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A Study on Generalized Einstein Tensor for an Almost Pseudo-Ricci Symmetric Manifold

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Abstract

The object of the paper is to study the generalized Einstein tensor $G(X,Y)$ on almost pseudo-Ricci symmetric manifolds, $A(PRS)_n$. Considering the generalized Einstein tensor $G(X,Y)$ as conservative, cyclic parallel and Codazzi type, it is investigated the properties of such a manifold.

Keywords: *Almost pseudo-Ricci symmetric manifold; Codazzi tensor; Conservative; Cyclic parallel; Generalized Einstein tensor; Torqued vector field.*
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1. Introduction

The class of the most important Riemannian manifolds is symmetric Riemannian spaces. This class includes many significant examples to be of great importance for different branches of mathematics. Moreover, any symmetric space has its special geometry, like Euclidean geometry, elliptic geometry, hyperbolic geometry, etc... Further, spaces admitting some sense of symmetry play a prominent role in general relativity. For instance, without imposing a symmetry condition, solving Einstein's field equations of gravitation becomes a difficult task.

Let (M,g) be an n-dimensional Riemannian manifold and let ∇ be the Levi-Civita connection of (M,g). In 1926, the concept of locally symmetry was introduced by E. Cartan [2], namely, if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M,g), then this manifold is called locally symmetric. Later this notion has been weakened by many authors in several ways a different extend [9, 17], etc...

In 1987, the concept of the pseudo symmetric manifold was introduced by M. C. Chaki [4]. In 1992, the concept of the pseudo symmetric manifold studied by R. Deszcz [12]. But it is noted that this concept is different from that of Chaki. Recently, some necessary and sufficient conditions for a Chaki pseudo symmetric (respectively pseudo-Ricci symmetric [5]) manifold to be Deszcz pseudo symmetric (respectively Ricci-pseudo symmetric [11]) have been examined in [18].

In 1988, M. C. Chaki [5] introduced the concept of pseudo-Ricci symmetric manifolds. Based on M. C. Chaki [5], a non-flat n-dimensional Riemannian manifold (M,g), $(n>3)$ is called a *pseudo-Ricci symmetric manifold* if the Ricci tensor S of type (0,2) is not identically equal to zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2\pi(X)S(Y,Z) + \pi(Y)S(X,Z) + \pi(Z)S(Y,X), \tag{1.1}$$

where π is a non-zero 1-form, ρ is a vector field by

$$g(X,\rho) = \pi(X)$$

for all vector fields X and ∇ indicates the operator of covariant differentiation with respect to the metric g. The 1-form π is called the associated 1−form of the manifold. If $\pi = 0$, then the manifold reduces to a Ricci symmetric manifold in the sense of Chaki. In [5], Chaki showed that if the scalar curvature r is constant, then it must be equal to zero or otherwise if $r \neq 0$ then its associated 1-form π must be closed.

In 2007, Chaki and Kawaguchi [6] introduced the notion of almost pseudo-Ricci symmetric manifolds. A non-flat n-dimensional Riemannian (or Semi-Riemannian) manifold (M,g) $(n>3)$ is called an almost pseudo-Ricci symmetric manifold [6] if the Ricci tensor S of type (0,2) is not identically equal to zero and satisfies the following condition

$$(\nabla_Z S)(X,Y) = [\pi(Z) + \omega(Z)]S(X,Y) + \pi(X)S(Z,Y) + \pi(Y)S(X,Z), \tag{1.2}$$

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where π, ω are non-vanishing 1-forms such that

$$g(X,\rho)=\pi(X) \quad \text{and} \quad g(X,\nu)=\omega(X) \quad \text{for all} \quad X, \tag{1.3}$$

