



# Existence Results for BVP of a Class of Generalized Fractional-Order Implicit Differential Equations

Kadda Maazouz<sup>1</sup>, Dvivek Vivek<sup>2\*</sup>, Elsayed M. Elsayed<sup>3</sup>

## Abstract

In this paper, we study and deal with the existence of solutions to boundary value problem for implicit differential equations involving generalized fractional derivative, this study is based on the approach of Nonlinear alternative and Krasnoselskii fixed points.

**Keywords:** Boundary value problem, Fixed point, Generalized fractional derivative, Integral equation

**2010 AMS:** 26A33, 34A08, 34B15

<sup>1</sup> Department of Mathematics, University of Tiaret, Algeria

<sup>2</sup> Department of Mathematics, PSG College of Arts & Science, Coimbatore, India-641 014, ORCID: 0000-0003-0951-8060

<sup>3</sup> Department of Mathematics, Faculty of Science, Mansura 35516, Egypt, ORCID: 0000-0003-0894-8472

\*Corresponding author: peppyvivek@gmail.com

Received: 7 February 2022, Accepted: 12 September 2022, Available online: 30 September 2022

## 1. Introduction

Because of its wide applicability in biology, medicine and in more and more fields, the theory of fractional differential equations has recently been attracting increasing interest. Especially, many research papers had devoted to generalized fractional differential operator, this concept of generalized integral and derivative was given through Katugampola [11, 12]. The use of Katugampola fractional derivative (KFD) is to generalize the Hadamard and Riemann-Liouville integrals and derivatives which widely discussed by many researchers, one can refer to [8, 11, 12, 22]. Anderson et al. [1] studied some properties of KFD with potential application in quantum mechanics. In [8], Janaki et al. established existence and uniqueness of solutions to the impulsive differential equations with inclusions, and the authors also established some conditions for the uniqueness and existence of solutions for a class of fractional implicit differential equations with KFD [9]. Recently, Vivek et al. [22] investigated existence and stability of solutions for impulsive type integro-differential equations. Followed by the work, the existence and Ulam stability of solutions for impulsive type pantograph equations was considered in [23].

As a result of unifying different techniques for initial or boundary conditions, nonlinear boundary conditions received more and more attention, see [5, 6, 10], [13]-[18].

In this paper, we consider the following boundary value problem for implicit differential equations with KFD of the form

$$\begin{cases} {}^{\rho}D^{\alpha}u(t) = \Psi(t, u(t), {}^{\rho}D^{\alpha}u(t)), & t \in J := [a, b], & 1 < \alpha < 2, & \rho > 0, \\ c_1u(a) - d_1u'(a) = u_1, \\ c_2u(b) - d_2u'(b) = u_2, \end{cases} \quad (1.1)$$

where  ${}^{\rho}D^{\alpha}$  is the generalized fractional derivative of order  $\alpha$ ,  $\Psi : J \times R \times R \rightarrow R$ , is given function,  $c_1, c_2, d_1, d_2, u_1, u_2 \in R$  and  $0 \leq a < b < \infty$ .

The paper is organized as follows: In Section 2, we present definitions, lemmas, and some results. Section 3 is devoted to establish our main results. Finally, two explanatory examples are given to illustrate the theoretical results.

## 2. Fundamental Results

We now introduce some definitions, preliminary facts about the fractional calculus, notations, and some auxiliary results, which will be used later.

**Definition 2.1.** [12] *The generalized left-sided fractional integral of order  $\alpha \in \mathbb{C}$ , ( $Re(\alpha) > 0$ ) is defined for  $t > a$  by*

$${}^\rho I^\alpha h(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} h(s) ds, \tag{2.1}$$

if the integral exists, where  $\Gamma(\cdot)$  is the Gamma function.

**Definition 2.2.** [12] *The generalized left-sided fractional derivative, corresponding to the generalized fractional integral (2.1) is defined for  $t > a$  by*

$${}^\rho D^\alpha h(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( t^{1-\rho} \frac{d}{dt} \right) \int_a^t (t^\rho - s^\rho)^{n-\alpha-1} s^{\rho-1} h(s) ds,$$

where  $n = [\alpha] + 1$ , if the integral exists.

