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The Finiteness of Smooth Curves of Degree ≤ 11 and Genus ≤ 3 on a General Complete Intersection of a Quadric and a Quartic in $\mathbb{P}^5$

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Abstract

Let $W \subset \mathbb{P}^5$ be a general complete intersection of a quadric hypersurface and a quartic hypersurface. In this paper, we prove that W contains only finitely many smooth curves $C \subset \mathbb{P}^5$ such that $d := \deg(C) \leq 11$, $g := p_a(C) \leq 3$ and $h^1(\mathscr{O}_C(1)) = 0$.

1. Introduction

The aim of this paper is to prove the following result.

Theorem 1.1. *Let $W \subset \mathbb{P}^5$ be a general complete intersection of a quadric hypersurface and a quartic hypersurface. Then W contains only finitely many smooth curves $C \subset \mathbb{P}^5$ such that $d := \deg(C) \leq 11$, $g := p_a(C) \leq 3$ and $h^1(\mathscr{O}_C(1)) = 0$.*

We recall that W is a Calabi-Yau threefold and that there are several papers considering finiteness results for rational curves on certain Calabi-Yau threefolds (see [1]-[6] for the general quintic hypersurface of $\mathbb{P}^4$, the topic of the Clemens conjecture, which ask about the finiteness of rational curves of any fixed degree on such a general quintic). This finiteness result is not true for an arbitrary Calabi-Yau threefold [7, Remark 3.24]. For other complete intersection Calabi-Yau threefolds there are results of two types: existence results of good curves on the Calabi-Yau threefold [8, Theorem 2], [9, Theorem 1.2] and finiteness results in very restricted ranges. As in [4] our classical approach to Theorem 1.1 cannot be applied when $\binom{10}{5} \geq 4d+1-g$. There are also papers on 3-folds of general type ([10]-[12] and see [13] and references therein for arithmetically Cohen-Macaulay codimension 2 subvarieties).

The upper bound $d \leq 11$ comes from the proof at a few critical steps, but in many lemmas $d = 12$ or even $d = 13$ may be handled. The approach used in this paper (as the one for quintic 3-folds introduced in [4]) requires that $126 = h^0(\mathscr{O}_{\mathbb{P}^5}(4)) > 4d+1-g$ or, working with a fixed smooth quadric hypersurface $Q \subset \mathbb{P}^5$, $\binom{9}{5} - \binom{7}{5} = h^0(\mathscr{O}_Q(4)) > 4d+1-g$. The upper bound $g \leq 3$ may be weakened in certain steps, but we are sure that new idea are needed to handle pairs (d,g) such that $4d+1-g \geq 126$. Theorem 1.1 is a negative result, a non-existence result. We point out that similar statements are very important, higher genera cases of the count of rational curves of fixed degree on Calabi-Yau manifolds, which is related to Mirror Symmetry [6, 14, 15]. For the Calabi-Yau threefold $X \subset \mathbb{P}^4$, X a very general quintic hypersurface, there is an explicit integer n_d for the number of the degree d rational curves contained in X [14, 15]. At the moment nobody is able to prove the finiteness of such rational curves of a given degree d, except for very low d.

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1.1. A roadmap of the proof

For all integers $d > 0$ and $g \geq 0$ let $M_{d,g}$ denote the locally closed subscheme of the Hilbert scheme of $\mathbb{P}^5$ parametrizing all smooth curves $C \subset \mathbb{P}^5$ such that $\deg(C) = d$, $p_a(C) = g$ and $h^1(\mathscr{O}_C(1)) = 0$. The scheme $M_{d,g}$ is an irreducible quasi-projective variety of dimension $6d+2-2g$. Let $\mathbb{W}$ be the set of all smooth threefolds $W \subset \mathbb{P}^5$, which are the complete intersection of a hypersurface of degree 2 and a hypersurface of degree 4. For each $W \in \mathbb{W}$ we have $\mathrm{Pic}(W) = \mathbb{Z}\mathscr{O}_W(1)$, its normal bundle $N_{W,\mathbb{P}^5}$ is isomorphic to $\mathscr{O}_W(2) \oplus \mathscr{O}_W(4)$, and the quadric hypersurface, Q, containing W is unique. Standard exact sequences give $h^0(\mathscr{O}_W(2)) \oplus \mathscr{O}_W(4)) = 1 + h^0(\mathscr{O}_W(4)) = 20 + h^0(\mathscr{O}_Q(4)) - h^0(\mathscr{O}_Q(2)) = \binom{9}{4} - \binom{7}{2} = 124$. Since $h^1(N_{W,\mathbb{P}^5}) = 0$, the set $\mathbb{W}$ is a smooth variety of dimension 124. The set $\mathbb{W}$ is obviously irreducible. For a general $W \in \mathbb{W}$ the quadric associated to W is smooth. Since all smooth quadric hypersurfaces of $\mathbb{P}^5$ are projectively equivalent, we may fix a smooth quadric hypersurface Q and look only at the set $M_{d,g}(Q) := \{C \in M_{d,g} \mid C \subset Q\}$. To prove Theorem 1.1 we see which elements of $M_{d,g}(Q)$ are contained in a smooth element of $|\mathscr{O}_Q(4)|$. Let $\mathbb{W}$ denote the set of all smooth elements of $|\mathscr{O}_Q(4)|$. To prove Theorem 1.1 for the pair (d,g) it is sufficient to prove that a general element of $|\mathscr{O}_Q(4)|$ contains only finitely many elements of $M_{d,g}(Q)$. We need to study the schemes $M_{d,g}(Q)$ and this is done in Section 3 (see in particular Remark 3.3).

A key idea in this paper is that the smooth quadric hypersurface $Q \subset \mathbb{P}^5$ is isomorphic to the Grassmannian $G(2,4)$ of all 2-dimensional linear subspace of a 4-dimensional vector spaces. By the universal properties of the Grassmannians each map $C \to Q$, $C \in M_{d,g}$, corresponds to a pair (E,V) with E a rank 2 spanned vector bundle on C and $V \subseteq H^0(E)$ a linear subspace spanning E. Section 3 shows how to use this correspondence between embeddings $C \subset Q$ and rank 2 vector bundles on C. Remark 3.3 first gives some elementary statements on rank 2 vector bundles and relate them to our main idea. Then (again in Remark 3.3) we consider separately each low genus. In part (a) we finish the known case $g = 0$. Steps (b), (c) and (d) considers curves of genus 1, 2 and 3, respectively. Lemmas in later sections prove key statements for these genera, but Remark 3.3 is the key first step for them. Thus the proof is done as a case by case proof in which for any smooth curve $C \subset \mathbb{P}^5$ we distinguish the genus of C and the dimension (at most 5) of the linear space $\langle C\rangle$ spanned by C. If $\langle C\rangle$ is a plane we also distinguish if $\langle C\rangle$ is contained in Q or not. If (E,V) is the pair giving the embedding $C \hookrightarrow Q$ the integer $\dim\langle C\rangle$ is the dimension of the image of $\wedge^2(V)$ into $H^0(\mathscr{O}_C(1))$.

Using this section and later lemmas we prove that all $M_{d,g}(Q)$ are irreducible of dimension $4d+1-g$, smooth if $g \leq 2$, while we describe the singular locus of $M_{d,3}(Q)$ (it contains only hyperelliptic curves). We stress again that to prove these results we use that Q is isomorphic to the Grassmannian $G(2,4)$ of all 2-dimensional linear subspaces of $\mathbb{C}^4$. In the case $(d,g) = (6,3)$ we see that all curves $C \subset W$ are hyperelliptic and that they have $h^1(\mathscr{I}_C(2)) = 1$, although $2d+1-g < \binom{7}{2}$ (Remark 4.5). In section 2 we study $M_{d,g}(Q)$, $g \leq 3$, and check all cases with $d \leq 7$ (Lemmas 4.3, 4.4, 4.6, 4.7) and all curves spanning a linear subspace of $\mathbb{P}^5$ of dimension ≤ 3. In section 5 we prove that if $d \leq 14$ a general element of $M_{d,g}(Q)$ has $h^1(\mathscr{I}_C(4)) = 0$ (Lemma 5.5). Lemma 5.3 do the same for a smooth hyperplane section of Q and its proof may be adapted to a singular hyperplane section of Q. In section 6 we handle the non-degenerate curves $C \in M_{d,g}$ with $h^1(\mathscr{I}_C(4)) > 0$. In the last section we handle the curves $C \in M_{d,g}$ with $h^1(\mathscr{I}_C(4)) > 0$ and spanning a hyperplane of $\mathbb{P}^5$.

2. Notation

For any $r \in \{1,2,3,4,5\}$ set $M_{d,g}(r) := \{C \in M_{d,g} : \dim(\langle C\rangle) = r\}$, where for any set $S \subset \mathbb{P}^5$, $\langle S\rangle$ denote the linear span of S. Let $\mathbb{W}$ be the set of all smooth complete intersection $W \subset \mathbb{P}^5$ of a quadric hypersurface and a quartic hypersurface. If we fix a smooth quadric hypersurface $Q \subset \mathbb{P}^5$, then we call $\mathbb{W}$ the set of all smooth elements of $|\mathscr{O}_Q(4)|$.