ρ and ν are called the basic vector fields of the manifold. The 1-forms π and ω are called the associated 1-forms of the manifold. Such a manifold will be denoted by $A(PRS)_n$. If $\omega=\pi$, then an almost pseudo-Ricci symmetric manifold reduces to a pseudo-Ricci symmetric manifold.
Let Q indicate the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S, i.e.,

$$S(X,Y)=g(QX,Y). \tag{1.4}$$

By substituting $X=Y=e_i$ into (1.2) where $\{e_i\}$ is an orthonormal bases of the tangent space of (M,g) and taking summation over i, $(1\leq i\leq n)$, we obtain

$$(\nabla_Z r)=[\pi(Z)+\omega(Z)]r+2\pi(QZ), \tag{1.5}$$

where r denotes the scalar curvature of (M,g).
In 1998, Gray [14] introduced two classes of Riemannian manifolds, which he determined by the covariant differentiation of the Ricci tensor S. One of these classes is the class including of Riemannian manifolds whose Ricci tensor S is of Codazzi type, that is to say,

$$(\nabla_X S)(Y,Z)=(\nabla_Y S)(X,Z). \tag{1.6}$$

The other is the class including of Riemannian manifolds whose Ricci tensor S is cyclic parallel, that is to say,

$$(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)=0. \tag{1.7}$$

The Einstein tensor $E(X,Y)$ for a Riemannian manifold is expressed as

$$E(X,Y)=S(X,Y)-rg(X,Y), \tag{1.8}$$

where $S(X,Y)$ and r are respectively the Ricci tensor and the scalar curvature of the manifold, plays an important part in Einstein's theory of gravitation as well as in proving some theorems in Riemannian geometry, [1, 15]. Further, the Einstein tensor can be obtained from Yano's tensor of concircular curvature. In [16], by using this approach was achieved some generalizations of the Einstein tensor.
The tensor is

$$G(X,Y)\overset{\text{def}}{=}S(X,Y)-\kappa rg(X,Y), \tag{1.9}$$

is referred to as the generalized Einstein tensor G where κ is constant [3].
This paper studies $A(PRS)_n$ manifolds admitting a generalized Einstein tensor and has organized as follows: In the premier section, it deals with $A(PRS)_n$ manifolds admitting a generalized Einstein tensor that equals zero. In the next section, it is proved some theorems about $A(PRS)_n$ manifolds admitting a generalized Einstein tensor that is conservative, cyclic parallel and Codazzi tensor, respectively.

2. G-flat $A(PRS)_n$

In this section, we consider an $A(PRS)_n$ manifold admitting a generalized Einstein tensor $G(X,Y)$ which identically equals zero.
Thus, from (1.9), we obtain

$$S(X,Y)=\kappa rg(X,Y). \tag{2.1}$$

Contraction over X and Y yields $\kappa=\frac{1}{n}$. Thus, the manifold is Einstein.
Using (2.1), we achieve

$$(\nabla_Z S)(X,Y)=\frac{1}{n}g(X,Y)dr(Z). \tag{2.2}$$

By contracting over Z and Y, from $divS=\frac{1}{2}dr(X)$ [10], we obtain,

$$dr(X)=0. \tag{2.3}$$

Combining equations (2.2) and (2.3) yields

$$(\nabla_Z S)(X,Y)=0. \tag{2.4}$$

Let us now assume that ρ is a unit torqued vector field [7, 8],

$$\nabla_X\rho=\phi X+\alpha(X)\rho, \qquad \alpha(\rho)=0, \tag{2.5}$$

where ϕ is a torqued function and a 1-form α is called a torqued form of ρ.
Due to (2.4), we obtain

$$(\nabla_Z S)(X,\rho) = 0. \tag{2.6}$$

For the left-side of the above equation, writing $(\nabla_Z S)(X,\rho) = \nabla_Z S(X,\rho) - S(\nabla_Z X,\rho) - S(X,\nabla_Z \rho)$, we get

$$\kappa r\left[(\nabla_Z \pi)(X) - g(X,\nabla_Z \rho)\right] = 0, \tag{2.7}$$

in which (1.3) and (2.1) have been used.
Let us substitute (2.5) in (2.7) and use (1.3). Since $\kappa \neq 0, r \neq 0$, we have

$$(\nabla_Z \pi)(X) - \phi g(X,Z) - \alpha(Z)\pi(X) = 0. \tag{2.8}$$

Substituting $X = \rho$ into the above equation and using $g(\rho,\rho) = 1$, thus the above equation takes the following form