**Lemma 2.3.** *Let  $\alpha > 0$  and  $\rho > 0$ , then the differential equation*

$${}^\rho D^\alpha f(t) = 0,$$

has solutions

$$f(t) = a_0 + \sum_{k=1}^{n-2} a_k \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-k}, \quad a_k \in \mathbb{R}, k = 0, 1, 2, \dots, n-2; \quad n = [\alpha] + 1.$$

**Lemma 2.4.** *Let  $\alpha > 0$  and  $\rho > 0$ , then*

$${}^\rho I^\alpha \left( {}^\rho D^\alpha f(t) \right) = f(t) + a_0 + \sum_{k=1}^{n-2} a_k \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-k},$$

for some

$$a_k \in \mathbb{R}, k = 0, 1, 2, \dots, n-2; \quad n = [\alpha] + 1.$$

**Theorem 2.5.** [7] *(Nonlinear alternative)*

Let  $X$  be a Banach space with  $C \subset X$  closed and convex. Assume  $U$  is a relatively open subset of  $C$  with  $0 \in U$  and  $T : \bar{U} \rightarrow C$  is a compact. Then either,

1.  $T$  has a fixed point in  $\bar{U}$ , or
2. there is a point  $u \in \partial U$  and  $\lambda \in (0, 1)$  with  $u = \lambda Tu$ .

**Theorem 2.6.** [19] *(Krasnoselskii's fixed point theorem)*

Let  $E$  be a bounded closed convex and nonempty subset of a Banach space  $X$ . Let  $A, B$  two operators such that  $Ax + By \in E$  for every pair  $x, y \in E$ . If  $A$  is a contraction and  $B$  is completely continuous then there exists  $z \in E$  such that  $Az + Bz = z$ .

## 3. Main Results

The following lemma is essential to state and prove our main result

**Lemma 3.1.** *Let  $1 < \alpha < 2$ ,  $\rho > 0$  and  $\psi \in C(J, \mathbb{R})$  be a continuous function. Then the following boundary value problem*

$$\begin{cases} {}^\rho D^\alpha u(t) = \psi(t), & t \in J, \\ c_1 u(a) - d_1 u'(a) = u_1, \\ c_2 u(b) - d_2 u'(b) = u_2, \end{cases} \tag{3.1}$$

has a unique solution given by

$$u(t) = \frac{c_2 u_1}{c_1} - u_2 + \phi_{a,b} \sigma_t + \int_a^t K_t^\alpha(s) \psi(s) ds,$$

where

$$K_t^\alpha(s) = \frac{\rho^{\alpha-1}}{\Gamma(\alpha)} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1}, \quad \sigma_t = \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1},$$

$$\phi_{a,b} = \frac{1}{\delta} \left(\frac{c_2 u_1}{c_1} - u_2 + \int_a^b K(s) \psi(s) ds\right), \quad K(s) = c_2 K_b^\alpha(s) - d_2 \rho b^{\rho-1} K_b^{\alpha-1}(s),$$

and

$$\delta = d_2(\alpha - 1) b^{\rho-1} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-2} - c_2 \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-1}.$$

**Proof.** Let  $u$  satisfies the problem (3.1) then, by Lemmas 2.3 and 2.3 we have

$$\begin{aligned} u(t) &= a_0 + a_1 \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} + \frac{\rho^{\alpha-1}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \psi(s) ds \\ &= a_0 + a_1 \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} + \int_a^t K_t^\alpha(s) \psi(s) ds. \end{aligned}$$

Then

$$u'(t) = a_1(\alpha - 1) t^{\rho-1} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-2} + \rho t^{\rho-1} \int_a^t K_t^{\alpha-1}(s) \psi(s) ds.$$

Therefore

$$u(a) = a_0 \quad \text{and} \quad u'(a) = 0,$$

so we have

$$c_1 u(a) - d_1 u'(a) = c_1 a_0 = u_1$$

it follows that

$$a_0 = \frac{u_1}{c_1}.$$

On the other hand, we have

$$c_2 u(b) = c_2 a_0 + c_2 a_1 \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-1} + c_2 \int_a^b K_b^\alpha(s) \psi(s) ds,$$

and

$$d_2 u'(b) = d_2 a_1(\alpha - 1) b^{\rho-1} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-2} + d_2 \rho b^{\rho-1} \int_a^b K_b^{\alpha-1}(s) \psi(s) ds.$$