3. Uses of vector bundles

The 4-dimensional smooth quadric hypersurface Q is isomorphic to the Grassmannian $G(2,4)$ of all 2-dimensional linear subspaces of $\mathbb{C}^4$. Hence for any projective curve X to get a morphism $\phi : X \to Q$ we need to take a rank 2 vector bundle E on X and a linear map $u : \mathbb{C}^4 \to H^0(E)$ such that $u(\mathbb{C}^4)$ spans E. To explain the proof here we assume that u is injective and instead of (E,u) we use (E,V) with $V := u(\mathbb{C}^4)$ (see Remark 3.1 for the case in which u is not injective). Assume that X is smooth. It is easy to check if ϕ is an embedding; indeed if we know that V spans E the map ϕ is an embedding if and only if $\dim(H^0(E(-Z)) \cap V) \leq 1$ for every degree 2 zero-dimensional scheme $Z \subset C$. Assume that ϕ is an embedding and call C its image. Let

$$0 \to \mathscr{F}^\vee \to \mathscr{O}_Q^{\oplus 4} \to \mathscr{E} \to 0$$

denote the tautological exact sequence of $Q = G(2,4)$ with $\mathrm{rank}(\mathscr{E}) = \mathrm{rank}(\mathscr{F}) = 2$ and $\det(\mathscr{E}) \cong \det(\mathscr{F}) \cong \mathscr{O}_Q(1)$. Identifying X and C, i.e. seeing E as a vector bundle on C, we have $E = \mathscr{E}_{|C}$, while $F^\vee := \mathscr{F}^\vee_{|C}$ is the kernel of the surjection $V \otimes \mathscr{O}_C \to E$. Note that $\mathscr{F}$ and F are spanned.

Remark 3.1. *Assume that $u : \mathbb{C}^4 \to H^0(E)$ is not injective, but that $V := \mathrm{Im}(u)$ spans E. Since E has rank 2, then $2 \leq \dim(V) \leq 3$ and $\dim(V) = 2$ if and only if $E \cong \mathscr{O}_X^{\oplus 2}$ and hence the associated map $\phi : X \to Q$ is constant. If $\dim(V) = 3$, then $\mathrm{Im}(\phi)$ is contained in a plane with $T\mathbb{P}^2(-1)$ as universal rank 2 quotient bundle and $\mathscr{O}_{\mathbb{P}^2}(-1)$ as universal rank 1 subbundle. Hence $\phi(X) \in M_{d,g}(2)$. This case is settled in Lemma 4.4.*

Remark 3.2. *Assume $E \cong \mathscr{O}_C \oplus L$ for some line bundle L. In this case $L \cong \mathscr{O}_C(1)$. Write $V = \mathbb{C} \oplus V_1$ with $\mathbb{C} = H^0(\mathscr{O}_C)$. Hence C is contained in a certain Schubert cell of Q, i.e., a 2-dimensional linear subspace contained in Q. Hence $C \in M_{d,g}(2)$. This case is solved in Lemma 4.4. If $F \cong \mathscr{O}_C \oplus \mathscr{O}_C(1)$, then C is contained in the other family of planes contained in Q and so $C \in M_{d,g}(2)$.*

In the next remark we point out some irreducibility and smoothness results for $M_{d,g}(Q)$.

Remark 3.3. *Since $TQ \cong \mathscr{E} \otimes \mathscr{F}$, we have $TQ_{|C} \cong E \otimes F$. In many cases with low g we have $h^1(E \otimes F) = 0$. In this case we have $h^1(N_{C,Q}) = 0$ and hence the Hilbert scheme $\mathrm{Hilb}(Q)$ of Q at $[C]$ is smooth of dimension $4d+1-g$, where $d := \deg(C)$ and $g := p_a(C)$.*

Claim 1: *If either $h^1(E) = 0$ or $h^1(F) = 0$, then $h^1(E \otimes F) = 0$.*

Proof of Claim 1: *Assume for instance $h^1(E) = 0$. Since F is spanned, the evaluation map $e_F : H^0(F) \otimes \mathscr{O}_C \to F$ is surjective. Set $K := \ker(e_F)$. Since $\dim C = 1$, $h^2(K \otimes E) = 0$. Hence the exact sequence*

$$0 \to K \otimes E \to H^0(F) \otimes E \to E \otimes F \to 0$$

proves Claim 1.

Claim 2: *In any genus $g \geq 2$ the set of all $C \in M_{d,g}(Q)$ with $h^1(E) = 0$ is an open, smooth and irreducible subset of $M_{d,g}(Q)$ with dimension $4d+1-g$.*

Proof of Claim 2: *The openness part follows from the semicontinuity of cohomology. Since C is a curve and F is spanned, the vanishing of $h^1(E)$ implies the vanishing of $h^1(E \otimes F)$. Hence this part of $M_{d,g}(Q)$ is smooth and everywhere of dimension $4d+1-g$. Since $g \geq 2$, any vector bundle on a smooth curve C is a flat limit of a family of stable bundles [16, Proposition 2.6]. If $h^1(E) = 0$, then E is a flat limit of a family of stable bundles with vanishing cohomology. The claim follows from the irreducibility of $\mathscr{M}_g$ and the irreducibility of the set of all stable vector bundles with rank two and degree d on a fixed smooth curve of genus $g \geq 2$. This set has dimension $4g-3$.*

(a) If $g = 0$, then $h^1(E \otimes F) = 0$, because $E \otimes F$ is spanned and hence a direct sum of line bundles of degree ≥ 0. The scheme $M_{d,0}(Q)$ is irreducible, because both E and F are specializations with constant cohomology of the rigid bundle with rank 2 and degree d (the direct sum of the line bundle of degree $\lceil d/2 \rceil$ and the one of degree $\lfloor d/2 \rfloor$).

(b) Assume $g = 1$.

Claim 3: *We claim that $h^1(E \otimes F) = 0$, unless $E \cong \mathscr{O}_C \oplus \mathscr{O}_C(1)$ and $F \cong \mathscr{O}_C \oplus \mathscr{O}_C(1)$.*

Proof of Claim 3: *Since $E \otimes F \cong F \otimes E$, it is sufficient to prove that $E \cong \mathscr{O}_C \oplus \mathscr{O}_C(1)$. Since E is spanned, it is a direct sum of indecomposable and spanned vector bundles of degree ≥ 0 and if one of them has degree zero, it is a factor $\mathscr{O}_C$ of E. By Atiyah's classifications of vector bundles on elliptic curves ([17, Part II]) every indecomposable vector bundle G with $\deg(G) > 0$ satisfies $h^1(G) = 0$, concluding the proof of Claim 3.*

This part of $M_{d,1}(Q)$ is irreducible for the following reasons. By Atiyah's classification of vector bundles on an elliptic curve ([17, Part II]), E is a specialization with constant cohomology of semistable bundles. Therefore to check that $M_{d,1}(Q)$ is irreducible, it is sufficient to test the cases with E semistable. If E is semistable, then $h^1(E \otimes F) = 0$ for any spanned bundle F by Claim 3. If d is odd, then we use that any two stable bundle with same rank and degree only differ by a twist with an element of $\mathrm{Pic}^0(C)$. If d is even, then either $E \cong R \oplus L$ with $R, L \in \mathrm{Pic}^{(d/2)}(C)$ and $R \otimes L \cong \mathscr{O}_C(1)$ or E is a non-trivial extension of R by itself and $R^{\otimes 2} \cong \mathscr{O}_C(1)$. The latter case is a specialization of the former one (at least varying C), because $M_{d,1}(Q)$ is smooth and equidimensional and the indecomposable bundles have a smaller dimension.

(c) Assume $g = 2$. By Remark 3.2 and Lemma 4.4 we may assume $E \neq \mathscr{O}_C \oplus \mathscr{O}_C(1)$ and $F \neq \mathscr{O}_C \oplus \mathscr{O}_C(1)$.

Now assume $g = 2$ and $h^1(E) > 0$. By duality we get a non-zero map $v : E \to \omega_C$. Since E is spanned, $\mathrm{Im}(v)$ is spanned. Hence either v is surjective or $\mathrm{Im}(v) \cong \mathscr{O}_C$. The latter case is not possible, because (since E is spanned), it would give that E has $\mathscr{O}_C$ as a factor. Thus v is surjective. Set $A := \ker(v)$. We have $A \cong \mathscr{O}_C(1) \otimes \omega_C^\vee$. Since $\mathscr{O}_C(1)$ is very ample, we have $d > 4$. Hence $h^1(A) = 0$. If $d \geq 6$, A is spanned. If $d \geq 7$, then $h^1(A \otimes \omega_C^\vee) = 0$ and hence $E \cong A \oplus \omega_C$. Assume also $h^1(F) > 0$. We get that F is an extension of ω_C by $\mathscr{O}_C(1) \otimes \omega_C$. Since $h^1(\omega_C^{\otimes 2}) = 0$, we get $h^1(E \otimes F) = 0$ and so $h^1(N_{C,Q}) = 0$. Hence $M_{d,2}(Q)$ is smooth and of pure dimension $4d+1-g$. To check the irreducibility of $M_{d,2}$, it is sufficient to prove that the bundles with $h^1(E) > 0$ do not fill a connected component of $M_{d,2}$. If $d \leq 6$, see Lemma 4.6 and Lemma 4.8. If $d \geq 7$, then $E \cong A \oplus \omega_C$ and so on a fixed curve C this set is isomorphic to $\mathrm{Pic}^{d-2}(C)$; we write g for the genus, because the same argument is needed when $g = 3$. Fix $C \in \mathscr{M}_g$ and take $E \cong A \oplus \omega_C$ with $A \in \mathrm{Pic}^2(C)$. This family of bundles is irreducible and (since $M_{d,g}(Q)$ is smooth along all these bundles) we only need to exclude that $M_{d,g}(Q)$ has two connected components, one formed by bundles E_1 with $h^1(E_1) = 0$ and the other ones with bundles with $h^1(E) = 1$. We have $h^1(E) = 1$ and so $h^0(E) = d+3-2g$. If $h^1(E_1) = 0$, then $h^0(E_1) = d+2-2g$. We have $\dim(G(4, d+1+2(1-g))) = \dim(G(4, d+2(1-g)) + 4$. Thus each bundle E with $h^1(E) > 0$ has the property that $H^0(E)$ has a family of 4-dimensional linear subspaces with higher dimension. For $g \geq 3$ it is sufficient to note that for a fixed C the possible E depends on $A \in \mathrm{Pic}^{d-g}(C)$, the set of all rank 2 stable bundles on C with degree d have dimension $4g-3$ and $g+4 < 4g-3$. When $g = 2$ we also need to factorize the huge automorphism group of $A \oplus \omega_C$ (we have $h^0(A \otimes \omega_C^\vee) = d-5$).

