-z^*}(T_{y^*} M_{z^*} h)(x) = \exp(iy^* z - iy^* z^*) T_y M_z h(x),$$

where $z^* = (-y_1^* \cot\alpha_1, \cdots, -y_d^* \cot\alpha_d)$ for all $x \in \mathbb{R}^d$ by the proof of Theorem 2.17 (2) in [36]. Therefore, we may write

$$\left\|T_y M_z h - T_{y^*} M_{z^*} h\right\|_{A_{\alpha,w}^{B,Y}} = \left\|\exp(iy^* z^* - iy^* z) T_{y-y^*} M_{z-z^*}(T_{y^*} M_{z^*} h) - T_{y^*} M_{z^*} h\right\|_{A_{\alpha,w}^{B,Y}}.$$

Let us take $T_{y^*} M_{z^*} h = g$. Then $g \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ by the first part of this theorem. Thus we have

$$\|T_yM_zh - T_{y^*}M_{z^*}h\|_{A_{\alpha,w}^{B,Y}} = \|\exp(iy^*z^* - iy^*z)T_{y-y^*}M_{z-z^*}g - g\|_{A_{\alpha,w}^{B,Y}}$$
$$\leq \|T_{y-y^*}M_{z-z^*}g - g\|_{A_{\alpha,w}^{B,Y}}$$
$$+\|g\|_{A_{\alpha,w}^{B,Y}}|\exp(iy^*z) - \exp(iy^*z^*)|.$$

Let $\varepsilon > 0$ be given. By using continuity of the function $y \to \exp(iy^*z)$ from $\mathbb{R}^d$ into $\mathbb{C}$ and continuity at zero of the function $y \to T_yM_zh$, there exists $\delta > 0$ such that

$$\|T_yM_zh - T_{y^*}M_{z^*}h\|_{A_{\alpha,w}^{B,Y}} < \varepsilon$$

when $\|y - y^*\| < \delta$. Since y^* is an arbitrary fixed point, then the function $y \to T_yM_zh$ is continuous on $\mathbb{R}^d$.

Proposition 2.8. Assume that all the hypotheses given in Theorem 2.7 are satisfied. Let $C_c(\mathbb{R}^d)$ be a dense subset of $W(B,Y)(\mathbb{R}^d)$ and w be a weight function of regular growth on $\mathbb{R}^d$. Then $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ has an approximate identity with compactly supported fractional Fourier transforms.

Proof. Let us define a set $H = \{h_1, h_2, \cdots, h_k\}$ such that $h_j \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ for all $j = 1,2,\cdots,k$. Let $h \in A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ and $z = (-y_1\cot\alpha_1, \cdots, -y_d\cot\alpha_d)$ for all $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$. It is shown that the function $y \to T_yM_z$h from $\mathbb{R}^d$ into $A_{\alpha,w}^{B,Y}(\mathbb{R}^d)$ is continuous by Theorem 2.7. Let $\varepsilon > 0$ be given. By the continuity of $y \to T_yM_z$h, there exist positive δ_j such that

$$\|T_yM_zh_j - h_j\|_{A_{\alpha,w}^{B,Y}} < \frac{\varepsilon}{2}$$

whenever $\|y\| < \delta_j$ for all $j = 1,2,\cdots,k$. Let $\delta = \min\{\delta_j | j = 1,2,\cdots,k\}$. Then we get

$$\|T_yM_zh_j - h_j\|_{A_{\alpha,w}^{B,Y}} < \frac{\varepsilon}{2} \tag{12}$$

whenever $\|y\| < \delta$ for all $j = 1,2,\cdots,k$. Let $g \in C_c(\mathbb{R}^d) \subset L_w^1(\mathbb{R}^d)$ be a positive function that

$$\text{supp}g \subset \{x \in \mathbb{R}^d | \|x\| < \delta\}$$

and $\int_{\mathbb{R}^d} g(x)\,dx = 1$. Therefore, by the definition of Θ convolution, we shall write

$$(g\Theta h_j)(x) - h_j(x) = \int_{\mathbb{R}^d} g(y)T_yM_zh_j(x)\,dy - h_j(x) = \int_{\mathbb{R}^d} g(y)\big(T_yM_zh_j(x) - h_j(x)\big)\,dy$$