Then we obtain

$$\begin{aligned} c_2 u(b) - d_2 u'(b) &= c_2 a_0 + c_2 a_1 \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-1} - d_2 a_1(\alpha - 1) b^{\rho-1} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-2} \\ &\quad + \int_a^b [c_2 K_b^\alpha(s) - d_2 \rho b^{\rho-1} K_b^{\alpha-1}(s)] \psi(s) ds = u_2 \\ &= \frac{c_2 u_1}{c_1} + c_2 a_1 \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-1} - d_2 a_1(\alpha - 1) b^{\rho-1} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-2} \\ &\quad + \int_a^b [c_2 K_b^\alpha(s) - d_2 \rho b^{\rho-1} K_b^{\alpha-1}(s)] \psi(s) ds = u_2 \\ &= \frac{c_2 u_1}{c_1} - a_1 \left( d_2(\alpha - 1) b^{\rho-1} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-2} - c_2 \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-1} \right) \\ &\quad + \int_a^b [c_2 K_b^\alpha(s) - d_2 \rho b^{\rho-1} K_b^{\alpha-1}(s)] \psi(s) ds. \end{aligned}$$

From  $c_2u(b) - d_2u'(b) = u_2$  we deduce that

$$\begin{aligned} a_1 &= \frac{1}{\delta} \left( \frac{c_2u_1}{c_1} - u_2 + \int_a^b [c_2K_b^\alpha(s) - d_2\rho b^{\rho-1}K_b^{\alpha-1}(s)] \psi(s) ds \right) \\ &= \frac{1}{\delta} \left[ \frac{c_2u_1}{c_1} - u_2 + \int_a^b (c_2K_b^\alpha(s) - d_2\rho b^{\rho-1}K_b^{\alpha-1}(s)) \psi(s) ds \right] \\ &= \frac{1}{\delta} \left( \frac{c_2u_1}{c_1} - u_2 + \int_a^b K(s) \psi(s) ds \right) = \phi_{a,b}. \end{aligned}$$

Then we obtain

$$\begin{aligned} u(t) &= \frac{c_2u_1}{c_1} - u_2 + \phi_{a,b} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \int_a^t K_t^\alpha(s) \psi(s) ds \\ &= \frac{c_2u_1}{c_1} - u_2 + \phi_{a,b} \sigma_t + \int_a^t K_t^\alpha(s) \psi(s) ds. \end{aligned}$$

Then we can accomplish the purpose desired, which complete the proof.

For sake of brevity, we need the following proposition which is very useful in what follows.

**Proposition 3.2.** For  $1 < \alpha < 2$ ,  $\rho > 0$ , and  $t, s \in J$  we have

- (i)  $\int_a^t K_t^\alpha(s) ds \leq \int_a^b K_b^\alpha(s) ds = \frac{\rho^{\alpha-2}}{\Gamma(\alpha+1)} (b^\rho - a^\rho)^\alpha$
- (ii)  $\int_a^b K_b^{\alpha-1}(s) ds = \frac{\rho^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)} (b^\rho - a^\rho)^{\alpha-1}$
- (iii)  $\int_a^b |K(s)| ds \leq \frac{(b^\rho - a^\rho)^{\alpha-1} \rho^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)} (|c_2|(b^\rho - a^\rho) + |d_2|\rho b^{\rho-1}) := K^*$ .

**Proof.** The proof of (i) and (ii) is immediate, it remains to prove (iii). Indeed, we have

$$\begin{aligned} \int_a^b |K(s)| ds &= \int_a^b |c_2K_b^\alpha(s) - d_2\rho b^{\rho-1}K_b^{\alpha-1}(s)| ds \\ &\leq \frac{|c_2|\rho^{\alpha-2}}{\Gamma(\alpha+1)} (b^\rho - a^\rho)^\alpha + \frac{|d_2|\rho b^{\rho-1}\rho^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)} (b^\rho - a^\rho)^{\alpha-1} \\ &\leq \frac{(b^\rho - a^\rho)^{\alpha-1} \rho^{\alpha-2}}{\Gamma(\alpha)} \left( |c_2| \frac{b^\rho - a^\rho}{\alpha} + |d_2| \frac{\rho b^{\rho-1}}{\alpha-1} \right) \\ &\leq \frac{(b^\rho - a^\rho)^{\alpha-1} \rho^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)} (|c_2|(b^\rho - a^\rho) + |d_2|\rho b^{\rho-1}) = K^*. \end{aligned}$$

### 3.1 Existence results

Now, we are in position to first result which is based on Theorem 2.5.

**Theorem 3.3.** Assume that

- ( $\mathcal{A}_1$ )  $\Psi$  is continuous.
- ( $\mathcal{A}_2$ ) There exist constants  $k > 0$  and  $0 < l < 1$  such that

$$|\Psi(t, u_2, v_2) - \Psi(t, u_1, v_1)| \leq k|u_2 - u_1| + l|v_2 - v_1|$$

for any  $u_1, v_1, u_2, v_2 \in \mathbb{R}$ , and  $t \in J$ .