(d) Assume $g = 3$. By Remark 3.2 and Lemma 4.4 we may assume $E \neq \mathscr{O}_C \oplus \mathscr{O}_C(1)$ and $F \neq \mathscr{O}_C \oplus \mathscr{O}_C(1)$. We also assume $d \geq 8$, leaving the cases $d \leq 7$ to Remark 4.7. All cases with $h^1(E) = 0$ are done as in Claim 2. Assume $h^1(E) > 0$ and $h^1(F) > 0$. As in step (b) we get non-zero maps $v_1 : E \to \omega_C$ and $v_2 : F \to \omega_C$ with $\mathrm{Im}(v_i)$ a non-trivial and spanned line bundle. Hence either v_i is surjective or C is not hyperelliptic and $\mathrm{Im}(v_i) = \omega_C(-p)$ for some $p \in C$ or C is hyperelliptic

and $\mathrm{Im}(v_i)$ *is the* g^1_2 *of* C*. In all cases* $\ker(v_i)$ *is spanned and non-special, because we assumed* $d \geq 9$*. The case in which* $E \cong A \oplus \omega_C$ *is handled as in step (c). If either* C *is not hyperelliptic or at least one among* $\mathrm{Im}(v_1)$ *and* $\mathrm{Im}(v_2)$ *is not the* g^1_2 *on* C*, we have* $h^1(E \otimes F) = 0$ *and so* $h^1(N_{C,Q}) = 0$*. So* $M_{d,3}(Q)$ *is smooth and of dimension* $4d+1-g = 4d-2$ *at* $[C]$*. Hence* $h^1(E \otimes F) > 0$ *if and only if* C *is hyperelliptic and* $\mathrm{Im}(v_1)$ *and* $\mathrm{Im}(v_2)$ *are the* g^1_2*,* R*, on* C*. In this case we have* $E \cong A \oplus R$ *and* $F \cong B \oplus R$ *with* $\deg(A) = \deg(B) = d-2$ *and so* $h^1(E \times F) = 1$*. Therefore every irreducible component of* $M_{d,3}(Q)$ *containing* $[C]$ *has dimension at least* $4d+1-g$ *and at most* $4d+2-g$*. To check that these points are singular points of* $M_{d,3}(Q)$ *and hence that* $M_{d,3}(Q)$ *has pure dimension* $4d-2$*, it is sufficient to prove that these bundles do not fill a subset of* $M_{d,3}(Q)$ *of dimension* $\geq 4d-2$*; we will prove that these bundles fill in a family of dimension* $\leq 4d-3$*, because this is needed to prove the irreducibility of* $M_{d,3}(Q)$*. The set of these bundles only depends on the choice of a hyperelliptic curve* C*, the choice of* $A \in \mathrm{Pic}^{d-2}(C)$ *and the choice of a* 4*-dimensional linear subspace of* $H^0(A \oplus R)$*. We have* $h^1(A \oplus R) = h^1(R) = 1$ *and so* $h^0(A \oplus R) = d+2-2g$*. Since there* ∞^5 *hyperelliptic curves and* $\mathrm{Pic}^{d-2}(C)$ *has dimension* 3*, it is sufficient to use that* $5+4+3 < 6+4g-3$*. Then the proof in step (c) handles all bundles of the form* $A \oplus \omega_C$*. It remains to handle the bundles* E *with* C *not hyperelliptic and* $\mathrm{Im}(v_1) \cong \omega_C(-p)$ *for some* $p \in C$*. Set* $A := \ker(v_1) \in \mathrm{Pic}^{d-3}(C)$*. Note that* $h^1(E) = 1$ *and* $h^1(F) = 0$*. Hence these bundles are in the smooth part of* $M_{d,3}(Q)$*. We have* $h^0(E) = h^0(E_1) + 1$ *when* $h^1(E_1)$ *and so the Grassamannian of all* 4*-dimensional linear subspaces has dimension* $4+z$*, where* z *is the dimension of all* 4*-dimensional linear subspaces of* $H^0(E_1)$*. The bundles* E_1 *depends on* $4g-3 = 9$ *parameters. The bundles* E *depends on* A *(*$g = 3$*) parameters, on* $p \in C$ *(one parameter) and an extension classes of* $\omega_C(-p)$ *by* A*. For the trivial extensions we use that* $4+g+1 < 4g-3$*. Two non-trivial, but proportional extensions, give the same bundle, up to isomorphisms. Hence the bundles* E *with* $h^1(A \otimes \omega_C^\vee(p)) \leq 1$*, do not fill a connected component of* $M_{d,3}(Q)$*. We have* $\deg(A \otimes \omega_C^\vee) = d-6$*. Since* C *is not hyperelliptic, we have* $h^1(A \otimes \omega_C^\vee(p)) \leq 1$ *for all* $d \geq 8$*. See Remark 4.7 for the case* $d \leq 7$*.*

4. Preliminary lemmas

The following lemma is proved as in [6, page 153].

Lemma 4.1. *Fix* (d,g) *such that* $2d \leq 19+g$ *and* $h^1(\mathscr{I}_C(2)) = 0$ *for all* $C \in M_{d,g}$*. Then a general* $W \in \mathbb{W}$ *contains finitely many elements of* $M_{d,g}$ *and the incidence variety* $I_{d,g} \subset M_{d,g} \times \mathbb{W}$ *is irreducible.*

Remark 4.2. *Unfortunately in several interesting cases many curves satisfies* $h^1(\mathscr{I}_C(2)) > 0$ *(e.g. if* $2d+1-g > 15$ *this is the case for all curves spanning a hyperplane of* $\mathbb{P}^5$*). Working with* $M_{d,g}(Q)$ *we only need to check if* $h^1(\mathscr{I}_C(4)) = 0$*. This is true for all* $C \in M_{d,g}(Q)$ *for some more pairs* (d,g)*. We divide* $M_{d,g}(Q)$ *in the one with* $h^1(\mathscr{I}_C(4)) = 0$ *and in the ones with* $h^1(\mathscr{I}_C(4)) > 0$*. We need to prove that for* C *in a non-empty open subset of* $M_{d,g}(Q)$ *we have* $h^1(\mathscr{I}_C(4)) = 0$ *(Lemma 5.5). The last two sections of this paper tackle the case* $h^1(\mathscr{I}_C(4)) > 0$*.*

Remark 4.3. $M_{d,g}(1) \neq \emptyset$ *if and only if* $d = 1$ *and* $g = 0$*. By Lemma 4.1 a general* W *has only finitely many lines.*

Lemma 4.4. $M_{d,g}(2) \neq \emptyset$ *if and only if either* $d = 2$ *and* $g = 0$ *or* $d = 3$ *and* $g = 1$*. In the cases* $(d,g) \in \{(2,0),(3,1)\}$ *a general* W *contains finitely many elements of* $M_{d,g}(2)$*.*

Proof. Since the curves in $M_{d,g}$ are non-special, $M_{d,g}(2) \neq \emptyset$ if and only if either $d = 2$ and $g = 0$ or $d = 3$ and $g = 1$. The second assertion follows from Lemma 4.1. □