$$(\nabla_Z \pi)(\rho) = \phi \pi(Z) + \alpha(Z). \tag{2.9}$$

Since ρ is a unit vector, it is

$$(\nabla_Z \pi)(\rho) = -\pi(\nabla_Z \rho). \tag{2.10}$$

From (2.5) and (2.10), we achieve

$$(\nabla_Z \pi)(\rho) = -\phi \pi(Z) - \alpha(Z). \tag{2.11}$$

Combining (2.9) and (2.11), we obtain

$$\phi \pi(Z) + \alpha(Z) = 0. \tag{2.12}$$

Substituting $Z = \rho$ into (2.12) and using $\pi(\rho) = 1, \alpha(\rho) = 0$, we obtain

$$\phi = 0. \tag{2.13}$$

Moreover, considering (2.5) and (2.13), we achieve

$$\nabla_X \rho = \alpha(X)\rho, \ \alpha(\rho) = 0. \tag{2.14}$$

Substituting $X = \rho$ into (2.14) , we reach the following equation

$$\nabla_\rho \rho = 0. \tag{2.15}$$

In this way, we prove the following theorem.

Theorem 2.1. *In G-flat $A(PRS)_n$ with non-zero scalar curvature, let the basic vector field ρ be a unit torqued vector field. Then ρ becomes a recurrent vector field with recurrency vector field $\alpha(\rho) = 0$ and the integral curves of the basic vector field ρ are also geodesics.*

Let us now consider (1.2) and (2.4). Then we obtain

$$[\pi(Z) + \omega(Z)]S(X,Y) + \pi(X)S(Z,Y) + \pi(Y)S(X,Z) = 0.$$

Substituting (2.1) into the above equation yields

$$[\pi(Z) + \omega(Z)]\kappa r g(X,Y) + \pi(X)\kappa r g(Z,Y) + \pi(Y)\kappa r g(X,Z) = 0. \tag{2.16}$$

Substituting $X = Y = e_i$ into (2.16) and taking summation over i, $(1 \leq i \leq n)$, from the above equation and (1.3), we achieve

$$\kappa r\left[(n+2)\pi(Z) + n\omega(Z)\right] = 0. \tag{2.17}$$

Since $\kappa \neq 0, r \neq 0$, we find

$$\omega(Z) = -\frac{(n+2)}{n}\pi(Z). \tag{2.18}$$

We can then state the following theorem

Theorem 2.2. *In G-flat $A(PRS)_n$ with non-zero scalar curvature, the associated $1-$forms π and ω are linearly dependent.*

Moreover, from (1.3) and (2.18), we obtain

$$\nu = -\frac{(n+2)}{n}\rho. \tag{2.19}$$

Remark 2.3. *By using (2.14) and (2.19), the vector field ν is also a torqued vector field, $\alpha(\nu) = 0$ and since the torqued fuction vanishes, ν becomes a recurrent vector field. Further, from (2.15) and (2.19), the integral curves of the basic vector field ν are also geodesics.*

3. $A(PRS)_n$ with non-zero generalized Einstein tensor

In this section, for the non-zero generalized Einstein tensor on $A(PRS)_n$ is studied different cases like to be conservative, cyclic parallel, and Codazzi type, respectively.
By taking covariant derivative (1.9) with respect to Z, we achieve

$$(\nabla_Z G)(X,Y) = (\nabla_Z S)(X,Y) - \kappa(\nabla_Z r)g(X,Y). \tag{3.1}$$

Substituting (1.2) into (1.5) provides

$$\begin{aligned}(\nabla_Z G)(X,Y) = {} & \{[\pi(Z)+\omega(Z)]S(X,Y)+\pi(X)S(Z,Y)+\pi(Y)S(X,Z)\} \\ & - \kappa\{[\pi(Z)+\omega(Z)]r + 2\pi(QZ)\}g(X,Y).\end{aligned} \tag{3.2}$$

By contracting over X and Z, we obtain

$$(\operatorname{div} G)(Y) = (1-\kappa)[2\pi(QY)+r\pi(Y)] + \omega(QY) - \kappa r\omega(Y). \tag{3.3}$$