Then the problem (1.1) has at least one solution.

**Proof.** Let us consider the operator  $\chi : \mathcal{C}(J, \mathbb{R}) \rightarrow \mathcal{C}(J, \mathbb{R})$  defined by

$$(\chi u)(t) = \frac{c_2u_1}{c_1} - u_2 + \phi_{a,b} \sigma_t + \int_a^t K_t^\alpha(s) \psi(s) ds$$

where

$$\psi(s) = \Psi(s, u(s), \psi(s)).$$

**Step 1:**  $\chi$  is continuous.

Let  $\{u_n\}$  be a sequence such that  $u_n \rightarrow u$  in  $\mathcal{C}(J, R)$ . Then for each  $t \in J$ , we have

$$\begin{aligned} |(\chi u_n)(t) - (\chi u)(t)| &= \left| \frac{\sigma_t}{\delta} \int_a^b K(s)(\psi_n(s) - \psi(s)) ds \right. \\ &\quad \left. + \int_a^t K_t^\alpha(s)(\psi_n(s) - \psi(s)) ds \right| \\ &\leq \frac{\sigma_b}{|\delta|} \int_a^b |K(s)| |\psi_n(s) - \psi(s)| ds \\ &\quad + \int_a^t |K_t^\alpha(s)| |\psi_n(s) - \psi(s)| ds \end{aligned}$$

where

$$\psi_n(s) = \Psi(s, u_n(s), \psi_n(s)).$$

In virtue of  $(\mathcal{A}_2)$ , we have

$$|\psi_n(s) - \psi(s)| \leq \frac{k}{1-l} |u_n(s) - u(s)|.$$

It follows that

$$\begin{aligned} |\chi u_n(t) - \chi u(t)| &\leq \frac{k}{1-l} \left( \frac{\sigma_b K^*}{|\delta|} + \int_a^b K_b^\alpha(s) ds \right) |u_n(s) - u(s)| \\ &\leq \frac{k}{1-l} \left( \frac{\sigma_b K^*}{|\delta|} + \frac{\rho^{\alpha-2}}{\Gamma(\alpha+1)} (b^\rho - a^\rho)^\alpha \right) \|u_n - u\|_\infty. \end{aligned}$$

Since  $u_n \rightarrow u$ , we get that  $\|\chi u_n - \chi u\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . Hence  $\chi$  is continuous.

**Step 2:**  $\chi$  maps bounded sets into bounded sets in  $\mathcal{C}(J, R)$ .

It is enough to show that there exists a positive constant  $m$  for  $r > 0$  such that for each  $u \in \mathcal{D}_r = \{u \in \mathcal{C}(J, R) : \|u\|_\infty \leq r\}$  we have  $\|\chi u\|_\infty \leq m$ . Indeed for each  $t \in J$ , and  $u \in \mathcal{D}_r$  we have

$$\begin{aligned} |(\chi u)(t)| &= \left| \frac{c_2 u_1}{c_1} - u_2 + \phi_{a,b} \sigma_t + \int_a^t K_t^\alpha(s) \psi(s) ds \right| \\ &\leq \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + |\phi_{a,b}| \sigma_b + \int_a^t |K_t^\alpha(s)| |\psi(s)| ds. \end{aligned}$$

According to  $(\mathcal{A}_2)$  we have

$$\begin{aligned} |\psi(s)| &= |\Psi(s, u(s), \psi(s)) - \Psi(s, 0, 0) + \Psi(s, 0, 0)| \\ &\leq \frac{k \|u\|_\infty + \sup_{s \in J} |\Psi(s, 0, 0)|}{1-l} \\ &\leq \frac{kr + \Psi^*}{1-l}, \quad \text{where } \Psi^* = \sup_{s \in J} |\Psi(s, 0, 0)|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\phi_{a,b}| &= \left| \frac{1}{\delta} \left( \frac{c_2 u_1}{c_1} - u_2 + \int_a^b K(s) \psi(s) ds \right) \right| \\ &\leq \frac{1}{|\delta|} \left( \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + \int_a^b |K(s)| |\psi(s)| ds \right) \\ &\leq \frac{1}{|\delta|} \left( \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + \frac{kr + \Psi^*}{1-l} \int_a^b |K(s)| ds \right) \\ &\leq \frac{1}{|\delta|} \left( \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + \frac{(kr + \Psi^*) K^*}{1-l} \right) := \phi_{a,b}^*. \end{aligned}$$