Remark 4.5. *Set* $\Gamma := \{C \in M_{6,3} : C \text{ is hyperelliptic}\}$*.* Γ *is an irreducible divisor of the* 32*-dimensional variety* $M_{6,3}$*. Fix a smooth quadric hypersurface* $Q \subset \mathbb{P}^5$ *and set* $\Gamma' := \Gamma \cap M_{6,3}(Q)$*. Fix* $C \in M_{6,3}(Q)$*. We have* $\dim(\langle C \rangle) = 3$*. Since* Q *is smooth,* $\langle C \rangle \nsubseteq Q$ *and so* $Q' := \langle C \rangle$ *is an irreducible quadric surface containing* C*. Since all even degree smooth curves of a quadric cone of* $\mathbb{P}^3$ *are complete intersection [18, V Ex. 2.9],* Q' *is a smooth quadric. Since* $(d,g) = (6,3)$*, then* $C \in |\mathscr{O}_{Q'}(2,4)| \cup |\mathscr{O}_{Q'}(4,2)|$ *and so* C *is hyperelliptic. Hence no* $C \in M_{6,3}(Q) \setminus \Gamma'$ *is contained in some* $W \in \mathbb{W}$*. Conversely, any hyperelliptic curve* X *may be embedded in* $Q' = \mathbb{P}^1 \times \mathbb{P}^1$ *as an element of* $|\mathscr{O}_{Q'}(2,4)|$ *using the* g^1_2*,* R*, of* X *to get one morphism* $X \to \mathbb{P}^1$ *and a general* $A \in \mathrm{Pic}^4(X)$ *for the other map* $X \to \mathbb{P}^1$ *so that* $A \otimes R$ *is very ample). Hence for a fixed* X *the set of all such embeddings is parametrized by an irreducible variety of dimension* 3*. Fix* $C \in \Gamma'$*, say with* $C \in |\mathscr{O}_{Q'}(2,4)|$*. We have* $N_{C,Q} \cong \mathscr{O}_C(1)^{\oplus 2} \oplus \mathscr{O}_C(2,4)$ *and hence* $h^1(N_{C,Q}) = 0$*. So* $M_{6,3}(Q)$ *is smooth at* $[C]$ *and of dimension* $4d+1-g = 22$*. Since* $|\mathscr{O}_{Q'}(2,4)|$ *is irreducible and as* $\langle C \rangle$ *we may take any* $\mathbb{P}^3 \subset \mathbb{P}^5$ *transversal to* Q*,* $M_{6,3}(Q)$ *is irreducible. Call* $\mathscr{I} \subset \Gamma' \times \mathbb{W}$ *the incidence correspondence and let* $\pi_1 : \mathscr{I} \to \Gamma'$ *and* $\pi_2 : \mathscr{I} \to \mathbb{W}$ *denote the projections. We have* $h^1(Q, \mathscr{I}_{C,Q}(4)) = 0$*, because* $h^1(Q', \mathscr{I}_{C,Q'}(4)) = h^1(Q', \mathscr{O}_{Q'}(2,0)) = 0$*. Lemma 4.1 concludes the proof of the theorem for* $(d,g) = (6,3)$*. In this case the incidence correspondence is irreducible, because the set of all hyperelliptic curves is irreducible and all these curves* C *have the same* $h^0(\mathscr{I}_C(2))$ *and* $h^1(\mathscr{I}_C(4)) = 0$ *(and so the incidence correspondence for* $M_{6,3}(Q)$ *is irreducible).*

Lemma 4.6. *We have* $M_{d,g}(3) \neq \emptyset$ *if and only if* $d \geq g+3$*. If* $g \leq 3$*, then a general* $W \in \mathbb{W}$ *contains some* $C \in M_{d,g}(3)$ *only if* $(d,g) \in \{(3,0),(4,1),(5,2),(6,3)\}$ *and in each of these cases* W *contains only finitely many curves* C*.*

Proof. Fix a smooth hyperquadric Q, $C \in M_{d,g}(3)$ and $W \in \mathbb{W}$ containing C. Set $U := \langle C \rangle$. Since Q is smooth, $U \nsubseteq Q$ and hence $Q' := Q \cap U$ is a quadric surface containing C. Since the irreducible curve C spans U and $C \subset Q'$, Q' is irreducible. If Q' is a quadric cone, then C is arithmetically normal [18, V Ex. 2.9] and hence $h^1(\mathscr{I}_C(t)) = 0$ for $t = 2,4$, so that we may apply Lemma 4.1 to these curves) and we find pairs $(d,g) \in \{(3,0),(4,1),(5,2)\}$. If Q', up to a change of the ruling of Q' we get all $C \in |\mathscr{O}_{Q'}(2,g+1)|$ and so $d = g+3$. If $g \leq 4$ we have $h^1(\mathscr{I}_C(4)) = h^1(Q', \mathscr{I}_{C,Q'}(4)) = h^1(Q', \mathscr{O}_{Q'}(2,4-g-1)) = 0$. □

Lemma 4.7. *Theorem 1.1 is true for $g=3$ and $d\le 7$.*

Proof. Take $g=3$ and $d\le 7$. Since $h^1(\mathscr{O}_C(1))=0$, we have $6\le d\le 7$. Remark 4.5 and Lemma 4.6 solve the case $d=6$ and the case $d=7$ in which $C\in M_{7,3}(3)$. Hence we may assume $d=7$ and $\dim(\langle C\rangle)=4$. In this case C is linearly normal in its linear span and so $h^1(\mathscr{I}_C(t))=0$ for all $t\in\mathbb{N}$. Apply Lemma 4.1. □

Lemma 4.8. *Fix $C\in M_{d,g}(Q)(r)$ with $d\le 7$, $g\le 2$ and $r=4,5$. Then $h^1(N_{C,Q})=h^1(\mathscr{I}_C(4))=0$. Moreover, these cases only contribute finitely many smooth curves to a general $W\in\mathbb{W}$.*

Proof. Since $g\le 2$, we have $h^1(N_{C,Q})=0$. Since $d<4+r$, we have $h^1(\mathscr{I}_C(4))=0$ [19, Theorem at page 492] and hence these cases contributes only finitely smooth curves to a general $W\in\mathbb{W}$. □

Lemma 4.9. *A general $W\in\mathbb{W}$ contains no singular conic (reducible or a double line).*

Proof. Take any conic $D\subset W$. Since $h^1(\mathscr{I}_{D,\mathbb{P}^5}(4))=0$, we have $h^1(Q,\mathscr{I}_{D,Q}(4))=0$ and hence $h^0(Q,\mathscr{I}_{D,Q}(4))=h^0(D,\mathscr{I}_{D,Q}(4))$. Either D is contained in a plane contained in Q or it is the complete intersection of Q and a plane. In both cases we have $h^1(N_{D,Q})=0$. Thus a dimensional count gives that a general $W\in\mathbb{W}$ contains only finitely many conics and that all these conics are smooth. □

We recall the following well-known consequence of the bilinear lemma (it is a key tool in [2]).

Lemma 4.10. *Fix integers $t\ge 2$, $r\ge 3$ and an integral and non-degenerate curve $T\subset\mathbb{P}^r$ such that $h^1(\mathscr{I}_T(t))>0$. Fix a linear subspace $V\subseteq H^0(\mathscr{O}_{\mathbb{P}^r}(1))$. Assume that $h^1(M,\mathscr{I}_{M\cap T,M}(t))=0$ for every hyperplane $M\in|V|$. Then $h^1(\mathscr{I}_T(t-1))\ge h^1(\mathscr{I}_T(t))+\dim(V)-1$.*

Proof. For any hyperplane $M\subset\mathbb{P}^r$ we have an exact sequence

$$0\to\mathscr{I}_T(t-1)\to\mathscr{I}_T(t)\to\mathscr{I}_{T\cap M,M}(t)\to 0$$

Now assume that V contains an equation of M. Since $h^1(M,\mathscr{I}_{T,M}(t))=0$, the map $H^1(\mathscr{I}_T(t-1))\to H^1(\mathscr{I}_T(t))$ is surjective and hence its dual $e_M: H^1(\mathscr{I}_T(t))^\vee\to H^1(\mathscr{I}_T(t-1))^\vee$ is injective. Taking the equations of all hyperplanes we get a bilinear map map $u: H^1(\mathscr{I}_T(t))^\vee\times V\to H^1(\mathscr{I}_T(t-1))^\vee$, which is injective with respect to the second variables, i.e. for every non-zero linear form ℓ the map $u_{|H^1(\mathscr{I}_T(t))^\vee\times\{\ell\}}$ is injective (it is e_M with $M:=\{\ell=0\}$). Hence if $(a,\ell)\in H^1(\mathscr{I}_T(t))^\vee\times V$ with $a\ne 0$ and $\ell\ne 0$, then $u(a,\ell)=e_M(a)\ne 0$. Therefore the bilinear map u is non-degenerate in each variable. Hence $h^1(\mathscr{I}_T(t-1))\ge h^1(\mathscr{I}_T(t))+\dim(V)-1$ by the bilinear lemma. □

5. Good postulation in degree 4

In this section we prove for certain d,g the existence of a non-degenerate $C\in M_{d,g}(Q)$ with $h^1(\mathscr{I}_C(4))=0$.

Lemma 5.1. *Fix $C\in M_{d,g}(Q)$ such that $h^1(N_{C,Q})=0$. Take an integer $t>0$ and a smooth rational curve $T\subset Q$ such that $\deg(C\cap T)=1$ and $\deg(T)=t$. Then $h^1(N_{C\cup T,Q})=0$ and $C\cup T$ is a flat limit of elements of $M_{d+t,g}(Q)$.*

Proof. Set $\{p\}:=C\cap T$. By assumption $h^1(\mathscr{O}_C(1))=0$. Since Q is homogeneous, its tangent bundle is spanned. Hence $N_{T,Q}$ is a direct sum of line bundles of degree ≥ 0. Therefore $h^1(N_{T,Q}(-p))=0$. A Mayer-Vietoris exact sequence gives $h^1(\mathscr{O}_{C\cup T}(1))=0$. Hence if $C\cup T$ is smoothable inside Q, then it is a flat limit of a family of elements of $M_{d+t,g}(Q)$. Since $h^1(N_{T,Q}(-p))=0$, as in [20, Theorem 4.1] we get that $C\cup T$ is smoothable inside Q and $h^1(N_{C\cup T,Q})=0$. □

Lemma 5.2. *For all $g\in\{0,1,2,3\}$ there is a non-degenerate $C\in M_{g+5,g}(Q)$ and any such C is projectively normal.*

Proof. Let $X\subset\mathbb{P}^5$ be a linearly normal smooth curve of genus $g\le 3$ and degree $g+5$. Since $g+5\ge 2g+1$, X is projectively normal [21]. It is sufficient to prove that some X is contained in a smooth quadric hypersurface. Since $g\le 3$, we start with a smooth quadric surface $Q_1\subset Q$, a smooth curve $A\in|\mathscr{I}_{Q_1}(2,g+1)|$ and then we apply the case $t=2$ of Lemma 5.1. □

Lemma 5.3. *Let $Q'\subset\mathbb{P}^4$ be a smooth quadric hypersurface. Fix integers d,g such that $0\le g\le 3$ and $d\ge g+4$. Let $M_{d,g}(Q')$ be the set of all non-special smooth curves $C\subset Q'$ of genus g and degree d.*

(a) There is $C\in M_{g+4,g}(Q')$ which is projectively normal.