Let us now suppose that the generalized Einstein tensor $G(X,Y)$ is conservative [13], i.e. $divG = 0$, (3.3) becomes

$$(1-\kappa)[2\pi(QY)+r\pi(Y)] + \omega(QY) - \kappa r\omega(Y) = 0.$$

By considering (1.3) and (1.4), we can write the above equation in the form

$$(1-\kappa)[2S(Y,\rho)+rg(Y,\rho)] + S(Y,\nu) - \kappa rg(Y,\nu) = 0. \tag{3.4}$$

According to the above equation, if $-\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ, then κr is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ν.
Conversely, if (3.4) holds, then from (3.3), the generalized Einstein tensor is conservative.
Hence, we prove the following theorem

Theorem 3.1. *In $A(PRS)_n$, for the generalized Einstein tensor $G(X,Y)$ to be conservative, the necessary and sufficient condition is that $-\frac{r}{2}$ and κr are eigenvalues of the Ricci tensor S corresponding to the eigenvectors ρ and ν, respectively.*

Let the generalized Einstein tensor $G(X,Y)$ be cyclic parallel. From (1.7), it follows that

$$(\nabla_X G)(Y,Z) + (\nabla_Y G)(Z,X) + (\nabla_Z G)(X,Y) = 0. \tag{3.5}$$

By using (1.2) and (3.1), we obtain

$$\begin{aligned}&[3\pi(Z)+\omega(Z)]S(X,Y)+[3\pi(X)+\omega(X)]S(Z,Y)+[3\pi(Y)+\omega(Y)]S(X,Z) = \\ &\qquad \kappa\{g(X,Y)\nabla_Z r + g(Y,Z)\nabla_X r + g(Z,X)\nabla_Y r\}.\end{aligned} \tag{3.6}$$

By contracting over X and Y, we achieve

$$[3\pi(Z)+\omega(Z)]r + 2[3\pi(QZ)+\omega(QZ)] = \kappa(n+2)\nabla_Z r. \tag{3.7}$$

Substituting (1.5) into the above equation yields

$$(3-\kappa(n+2))\left\{\frac{r}{2}\pi(Z)+\pi(QZ)\right\} + (1-\kappa(n+2))\frac{r}{2}w(Z) + w(QZ) = 0. \tag{3.8}$$

Due to (1.3) and (1.4), the above equation can be written as

$$(3-\kappa(n+2))\left\{\frac{r}{2}g(Z,\rho)+S(Z,\rho)\right\} + (1-\kappa(n+2))\frac{r}{2}g(Z,\nu) + S(Z,\nu) = 0. \tag{3.9}$$

Based on the above equation, if $-\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ, then $(\kappa(n+2)-1)\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ν, and vice versa.
Then, the following theorem holds true:

Theorem 3.2. *In $A(PRS)_n$, let the generalized Einstein tensor $G(X,Y)$ be cyclic parallel. Then, $-\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ if and only if $(\kappa(n+2)-1)\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ν.*

Let us now assume that the generalized Einstein tensor G is of Codazzi type. Due to (1.6), it can be written

$$(\nabla_X G)(Y,Z) = (\nabla_Y G)(X,Z). \tag{3.10}$$

From (1.9) and (3.10), we achieve

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \kappa\left[g(X,Y)\nabla_Z r - g(Y,Z)\nabla_X r\right]. \tag{3.11}$$

By contracting over Y and Z, it is obtained

$$\left(\frac{1}{2} - \kappa(n-1)\right)\nabla_X r = 0.$$

Since $\kappa \neq 0$ and $n \neq 1$, the above equation implies that the scalar curvature is covariantly constant, i.e.,

$$\nabla_X r = 0. \tag{3.12}$$

Now, we substitute (1.2) into (3.11). By using (3.12), we achieve

$$w(Z)S(X,Y) = w(X)S(Z,Y). \tag{3.13}$$

By contracting over X and Y, we have

$$rw(Z) = w(QZ). \tag{3.14}$$

Hence, using (1.3) and (1.4), we obtain

$$rg(Z,\nu) = S(Z,\nu). \tag{3.15}$$

Equation (3.15) implies that the scalar curvature r is an eigenvalue of the Ricci symmetric tensor S corresponding to the eigenvector ν. We have thus proved that the following.

Theorem 3.3. *In $A(PRS)_n$, let the generalized Einstein tensor $G(X,Y)$ be of Codazzi type. Then r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ν.*

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