Then,

$$\begin{aligned} |(\chi u)(t)| &\leq \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + \phi_{a,b}^* \sigma_b + \frac{kr + \Psi^*}{1-l} \int_a^t K_t^\alpha(s) ds \\ &\leq \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + \phi_{a,b}^* \sigma_b + \frac{kr + \Psi^*}{1-l} \int_a^b K_b^\alpha(s) ds := m. \end{aligned}$$

It follows that

$$\|\chi u\|_\infty \leq m$$

which implies that  $\chi$  maps bounded sets into bounded sets of  $\mathcal{C}(J, R)$ .

**Step 3:**  $\chi$  maps bounded sets into a equicontinuous set of  $\mathcal{C}(J, R)$ .

Let  $u \in \mathcal{D}_r$  (as defined in **Step 2**), and  $t_1, t_2 \in J$  with  $t_1 < t_2$ , then

$$\begin{aligned} &|\chi u(t_2) - \chi u(t_1)| \\ &\leq |\phi_{a,b}^* \sigma_{t_2} - \sigma_{t_1}| + \left| \int_a^{t_2} K_{t_2}^\alpha(s) \psi(s) ds - \int_a^{t_1} K_{t_1}^\alpha(s) \psi(s) ds \right| \\ &\leq \phi_{a,b}^* |\sigma_{t_2} - \sigma_{t_1}| + \left| \int_a^{t_1} (K_{t_2}^\alpha - K_{t_1}^\alpha)(s) \psi(s) ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s) \psi(s) ds \right| \\ &\leq \phi_{a,b}^* |\sigma_{t_2} - \sigma_{t_1}| + \frac{(k\|u\|_\infty + \Psi^*) \rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)} \left| \int_a^{t_1} (K_{t_2}^\alpha - K_{t_1}^\alpha)(s) ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s) ds \right| \\ &\leq \phi_{a,b}^* |\sigma_{t_2} - \sigma_{t_1}| + \frac{(kr + \Psi^*) \rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)} \left[ 2(t_2^\rho - t_1^\rho)^\alpha + t_1^{\rho\alpha} - t_2^{\rho\alpha} \right]. \end{aligned}$$

As  $t_2 \rightarrow t_1$  the right-hand side of above inequality tends to zero. As a sequence of Steps 1 to 3 together with Arzelà-Ascoli theorem, we conclude that  $\chi$  is completely continuous.

**Step 4:** A priori bounds.

We show there exists an open set  $\mathcal{O} \subset \mathcal{C}(J, R)$  with  $u \neq \lambda \chi(u)$  where  $\lambda \in (0, 1)$  and  $u \in \partial \mathcal{O}$ . Let  $u \in \mathcal{C}(J, R)$  and  $u = \lambda \chi(u)$ , with  $\lambda \in (0, 1)$ , then for each  $t \in J$  we have

$$\begin{aligned} |u(t)| &= \lambda \left| \frac{c_2 u_1}{c_1} + u_2 + \phi_{a,b} \sigma_t + \int_a^t K_t^\alpha(s) \psi(s) ds \right| \\ &\leq \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + |\phi_{a,b} \sigma_b| + \int_a^b K_b^\alpha(s) |\psi(s)| ds \\ &\leq \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + \phi_{a,b}^* \sigma_b + \frac{kr + \Psi^*}{1-l} \int_a^b K_b^\alpha(s) ds. \end{aligned}$$

Thus

$$\|u\|_\infty \leq m.$$

Let

$$\mathcal{O} = \{u \in \mathcal{C}(J, R) : \|u\|_\infty < m + 1\}.$$

By our choosing of  $\mathcal{O}$ , there is no  $u \in \partial \mathcal{O}$ , such that  $u = \lambda \chi(u)$ , for  $\lambda \in (0, 1)$ . As a consequence of Theorem 3.3 and the nonlinear alternative of Leray-Schauder's fixed point theorem,  $\chi$  has a fixed point  $u \in \mathcal{O}$  which is a solution of our problem (1.1).

The second result is based on Theorem 2.6.

**Theorem 3.4.** Assume that  $(\mathcal{A}_1)$ ,  $(\mathcal{A}_2)$ , and

$$\theta = \frac{k\sigma_b K^*}{|\delta|(1-l)} < 1. \tag{3.2}$$

Then the problem (1.1) has at least one solution.