(b) If either $g+4\le d\le g+6$ or $g\le 2$ and $d=g+7$ or $g=0$ and $d=8$, then there is $C\in M_{d,g}(Q')$ such that $h^1(Q',\mathscr{I}_{C,Q'}(3))=0$.

(c) If either $g+4\le d\le g+9$, or $g\le 2$ and $d=g+10$ or $g=0$ and $d=11,12$, then there is $C\in M_{d,g}(Q')$ such that $h^1(Q',\mathscr{I}_{C,Q'}(4))=0$.

Proof. The proof of part (a) is similar to the one Lemma 5.2. The same proof also gives the case $d = g+4$ of part (b).

(i) Let $A \subset Q'$ be a smooth projectively normal curve of genus g and degree $g+4$. Let $Q_1 \subset Q'$ be a general hyperplane section. Q_1 is a smooth quadric surface and $S := A \cap Q_1$ is a subset of Q_1 with degree $g+4$, in uniform position and spanning the 3-dimensional linear space spanned by Q_1. Fix $p \in S$ and set $S' := S \setminus \{p\}$. Let B be a general element of $|\mathscr{I}_{p,Q_1}(1,2)|$. Lemma 5.1 shows that $A \cup B$ is smoothable inside Q'. Hence to prove the case $d = g+7$, $g \le 2$, of part (b) it is sufficient to prove that $h^1(Q', \mathscr{I}_{A\cup B,Q'}(3)) = 0$. We have $\mathrm{Res}_{Q_1}(A \cup B) = A$. Since $h^1(Q', \mathscr{I}_{A,Q'}(2)) = 0$, the case $t = 3$ of the residual sequence

$$0 \to \mathscr{I}_{A,Q'}(t-1) \to \mathscr{I}_{A\cup B,Q'}(t) \to \mathscr{I}_{(A\cup B)\cap Q_1,Q_1}(t) \to 0$$

shows that it is sufficient to prove that $h^1(Q_1, \mathscr{I}_{(A\cup B)\cap Q_1,Q_1}(3)) = 0$. We have $Q_1 \cap (A \cup B) = S' \cup B$ and hence it is sufficient to prove that $h^1(Q_1, \mathscr{I}_{S',Q'}(2,1)) = 0$. S' is a set of $g+3 \le 6$ points of Q_1. Assume $e := h^1(Q_1, \mathscr{I}_{S',Q_1}(2,1)) > 0$. Hence $h^0(Q, \mathscr{I}_{S',Q_1}(2,1)) = e+3-g$. Since S is in uniform position, we get $h^0(Q_1, \mathscr{I}_{S,Q_1}(2,1)) = e+g-3$. Fix a general $D \in |\mathscr{I}_{S,Q_1}(2,1)|$. First assume that D is irreducible. For any set $E \subset D$ with $\#(E) = 5$, we have $h^0(Q_1, \mathscr{I}_{D,Q_1}(2,1)) = h^0(Q_1, \mathscr{I}_{E,Q_1}(2,1))$ and hence $h^1(Q_1, \mathscr{I}_{E,Q_1}(2,1)) = 0$. If $g \le 2$ we may take $S' \subseteq E$. Now assume that D is reducible. Since S is in uniform position, we may assume that no 2 of the points of S are contained in a line of Q_1. Hence we get the existence of a smooth conic $D_1 \subset Q_1$ containing at least $g+4$ points of S'. Since S is in uniform position, we get $S \subset D_1$. If $g = 3$ we use instead of B a curve $B' \in |\mathscr{I}_{p,Q_1}(1,1)|$ (in this case the equality $h^1(Q_1, \mathscr{I}_{S',Q_1}(2,2)) = 0$ may be proved using an elliptic curve $D' \in |\mathscr{O}_{Q_1}(2,2)|$, because $h^1(D, \mathscr{I}_{S',D_1}(2,2)) = 0$ for any set $E \subset D$ with $\#(E) \le 7$. Now assume $g = 0$ and $d = 8$. Instead of B we take a general $B_1 \in |\mathscr{I}_{p,Q_1}(1,3)|$. It is sufficient to prove that $h^1(Q, \mathscr{I}_{S',Q_1}(2,0)) = 0$. We have $\#(S') = 3 = h^0(Q_1, \mathscr{O}_{Q_1}(0,2))$, and it is sufficient to use again by the uniform position that no two points of S are on a line of Q_1.

(ii) Now we prove part (c). Since in part (b) we get non-special curves, the same curves C have $h^1(Q', \mathscr{I}_{C,Q'}(4)) = 0$ by the Castelnuovo-Mumford's lemma. Hence we may assume that either $d \ge g+8$ and $g \le 2$, or $d \ge g+7$ and $g = 3$ or $g = 0$ and $d \ge 9$. Set $t := 8$ if $g = 0$, $t := g+7$ if $g = 1, 2$ and $t := 9$ if $g = 3$. By part (b) there is $A \subset M_{t,g}(Q')$ such that $h^1(Q', \mathscr{I}_{A,Q'}(3)) = 0$. Take a general hyperplane section Q_1 of Q' and set $S := Q_1 \cap S$. S' is a subset of Q_1 with cardinality t, spanning a $\mathbb{P}^3$ and in uniform position. Fix $p \in S$ and set $S' := S \setminus \{p\}$. Fix a general $B \in |\mathscr{I}_{p,Q_1}(1,2)|$. As in step (i) it is sufficient to prove that $h^1(Q_1, \mathscr{I}_{S',Q}(3,2)) = 0$. In all cases we have $t-1 \le 8$. The uniform position and the non-degeneracy of S' imply that no line of Q_1 contains at least 2 points of S' and no conic of Q_1 contains at least 4 points of S'.
Now take $g = 0$. In this case A may be dismantled into a union of lines. Fix a general line $L \subset Q'$. For each $q \in L$. The union of all lines of Q' trough q is the 2-dimensional quadric cone $T_q(Q') \cap Q'$. For a general $q \in L$ the curve $T_q(Q') \cap Q_1$ is a smooth element D_q of $|\mathscr{O}_{Q_1}(1,1)|$ and a general line in Q' passing through q meets Q_1 at a general point of Q_1. Hence we get $h^0(Q_1, \mathscr{I}_{S'}(3,1)) = 0$ if $\#S' \le 8$, i.e. if we start with a general $A \in_{d,0} (Q')$ with $d \le 9$. Thus we get the case $g = 0$ of part (c). □

Lemma 5.4. *Let $Q' \subset \mathbb{P}^4$ be a smooth quadric hypersurface. Fix a set $S \subset Q'$ with $\#S \le 10$ and S is in linearly general position. Take $p \in S$ and set $S' := S \setminus \{p\}$.*

(a) If $1 \le d \le 4$, then there is $C \in M_{d,0}(Q')$ such that $C \cap S = \{p\}$ and $h^1(Q', \mathscr{I}_{S'\cup C,Q'}(3)) = 0$.

(b) If $1 \le d \le 9$, then there is $C \in M_{d,0}(Q')$ such that $C \cap S = \{p\}$ and $h^1(Q', \mathscr{I}_{S'\cup C,Q'}(4)) = 0$.

Proof. Let Q_1 be a general hyperplane section of Q' containing p. Q_1 is smooth and $Q_1 \cap S = \{p\}$. We have $h^1(Q', \mathscr{I}_{S',Q'}(2)) = 0$, because $\#S' \le 9$ [22, Theorem 3.2]. To prove part (a) it is sufficient to take any smooth $C \in |\mathscr{I}_{p,Q_1}(1,3)|$. By Castelnuovo-Mumford's lemma to prove part (b) we may assume $d > 4$. Fix a general $A \in M_{4,0}(Q')$ containing p. Part (a) gives $h^1(Q', \mathscr{I}_{A\cup S',Q'}(3)) = 0$. Fix a general hyperplane section $Q_2 \subset Q'$. We have $Q_2 \cap S = \emptyset$ and the set $E := Q_2 \cap A$ is in linearly general position in the $\mathbb{P}^3$ spanned by Q_2. Fix $q \in E$ and set $E' := E \setminus \{q\}$. Fix a general $B \in |\mathscr{I}_{q,Q_2}(1,4)|$. By Lemma 5.1 it is sufficient to prove that $h^1(\mathscr{I}_{S'\cup A\cup B,Q'}(4)) = 0$. Since $\mathrm{Res}_{Q_1}(S' \cup A \cup B) = S' \cup A$ and $h^1(\mathscr{I}_{A\cup S',Q'}(3)) = 0$, it is sufficient to prove that $h^1(Q_1, \mathscr{I}_{E'\cup B,Q_1}(4)) = 0$, i.e. $h^1(Q', \mathscr{I}_{E'}(3,0)) = 0$. This is true, because E' is formed by 3 points in uniform position. □

Lemma 5.5. *(a) For all integers d, g such that $0 \le g \le 3$ and $g+5 \le d \le g+9$ there is a non-degenerate $C \in M_{d,g}(Q)$ such that $h^1(\mathscr{I}_C(3)) = 0$.*

(b) For all integers d, g such that either $0 \le g \le 3$ and $g+5 \le d \le 14$ there is a non-degenerate $C \in M_{d,g}(Q)$ such that $h^1(\mathscr{I}_C(4)) = 0$.