for all $x \in \mathbb{R}^d$ and $j = 1,2,\cdots,k$. By using (12), we obtain

$$\begin{aligned}\|g\Theta h_j - h_j\|_{A_{\alpha,w}^{B,Y}} &= \left\|\int_{\mathbb{R}^d} g(y)\big(T_yM_zh_j - h_j\big)\,dy\right\|_{A_{\alpha,w}^{B,Y}} \\ &\leq \int_{\text{supp}g} |g(y)|\|T_yM_zh_j - h_j\|_{A_{\alpha,w}^{B,Y}}\,dy \\ &< \frac{\varepsilon}{2}\int_{\text{supp}g}|g(y)|\,dy = \frac{\varepsilon}{2}\end{aligned} \tag{13}$$

for all $j = 1,2,\cdots,k$. Let $K = \max\{\|h_j\|_{A^{B,Y}_{\alpha,w}} \mid j = 1,2,\cdots,k\}$. Let us take the set $F^{\alpha}_{0,w}(\mathbb{R}^d) = \{g \in L^1_w(\mathbb{R}^d) | F_\alpha g \in C_c(\mathbb{R}^d)\}$. Then it is known that the set $F^{\alpha}_{0,w}(\mathbb{R}^d)$ is dense in $L^1_w(\mathbb{R}^d)$ by Corollary 2.14 in [36]. From this density, there exists a function $f \in F^{\alpha}_{0,w}(\mathbb{R}^d)$ where

$$\|g - f\|_{1,w} < \frac{\varepsilon}{2K}. \quad (14)$$

Since $C_c(\mathbb{R}^d)$ is a subset of $W(B,Y)(\mathbb{R}^d)$, then $f \in A^{B,Y}_{\alpha,w}(\mathbb{R}^d)$. By using (13) and (14), we get

$$\begin{aligned}\|f\Theta h_j - h_j\|_{A^{B,Y}_{\alpha,w}} &\leq \|f\Theta h_j - g\Theta h_j\|_{A^{B,Y}_{\alpha,w}} + \|g\Theta h_j - h_j\|_{A^{B,Y}_{\alpha,w}} \\ &\leq \|g - f\|_{1,w}\|h_j\|_{A^{B,Y}_{\alpha,w}} + \|g\Theta h_j - h_j\|_{A^{B,Y}_{\alpha,w}} < \varepsilon\end{aligned}$$

for all $j = 1,2,\cdots,k$. Hence, for every infinite subset $H = \{h_1, h_2, \cdots, h_k\}$ of $A^{B,Y}_{\alpha,w}(\mathbb{R}^d)$ and every $\varepsilon > 0$ there exists a function $f \in F^{\alpha}_{0,w}(\mathbb{R}^d)$ such that

$$\|f\Theta h_j - h_j\|_{A^{B,Y}_{\alpha,w}} < \varepsilon \quad (15)$$

for all $j = 1,2,\cdots,k$. Therefore, there exists an approximate identity of $A^{B,Y}_{\alpha,w}(\mathbb{R}^d)$ that is defined by functions $f \in F^{\alpha}_{0,w}(\mathbb{R}^d)$ which ensure inequality (15) for every infinite subset $H = \{h_1, h_2, \cdots, h_k\}$ of $A^{B,Y}_{\alpha,w}(\mathbb{R}^d)$ and every $\varepsilon > 0$, by Proposition 1.3 in [37]. This means $A^{B,Y}_{\alpha,w}(\mathbb{R}^d)$ has an approximate identity with compactly supported fractional Fourier transforms.

III. CONCLUSION

In this study, we investigate a subalgebra of $L^1_w(\mathbb{R}^d)$ (with Θ convolution operation) that fractional Fourier transforms of its elements belong to $W(B,Y)(\mathbb{R}^d)$. Let G be a locally compact abelian group and $\widehat{\mathrm{G}}$ be the dual group of G. It is known that the space $A^{B,Y}_w(\mathrm{G})$ consisting of all functions $h \in L^1_w(\mathrm{G})$ whose Fourier transforms belong to Wiener-type spaces $W(B,Y)$, [35]. Let us take $\alpha_j = \frac{\pi}{2}$ for all $j = 1,2,\cdots,d$ such that $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$. Therefore, the Θ convolution opeator and the fractional Fourier transform coincide the usual convolution and the Fourier transform, respectively. Hence the space $A^{B,Y}_{\alpha,w}(\mathbb{R}^d)$ corresponds the space $A^{B,Y}_w(\mathbb{R}^d)$ which is given in [35]. This means that this study extend some results of [35] for $\mathrm{G} = \mathbb{R}^d$.

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