**Proof.** Let

$$\mathcal{M} = \{u \in \mathcal{C}(J, \mathbb{R}) : \|u\|_\infty \leq r_1 + r_2 \leq r\},$$

where

$$r_1 = \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + \phi_{a,b}^* \sigma_b, \quad r_2 = \frac{(kr + \Psi^*)(b^\rho - a^\rho)^\alpha \rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)}.$$

We define two operators  $S_1$  and  $S_2$  by

$$S_1 u(t) = \frac{c_2 u_1}{c_1} - u_2 + \phi_{a,b} \sigma_t$$

$$S_2 u(t) = \int_a^t K_t^\alpha(s) \Psi(s) ds$$

where

$$\Psi(s) = \Psi(s, u(s), \psi(s)).$$

**Step 1:** We will show that  $S_1 u + S_2 v \in \mathcal{M}$ .

Let  $u, v \in \mathcal{M}$ , and  $t \in J$  so we have

$$\begin{aligned} |S_1 u(t)| &\leq \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + |\phi_{a,b}| \sigma_t \\ &\leq \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + |\phi_{a,b}| \sigma_b \\ &\leq \left| \frac{c_2 u_1}{c_1} \right| + |u_2| + \phi_{a,b}^* \sigma_b \\ &\leq r_1, \end{aligned}$$

and

$$\begin{aligned} |S_2 v(t)| &\leq \int_a^t K_t^\alpha(s) |\Psi(s)| ds \\ &\leq \frac{(kr + \Psi^*)}{1-l} \int_a^b K_b^\alpha(s) ds \\ &\leq \frac{(kr + \Psi^*)(b^\rho - a^\rho)^\alpha \rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)} \\ &\leq r_2. \end{aligned}$$

Therefore

$$\begin{aligned} \|S_1 u + S_2 v\|_\infty &\leq \|S_1 u\|_\infty + \|S_2 v\|_\infty \\ &\leq r_1 + r_2 \\ &\leq r. \end{aligned}$$

We deduce that  $S_1 u + S_2 v \in \mathcal{M}$ .

**Step 2:**  $S_1$  is a contraction on  $\mathcal{M}$ .

For each  $t \in J$ ,  $u, v \in \mathcal{M}$ ,  $\Psi(s) = \Psi(s, u(s), \psi(s))$ , and  $\phi(s) = \Psi(s, v(s), \phi(s))$ , we have

$$\begin{aligned} |S_1 u(t) - S_1 v(t)| &= \left| \frac{\sigma_t}{\delta} \int_a^b K(s) (\Psi(s) - \phi(s)) ds \right| \\ &\leq \frac{\sigma_b}{|\delta|} \int_a^b |K(s)| |\Psi(s) - \phi(s)| ds \\ &\leq \frac{k\sigma_b}{|\delta|(1-l)} \int_a^b |K(s)| |u(s) - v(s)| ds \\ &\leq \frac{k\sigma_b K^*}{|\delta|(1-l)} |u(s) - v(s)|. \end{aligned}$$

Therefore

$$\|S_1 u - S_1 v\|_\infty \leq \frac{k\sigma_b K^*}{|\delta|(1-l)} \|u - v\|_\infty.$$

By (3.2) we deduce that  $S_1$  is a contraction.

**Step 3:**  $S_2$  is compact.

It is clear that  $S_2$  is continuous and uniformly bounded on  $\mathcal{M}$  ( $\|S_2 u\|_\infty \leq r_2$ ).

It remains to show that  $S_2$  maps bounded sets into a equicontinuous set of  $\mathcal{C}(J, R)$ .

Let  $u \in \mathcal{M}$ , and  $t_1, t_2 \in J$  with  $t_1 < t_2$ , then

$$\begin{aligned} |S_2 u(t_2) - S_2 u(t_1)| &= \left| \int_a^{t_2} K_{t_2}^\alpha(s) \Psi(s) ds - \int_a^{t_1} K_{t_1}^\alpha(s) \Psi(s) ds \right| \\ &= \left| \int_a^{t_1} (K_{t_2}^\alpha - K_{t_1}^\alpha)(s) \Psi(s) ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s) \Psi(s) ds \right| \\ &\leq \frac{(k\|u\|_\infty + \Psi^*)\rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)} \left| \int_a^{t_1} (K_{t_2}^\alpha - K_{t_1}^\alpha)(s) ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s) ds \right| \\ &\leq \frac{(kr + \Psi^*)\rho^{\alpha-2}}{(1-l)\Gamma(\alpha+1)} [2(t_2^\rho - t_1^\rho)^\alpha + t_1^{\rho\alpha} - t_2^{\rho\alpha}]. \end{aligned}$$

It is obvious that since  $t_2 \rightarrow t_1$  we get  $|S_2 u(t_2) - S_2 u(t_1)| \rightarrow 0$ . It means that  $S_2$  is compact. By Theorem 3.4 we conclude that our problem (1.1) has a solution in  $\mathcal{C}(J, R)$ .