Proof. Fix a projectively normal $A \in M_{g+5,5}(Q)$. Fix a general hyperplane section $Q' \subset Q$. Since $h^1(Q, \mathscr{I}_{A,Q}(4)) = 0$, we may assume $d > g+5$. The set $S := A \cap Q_1$ is in linearly general position. Fix $p \in S$ and set $S' := S \setminus \{p\}$. Apply part (b) of Lemma 5.4 to get $T \in M_{d-g-5,0}(Q')$ such that $h^1(Q', \mathscr{I}_{S'\cup T}(4)) = 0$. Since $h^1(Q, \mathscr{I}_{A\cup T}(3)) = 0$ and $(A \cup T) \cap Q' = S' \cup T$, the residual sequence of Q' in Q gives $h^1(Q, \mathscr{I}_{A\cup B}(4)) = 0$. Use Lemma 5.1 and the semicontinuity theorem for cohomology to prove part (b). For part (a) we take T of degree ≤ 4 and use that $h^1(Q, \mathscr{I}_{A,Q}(2)) = 0$. □

Remark 5.6. *A general element of $M_{d,0}(Q')$ (resp. $M_{d,0}(Q)$) is a deformation of a tree contained in Q' (resp. Q). Using this observation we may improve parts (a) and (b) of Lemma 5.5, but for a range of integers d out of reach with our tools for the Clemen's conjecture.*

6. Non-degenerate curves

In this section we consider non-degenerate curves C of $M_{d,g}$ or of $M_{d,g}(Q)$. By [19, Theorem at page 492] we have $h^1(\mathscr{I}_C(4)) = 0$ if either $d \le 8$ or $d = 9$ and $g > 0$ or $d = 9$, $g = 0$ and there is no line $R \subset \mathbb{P}^5$ with $\deg(R\cap C) \ge 6$. By Lemma 5.5, the irreducibility of $M_{d,g}(Q)$ and the equality $\dim(M_{d,g}(Q)) = 4d+1-g$ we may assume $h^1(\mathscr{I}_C(4)) > 0$.

Lemma 6.1. *Assume $d \le 11$ and fix a non-degenerate $C \in M_{d,g}$ such that there is no line $R \subset \mathbb{P}^5$ with $\deg(R\cap C) \ge 6$. Then $h^1(M, \mathscr{I}_{C\cap M,M}(4)) = 0$ for every hyperplane $M \subset \mathbb{P}^5$.*

Proof. Fix a hyperplane $M \subset \mathbb{P}^5$. Since C spans $\mathbb{P}^5$, $Z := C\cap M$ is a curvilinear scheme spanning M. Assume $h^1(M, \mathscr{I}_{Z,M}(4)) > 0$. Let N be a hyperplane of N with maximal $a := \deg(Z\cap N)$. Since Z spans M, we have $a \ge 4$. Assume for the moment $a = 4$, i.e. assume that Z is in linearly general position. Since $d \le 17$, we have $h^1(M, \mathscr{I}_{Z,M}(4)) = 0$ [22, Theorem 3.2]. Hence we may assume $a \ge 5$.

(a) First assume $h^1(N, \mathscr{I}_{Z\cap N,N}(4)) > 0$. Since Z spans M, we have $a \le d-1 \le 10$. The maximality property of N implies that $Z\cap N$ spans N. Hence $\deg(Z\cap U) \le 9$ for every plane $U \subset N$. Fix a plane $U \subset N$ with $b := \deg(Z\cap U)$ is maximal. If $h^1(U, \mathscr{I}_{Z\cap U,U}(4)) > 0$, then there is a line $R \subset U$ with $\deg(R\cap Z) \ge 6$. Hence we may assume $h^1(U, \mathscr{I}_{Z\cap U,U}(4)) = 0$. The residual sequence of U in N gives $h^1(N, \mathscr{I}_{\mathrm{Res}_U(Z\cap N),N}(3)) > 0$. We have $\deg(\mathrm{Res}_U(Z\cap N)) \le 10-b \le 7$. By [23, Lemma 34] there is a line $L \subset N$ such that $\deg(L\cap \mathrm{Res}_U(Z)) \ge 5$. Hence $b \ge 6$. Hence $10-b > \deg(L\cap \mathrm{Res}_U(Z))$, a contradiction.

(b) Now assume $h^1(N, \mathscr{I}_{Z\cap N}(4)) = 0$. The residual exact sequence

$$0 \to \mathscr{I}_{\mathrm{Res}_N(Z),M}(3) \to \mathscr{I}_{Z,M}(4) \to \mathscr{I}_{Z\cap N,N}(4) \to 0$$

gives $h^1(M, \mathscr{I}_{\mathrm{Res}_N(Z),M}(3)) > 0$. Since $d-a \le 7$, then there is a line $L \subset M$ such that $\deg(\mathrm{Res}_N(Z)) \ge 5$ [23, Lemma 34]. By assumption we have $\deg(L\cap Z) = 5$. Since $\deg(Z\cap L) \ge 5$, the maximality property of a gives $a \ge 7$. Since $d-a \ge 5$, we get $d \ge 12$, a contradiction. □

Lemma 6.2. *Assume $d \le 11$ and fix a non-degenerate $C \in M_{d,g}$ such that there is no line $R \subset \mathbb{P}^5$ with $\deg(R\cap C) \ge 5$, no conic $D \subset \mathbb{P}^5$ with $\deg(D\cap C) \ge 8$, no plane cubic T with $\deg(T\cap C) = 9$ and $C\cap T \in |\mathscr{O}_T(3)|$. Then $h^1(M, \mathscr{I}_{C\cap M,M}(3)) = 0$ for every hyperplane $M \subset \mathbb{P}^5$.*

Proof. Fix a hyperplane $M \subset \mathbb{P}^5$. Since C spans $\mathbb{P}^5$, $Z := C\cap M$ is a curvilinear scheme spanning M. Assume $h^1(M, \mathscr{I}_{Z,M}(3)) > 0$. Let N be a hyperplane of N with maximal $a := \deg(Z\cap N)$. Since Z spans M, we have $a \ge 4$. Assume for the moment $a = 4$, i.e. assume that Z is in linearly general position. Since $d \le 13$, we have $h^1(M, \mathscr{I}_{Z,M}(3)) = 0$ [22, Theorem 3.2]. Hence we may assume $a \ge 5$.

(a) First assume $h^1(N, \mathscr{I}_{Z\cap N,N}(3)) > 0$. Since Z spans M, we have $a \le d-1 \le 10$. The maximality property of N implies that $Z\cap N$ spans N. Hence $\deg(Z\cap U) \le 9$ for every plane $U \subset N$. Let $U \subset N$ be a plane such that $b := \deg(U\cap Z)$ is maximal. If $h^1(U, \mathscr{I}_{Z\cap U,U}(3)) > 0$, then [24, Corollaire 2] shows the existence of either R or D or T. Now assume $h^1(U, \mathscr{I}_{U\cap Z,U}(3)) = 0$. The residual sequence of U gives $h^1(N, \mathscr{I}_{\mathrm{Res}_U(N\cap Z),N}(2)) > 0$. Since $\deg(\mathrm{Res}_U(N\cap Z)) \le 10-b \le 7$, either there is a line $L \subset N$ with $\deg(L\cap \mathrm{Res}_U(Z)) \ge 4$ or there is a conic $D \subset N$ with $\deg(D\cap Z) \ge 6$. The latter case is impossible, because it implies $a-b \ge 6$ and $b \ge 6$, a contradiction. Hence there is a line L with $\deg(L\cap \mathrm{Res}_U(Z)) \ge 4$. To prove the lemma we may assume $\deg(Z\cap L) = 4$. Let $E \subset N$ be a plane containing L and with maximal $c := \deg(E\cap Z)$ among the planes containing L. If $h^1(E, \mathscr{I}_{E\cap Z,E}(3)) > 0$, then [24, Corollaire 2] shows the existence of either R or D or T. Now assume $h^1(E, \mathscr{I}_{E\cap Z,E}(3)) = 0$. The residual sequence of E gives $h^1(N, \mathscr{I}_{\mathrm{Res}_E(Z\cap N),N}(2)) > 0$. Since $c \ge 5$, there is a line $R \subset N$ such that $\deg(R\cap \mathrm{Res}_U(Z\cap N) \ge 4$. To prove the lemma we may assume that $\deg(R\cap Z) = 4$. First assume $R\cap L = \emptyset$. Let $Q' \subset N$ be a general quadric containing $L\cup R$. Note that Q' is a smooth quadric. Since Z is curvilinear and $\mathscr{I}_{L\cup R,N}(2)$ is spanned, we have $Z\cap Q' = Z\cap (R\cup L)$. Since $h^1(Q', \mathscr{I}_{Z\cap(L\cup R,Q'}(3)) = 0$, we get $h^1(N, \mathscr{I}_{\mathrm{Res}_{Q'}(Z\cap N),N}(1)) > 0$, contradicting the inequality $\deg(\mathrm{Res}_{Q'}(Z\cap N)) \le 2$.

Now assume $R\cap L \ne \emptyset$ and $R \ne L$. Since $\deg(R\cap \mathrm{Res}_E(Z\cap N)) \ge 4$ and $E \supset L$, we have $\deg(Z\cap (R\cup L)) \ge 8$ and so we may take $D := R\cup L$.