## 4. Examples

**Example 4.1.** Let us consider the following boundary problem

$$\begin{cases} \frac{1}{3} D^{\frac{3}{2}} u(t) = \frac{|u(t)|}{5+|u(t)|} + \frac{1}{2} \tan \left| \frac{1}{3} D^{\frac{3}{2}} u(t) \right|, & t \in [0, \frac{\pi}{3}], \\ u(0) - u'(0) = \frac{3}{2}, \\ u(\frac{\pi}{3}) + u'(\frac{\pi}{3}) = \frac{\pi}{6}. \end{cases} \tag{4.1}$$

Let the function  $\Psi$  defined by

$$\Psi(t, u, v) = \frac{u}{5+u} + \frac{1}{2} \tan v, \quad u, v \in R^+, \quad t \in [0, \frac{\pi}{3}].$$

Obviously the function  $\Psi$  is continuous. Now we check assumption  $(\mathcal{A}_2)$ . Indeed for each  $t \in [0, \frac{\pi}{3}]$  and  $u, v \in R^+$ , we have

$$\begin{aligned} |\Psi(t, u_2, v_2) - \Psi(t, u_1, v_1)| &= \left| \frac{u_2}{5+u_2} - \frac{u_1}{5+u_1} + \frac{1}{2} (\tan v_2 - \tan v_1) \right| \\ &\leq \left| \frac{5(u_2 - u_1)}{(5+u_2)(5+u_1)} \right| + \frac{1}{2} |\tan v_2 - \tan v_1| \\ &\leq \frac{1}{5} |u_2 - u_1| + \frac{2}{3} |v_2 - v_1|. \end{aligned}$$

Therefore  $(\mathcal{A}_2)$  holds for  $k = \frac{1}{5}$ , and  $l = \frac{2}{3}$ . Then according to Theorem 3.3 the problem (4.1) has at least one solution.

**Example 4.2.** Let us consider the following boundary problem

$$\begin{cases} \frac{1}{4} D^{\frac{5}{2}} u(t) = \frac{|u(t)|}{3+|u(t)|} + \left| \frac{1}{4} D^{\frac{5}{2}} u(t) \right|, & t \in [0, 1], \\ u(0) - u'(0) = 1, \\ u(1) + u'(1) = \frac{1}{2}. \end{cases} \tag{4.2}$$

Set the function  $\Psi$  as

$$\Psi(t, u, v) = \frac{u}{3+v} + \frac{v}{3+u}, \quad u, v \in R^+, \quad t \in [0, 1].$$



It is easy to see that the function  $\Psi$  is continuous. On the other hand for each  $t \in [0, 1]$  and  $u, v \in R^+$ , we have

$$\begin{aligned} |\Psi(t, u_2, v_2) - \Psi(t, u_1, v_1)| &= \left| \frac{u_2}{3+v_2} + \frac{v_2}{3+u_2} - \frac{u_1}{3+v_1} - \frac{v_1}{3+u_1} \right| \\ &\leq \left| \frac{3u_2 + u_2v_1 - 3u_1 - u_1v_2}{(3+u_2)(3+v_2)} \right| + \left| \frac{3v_2 + v_2u_1 - 3v_1 - u_2v_1}{(3+v_1)(3+u_1)} \right| \\ &\leq \frac{1}{9} (|3u_2 - 3u_1| + |3v_2 - 3v_1|) \\ &\leq \frac{1}{3} (|u_2 - u_1| + |v_2 - v_1|). \end{aligned}$$

Therefore the assumption  $(\mathcal{A}_2)$  holds for  $k = l = \frac{1}{3}$ . On the other hand we have

$$\theta = \frac{\frac{1}{3} \times 8 \times \frac{16}{15\Gamma(\frac{5}{2})}}{11 \times \frac{2}{3}} = \frac{128}{495\sqrt{\pi}} < 1$$

By Theorem 3.4 we conclude that the problem (4.2) has at least one solution.

## 5. Conclusion

In this paper, we studied some existence results of certain type of differential fractional problem involving the concept of the generalized fractional derivative, in this study we focused on Nonlinear alternative and Krasnoselskii fixed points.

## Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

## Funding

There is no funding for this work.

## Availability of data and materials

Not applicable.

## Competing interests

The authors declare that they have no competing interests.

## Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

## References

- [1] D. R. Anderson, D. J. Ulness, *Properties of Katugampola fractional derivative with potential application in quantum mechanics*, J. Math. Phys., **56** (2015).
- [2] M. S. Abdo, *Further results on the existence of solutions for generalized fractional quadratic functional integral equations*, J. Math. Anal & Model, **1**(1) (2020), 33-46.
- [3] A. Bashir, S. Sivasundaram, *Some existence results for fractional integro-differential equations with nonlinear conditions*, Comm. Appl. Analysis, **12** (2) (2008), 107-112.
- [4] B. N. Abood, S. S. Redhwan, O. Bazighifan, K. Nonlaopon, *Investigating a generalized fractional quadratic integral equation*, Fractal Fract., **6** (2022), 251.

- [5] M. Benchohra, K. Maazouz, *Existence and uniqueness results for implicit fractional differential equations with integral boundary conditions*, *Comm. App. Analysis*, **20** (2016), 355-366.
- [6] A. Cabada, K. Maazouz, *Results for Fractional Differential Equations with Integral Boundary Conditions Involving the Hadamard Derivative*, In: Area I. et al. (eds) *Nonlinear Analysis and Boundary Value Problems, NABVP 2018*, Springer Proceedings in Mathematics & Statistics, vol 292. Springer, Cham, 2019.
- [7] G. J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.
- [8] M. Janaki, K. Kanagarajan, E. M. Elsayed, *Existence criteria for Katugampola fractional type impulsive differential equations with inclusions*, *J. Math. Sci. Model.*, **2**(1) (2019), 51-63.
- [9] M. Janaki, K. Kanagarajan, D. Vivek, *Analytic study on fractional implicit differential equations with impulses via Katugampola fractional Derivative*, *Int. J. Math. Appl.*, **6**(2-A) (2018), 53-62.
- [10] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
- [11] U. N. Katugampola, *New approach to generalized fractional integral*, *Appl. Math. Comput.*, **218**(3) (2011), 860-865.
- [12] U. N. Katugampola, *New approach to generalized fractional derivative*, *Bull. Math. Anal. Appl.*, **6** (4) (2014), 1-15.
- [13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differentiatl Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B. V. Amsterdam, 2006.
- [14] V. Lakshmikantham, S. Leela, J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.
- [15] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity. An Introduction to Mathematical Models*, Imperial College Press, London, 2010.
- [16] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
- [17] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [18] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.
- [19] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge 1974, *Bull. Math. Anal. Appl.*, **6** (4) (2014), 1-15.
- [20] S. S. Redhwan, S. L. Shaikh, M. S. Abdo, *Caputo-Katugamola type implicit fractional differential equation with two-point anti-periodic boundary conditions*, *Results in Nonlinear Analysis*, **5**(1) (2022), 12-28.
- [21] S. S. Redhwan, S. L. Shaikh, M. S. Abdo, W. Shatanawi, K. Abodayeh, M. A. Almalahi, T. Aljaaidi, *Investigating a generalized Hilfer-type fractional differential equation with two-point and integral boundary conditions*, *AIMS Math.*, **7**(2) (2021), 1856-187.
- [22] D. Vivek, E. M. Elsayed, K. Kanagarajan, *Dynamics and stability results for impulsive type integro-differential equations with generalized fractional derivative*, *Math. Nat. Sci.*, **4** (2019), 1-12.
- [23] D. Vivek, K. Kanagarajan, S. Harikrishnan, *Theory and analysis of impulsive type pantograph equations with Katugampola fractionl derivative*, *J. Vabration Testing and System Dynamic*, **2** (1) 2018, 9-20.
- [24] D. Vivek, E. M. Elsayed, K. Kanagarajan, *Theory of fractional implicit differential equations with complex order*, *J. Uni. Math.*, **2**(2) (2019), 154-165.
- [25] H. Wang, *Existence results for fractional functional differential equations with impulses*, *J. Appl. Math. Comput.*, **38** (2012), 85-101.
- [26] E. Zeidler, *Nonlinear functional Analysis and its Applications-I Fixed Point Theorem*, Springer, New-York, 1993.