Now assume $R = L$. We may take $Z' \subseteq Z\cap N$ minimal among the subschemes such that $h^1(N, \mathscr{I}_{Z',M}(3)) > 0$. Let Q' be a quadric surface containing L in its singular locus. Since $\deg(\mathrm{Res}_{Q'}(Z')) \le 10-4-4 = 2$, we have $h^1(M, \mathscr{I}_{\mathrm{Res}_Q(Z')}(1)) = 0$. Therefore the residual exact sequence of Q' gives $h^1(Q', \mathscr{I}_{Z'\cap Q',Q'}(t)) > 0$. The minimality of Z' gives $Z' \subset Q$. Since Z' is curvilinear we get $\deg(Z') = 8$ and that each connected component γ of Z' has even degree with $\deg(\gamma\cap L) = \deg(\gamma)/2$. Hence there is a plane $N' \supset L$ with $\deg(N\cap Z') > \deg(Z'\cap L) = 4$. We get $\deg(\mathrm{Res}_{N'}(Z')) \le 3$ and hence by a residual exact sequence of N' gives $h^1(N, \mathscr{I}_{Z',M}(3)) = 0$, a contradiction.

(b) Now assume $h^1(N, \mathscr{I}_{Z\cap N}(3)) = 0$. A twist of the residual exact sequence in step (b) of the proof of Lemma 6.1 gives $h^1(M, \mathscr{I}_{\mathrm{Res}_N(Z),M}(2)) > 0$. If $d-a \le 5$, then there is a line $L \subset M$ such that $\deg(\mathrm{Res}_N(Z)) \ge 4$ [23, Lemma 34]. By assumption we have $\deg(L\cap Z) = 4$. Since $\deg(Z\cap L) \ge 4$, the maximality property of a gives $a \ge 6$. Since $d-a \ge 5$, we also get $d = 11$. Let $U \subset M$ be a hyperplane such that $U \supset L$ and $\deg(U\cap Z)$ is maximal. If $h^1(U, \mathscr{I}_{U\cap Z,U}(3)) > 0$, then we may repeat part (a). Now assume $h^1(U, \mathscr{I}_{U\cap Z,U}(3)) = 0$. The residual sequence of U gives $h^1(N, \mathscr{I}_{\mathrm{Res}_U(Z),N}(2)) > 0$. Since $\deg(\mathrm{Res}_E(Z)) \le 4$, there is a line $R \subset N$ with $R \supset \mathrm{Res}_E(Z)$ and $\deg(\mathrm{Res}_E(Z)) = 4$. We conclude as in step (a). □

Lemma 6.3. *Let $X \subset \mathbb{P}^5$ be an integral and non-degenerate curve of degree $d \le 13$. Then $h^1(H, \mathscr{I}_{C\cap H,H}(t)) = 0$, $t = 3,4$, for a general hyperplane $H \subset \mathbb{P}^5$.*

Proof. The scheme $C\cap H$ spans H and it is in uniform position and in particular it is in linearly general position. Apply [22, Theorem 3.2]. □

Lemma 6.4. *Let $X \subset \mathbb{P}^5$ be an integral and non-degenerate curve of degree $d \ge 9$ (resp. $5 \le d \le 8$). Then $h^0(\mathscr{I}_X(2)) \le 6$ (resp. $h^0(\mathscr{I}_X(2)) \le 15-d$).*

Proof. Fix a general hyperplane $H \subset \mathbb{P}^5$. The scheme $S := X\cap H$ spans H and it is formed by d points in linearly general position in H. Hence $h^0(H, \mathscr{I}_{S,H}(2)) \le 6$ if $d \ge 9$ and $h^0(H, \mathscr{I}_{S,H}(2)) = 15-d$ if $d \le 8$. Use the exact sequence

$$0 \to \mathscr{I}_X(1) \to \mathscr{I}_X(2) \to \mathscr{I}_{X\cap H,H}(2) \to 0$$

and that X is non-degenerate, i.e., $h^0(\mathscr{I}_X(1)) = 0$. □

Lemma 6.5. *Assume $g \le 3$ and $d \le 11$. There is no non-degenerate $C \in M_{d,g}$ such that $h^1(\mathscr{I}_C(4)) > 0$ and there is no line $L \subset \mathbb{P}^5$ with $\deg(L\cap C) \ge 5$,no conic D with $\deg(C\cap D) \ge 8$ and no plane cubic T with $\deg(T\cap C) = 9$ and $C\cap T \in |\mathscr{O}_T(3)|$.*

Proof. Since $h^1(\mathscr{I}_C(4)) > 0$ and $\deg(R\cap C) \le 5$ for all lines R, we have $d \ge 9$ [19, Theorem at page 492]. By Lemmas 4.10, 6.1 and 6.2 we have $h^1(\mathscr{I}_C(3)) \ge 5 + h^1(\mathscr{I}_C(4)) \ge 10 + h^1(\mathscr{I}_C(5)) \ge 11$. By Lemma 6.3 we have $h^1(\mathscr{I}_C(2)) \ge h^1(\mathscr{I}_C(3))$. Hence $h^0(\mathscr{I}_C(2)) \ge 31+g-2d$. Use Lemma 6.4. □

Lemma 6.6. *Fix an integer $a > 0$ and assume $d \ge 2g-1+a$. Fix a zero-dimensional curvilinear scheme $Z \subset \mathbb{P}^5$ such that $\deg(Z) = a$. Set $E_Z := \{C \in M_{d,g} : Z \subset C\}$. Then every irreducible component of E_Z has dimension $\le 6d+2-2g-4a$.*

Proof. If $E_Z = \emptyset$, then the lemma is true. Hence we may assume $E_Z \neq \emptyset$. Fix $C \in E_Z$. By [25, Theoreme 1.5] it is sufficient to prove that $h^1(N_C(-Z)) = 0$. Since C is smooth, N_C is a quotient of $T\mathbb{P}^5_{|C}$ and hence by the Euler's sequence of $T\mathbb{P}^5$ the bundle N_C is a quotient of $\mathscr{O}_C(1)^{\oplus 6}$. Since $d \ge 2g-1+a$, we have $h^1(\mathscr{O}_C(1)(-Z)) = 0$. Use that $h^2(\mathscr{G}) = 0$ for every coherent sheaf $\mathscr{G}$ on C. □

Corollary 6.7. *Assume $d \ge 9$. Fix $a \in \{4,5,6\}$. Let $\mathscr{A}_a$ be the set of all non-degenerate $C \in M_{d,g}$ such that there is a line $R \subset \mathbb{P}^5$ such that $\deg(C\cap R) \ge a$. Then every irreducible component of $\mathscr{A}_a$ has dimension $\le 6d+2-2g+8-3a$*

Proof. Fix a line $R \subset \mathbb{P}^5$ and a zero-dimensional scheme $Z \subset R$ with $\deg(Z) = a$. First apply Lemma 6.6, then use that R has ∞^a zero-dimensional schemes of degree a and then use that $\mathbb{P}^5$ contains ∞^8 lines. □

Lemma 6.8. *Assume $0 \le g \le 3$ and $d \le 11$. Let $\mathscr{B}$ be the set of all non-degenerate $C \in M_{d,g}$ having a line R with $\deg(R\cap C) \ge 6$. Then a general element of $\mathbb{W}$ contains no element of $\mathscr{B}$.*

Proof. Fix $C \in \mathscr{B}$. The existence of R implies $d \ge 9$ and that $d \ge 10$ if $g > 0$. By Corollary 6.7 to prove the lemma it is sufficient to avoid all $C \in \mathscr{B}$ with $h^1(\mathscr{I}_C(4)) \ge 10$. Since $d \le 11$, Lemma 6.3 and the exact sequence in the proof of Lemma 6.4 for $X = C$ and $t = 3,4$ give $h^1(\mathscr{I}_C(2)) \ge 10$. Hence $h^0(\mathscr{I}_C(2)) \ge 30+g-2d$, contradicting Lemma 6.4. □

Lemma 6.9. *Assume $0 \le g \le 3$ and $d \le 11$. Let $\mathscr{B}'$ be the set of all non-degenerate $C \in M_{d,g}$ having a line R with $\deg(R\cap C) \ge 4$. Then a general element of $\mathbb{W}$ contains no element of $\mathscr{B}'$.*

Proof. By Corollary 6.7 it is sufficient to test all $C \in M_{d,g}$ with $h^1(\mathscr{I}_C(4)) \ge 4$. By Lemma 6.8 we may assume that C has no line R with $\deg(R\cap C) \ge 6$. Hence Lemmas 4.10 and 6.1 give $h^1(\mathscr{I}_C(3)) \ge 5 + h^1(\mathscr{I}_C(4)) \ge 9$. By Lemma 6.3 and the exact sequence in the proof of Lemma 6.4 for $t = 3$ and $X = C$ we have $h^1(\mathscr{I}_C(2)) \ge 9$ and so $h^0(\mathscr{I}_C(2)) \ge 31+g-2d$. Lemma 6.4 gives a contradiction. □

Lemma 6.10. *Assume $0 \le g \le 3$ and $d \le 11$. Let $\mathscr{B}_1$ be the set of all non-degenerate $C \in M_{d,g}$ having a conic D with $\deg(D\cap C) \ge 8$. Then a general element of $\mathbb{W}$ contains no element of $\mathscr{B}_1$.*

Proof. Fix $C \in \mathscr{B}_1$, say associated to the conic D, and take $W \in \mathbb{W}$ containing C (if any). By Lemma 6.9 we may assume the non-existence of lines L with $\deg(L\cap C) \ge 4$. Hence D is not a reducible conic. It is not a double conic, say with $L := A_{\mathrm{red}}$, because we would have $\deg(L\cap C) \ge \deg(A\cap C)/2 \ge 4$. Hence D is smooth. By Lemma 4.9 it is sufficient to test the curves C with $h^1(\mathscr{I}_C(4)) \ge 10$. Lemmas 4.10 and 6.1 give $h^1(\mathscr{I}_C(3)) \ge 15$. Lemma 6.3 and the cohomology exact sequence of the the exact sequence in the proof of Lemma 6.4) for $X = C$ and $t = 3$ give $h^1(\mathscr{I}_C(2)) \ge 15$ and so $h^0(\mathscr{I}_C(2)) \ge 14+g$, contradicting Lemma 6.4. □

Lemma 6.11. *Assume $0 \le g \le 3$ and $d \le 11$. Let $\mathscr{B}_2$ be the set of all non-degenerate $C \in M_{d,g}$ having a plane cubic T with $\deg(T\cap C) = 9$ and $C\cap T \in |\mathscr{O}_{C\cap T,T}(3)|$. Then a general element of $\mathbb{W}$ contains no element of $\mathscr{B}_2$.*

Proof. Take C for which T exists. We have $d = 11$. The set of all hyperplanes of $\mathbb{P}^5$ containing $\langle T\rangle$ induces a g^2_2 on C. Hence $g = 0$. Fix any scheme $Z \in |\mathscr{O}_T(3)|$. Since $g = 0$, Lemma 6.6 implies $h^1(N_C(-Z)) = 0$ and hence the set of all $C \subset \mathbb{P}^5$ containing Z has dimension $6d+1-4\deg(Z) = 31$. Since $\mathbb{P}^5$ has ∞^9 planes, each plane has ∞^9 plane cubics and each plane cubic T has ∞^9 elements of $|\mathscr{O}_T(3)|$, it is sufficient to exclude all $C \in \mathscr{B}_2$ with $h^1(\mathscr{I}_C(4)) \geq 9$. By Lemmas 6.9 and 6.10 we may assume the non-existence of line $R \subset \mathbb{P}^5$ with $\deg(C\cap R) \geq 4$ and of conics $D \subset \mathbb{P}^5$ with $\deg(C\cap D) \geq 8$. As in the proof Lemma 6.10 we get $h^1(\mathscr{I}_C(2)) \geq 14$, i.e $.h^0(\mathscr{I}_C(2)) \geq 13+g$, contradicting Lemma 6.4. □

By Lemma 5.5 at this point we proved that a general $W \in \mathbb{W}$ contains only finitely many non-degenerate $C \in M_{d,g}$.

7. Degenerate curves

In this section we prove that a general $W \in \mathbb{W}$ contains only finitely many degenerate $C \in M_{d,g}(Q)$, $d \leq 11$ and $g \leq 3$. By Remarks 4.3, 4.4 and Lemma 4.6 it is sufficient to test the curves $C \in M_{d,g}(4)$. By [19, Theorem at page 492] we may assume $d \geq 7$ and $d \geq 8$ if either $g > 0$ or C has genus 0 and no line R with $\deg(R\cap C) \geq 6$. By Remark 4.3 and Lemma 4.6 it is sufficient to test the degenerate $C \in M_{d,g}(Q)$. Fix a hyperplane $M \subset \mathbb{P}^5$ and set $Q' := Q\cap M$. Set $M'_{d,g}(Q') := \{C \in M_{d,g}(Q) : C \subset Q'$ and C spans $M\}$. Either Q' is smooth or Q' has a unique singular point, o. For any $C \in M'_{d,g}(Q')$ set $x(C) = 0$ if either Q' is smooth or Q' is a cone with vertex o and $o \notin C$, and set $x(C) := 1$ if Q' has vertex o and $o \in C$. Since $\omega_{Q'} \cong \mathscr{O}_{Q'}(-3)$, if $x(C) = 0$, then $\mathrm{Hilb}(Q')$ is smooth and of dimension $3d+2-2g$. Now assume that Q' is a cone with vertex o and that $x(C) = 1$, i.e. that $o \in C$. Let $u : \widetilde{Q}' \to Q'$ be the blowing up of o. Let $E := v^{-1}(o)$ be the exceptional divisor and let $\widetilde{C} \subset \widetilde{Q}'$ be the strict transform of C. Since C is smooth, v maps isomorphically $\widetilde{C}$. Let Ψ be closure in $\mathrm{Hilb}(\widetilde{Q}')$ of the strict transforms of all $A \in M_{d,g}(Q')$ with $x(A) = 1$. We claim that $\dim\Psi \leq 3d+1$. Fix $D \in \Psi$. Since $\mathrm{Aut}(\widetilde{Q}')$ acts transitively of $\widetilde{Q}'\setminus E$, the first part of the proof gives $h^1(N_{D,\widetilde{Q}}) = 0$. Hence it is sufficient to prove that $\deg(N_{D,\widetilde{Q}}) \leq 3d-1$, i.e. $\deg(\tau_{\widetilde{Q}|D}) \leq 3d+1$, i.e. $\deg(\omega_{\widetilde{Q}}|D) \geq -3d-1$. The group $\mathrm{Pic}(\widetilde{Q})$ is freely generated by E and the pull-back H of $\mathscr{O}_Q(1)$. We have $D\cdot H = d$ and $D\cdot E = x$. We have $\omega_{\widetilde{Q}} \cong \mathscr{O}_{\widetilde{Q}}(-3H-E)$ [26, Example 8.5 (2)]. Hence $\dim(M'_{d,g}(Q'))$ has dimension $\leq 3d+x(C)$ at C. Since Q has ∞^4 singular hyperplane sections and ∞^5 smooth hyperplane sections, to prove that a general $W \in \mathbb{W}$ has no (resp. finitely many) curves C spanning a hyperplane, it is sufficient to exclude the ones with $h^1(\mathscr{I}_C(4)) \geq d-4-g$. For all d,g for which we only use that $h^1(\mathscr{I}_C(4)) \geq d-5-g$, no degenerate $C \in M_{d,g}$ is contained in a general $W \in \mathbb{W}$. Fix a hyperplane $M \subset \mathbb{P}^5$. Let $M'_{d,g}(M)$ be the set of all $C \in M_{d,g}$ contained in M and spanning M.

Lemma 7.1. *A general $W \in \mathbb{W}$ contains no $C \in M_{d,g}$ such that there is a hyperplane M with $C \in M'_{d,g}(M)$ and $h^0(M,\mathscr{I}_C(2)) \geq 4$.*

Proof. Let $K \subset M$ denote the set-theoretic base locus of $|\mathscr{I}_{C,M}(2)|$ and A any irreducible component of K containing C. Note that $|\mathscr{I}_{C,M}(2)| = |\mathscr{I}_{A,M}(2)|$. Since C spans M, every element of $|\mathscr{I}_{C,M}(2)|$ is irreducible and A spans M. Hence $\dim(K) \leq 2$. First assume $\dim(A) = 2$. Since a complete intersection B of two quadrics of M has $h^0(M,\mathscr{I}_{B,M}(2)) = 2 < 4$ and A spans M, we get $\deg(A) = 3$. Hence either A is a smooth rational normal scroll or a cone over a rational normal curve of $\mathbb{P}^3$. In both cases we have $h^0(M,\mathscr{I}_{A,M}(2)) = 3$, a contradiction. Hence $\dim(A) = 1$, i.e. $A = C$. Fix two general elements Q_1, Q_2 of $|\mathscr{I}_{C,M}(2)|$ and let E be an irreducible component of $Q_1\cap Q_2$ containing C. Since $A = C$, there is a quadric hypersurface $Q_3 \subset M$, containing C, but not E. Since $C \subseteq E\cap Q_3$, we get $E = Q_1\cap Q_2$, $d \leq 8$, and that either $d = 8$ and $C = Q_1\cap Q_2\cap Q_3$ or $d = 7$ and C is linked to a line by the complete intersection $Q_1\cap Q_2\cap Q_3$. In both cases C is arithmetically Cohen-Macaulay and in particular $h^1(\mathscr{I}_C(4)) = 0$, a contradiction. □

Lemma 7.2. *A general $W \in \mathbb{W}$ contains no $C \in M_{11,g}$ such that there is a hyperplane M with $C \in M'_{11,g}(M)$ and $h^0(M,\mathscr{I}_{C,M}(2)) = 3$.*

Proof. Take K, A as in the proof of Lemma 7.1. Since $d > 8$, we only need to modify the proof of the case $\dim(A) = 2$. If $\dim(A) = 2$, then $\deg(A) = 3$ and A is either the cone of of a rational normal curves of $\mathbb{P}^3$ or it is a smooth rational normal curve isomorphic to the Hirzebruch surface F_1 embedded by the complete linear system $|h+2f|$. Write $C \in |ah+bf|$ with $a > 0$ and $b \geq a$. We have $11 = a+b$ and hence $b > a$. Since $\omega_{F_1} \cong \mathscr{O}_{F_1}(-2h-3f)$, the adjunction formula gives $2g-2 = (ah+bf)\cdot((a-2)h+(b-3)f) = -a(a-2)+a(b-3)+b(a-2) = (a-2)(b-a)+a(b-3)$. If $g = 0$ we get that either $a = 1$ (and hence $b = 10$) or $a = b = 2$, contradicting the equality $a+b = 10$. If $g > 0$, then $a \geq 2$. There is no solution with $a+b = 11$, $a \geq 2$, and $g \leq 3$. In the case $a = 1$ and $b = 10$ the curve C has $h^0(A,\mathscr{O}_A(4-C)) = 0$. Hence if $C \subset W$, then $A \subset W$, contradicting the fact that $\mathrm{Pic}(W)$ is generated by $\mathscr{O}_W(1)$.
Now assume that A is a cone over a rational normal curve. Let o be the vertex of A and call $u : F_2 \to A$ the blowing up of o. Set $h := u^{-1}(o)$. F_2 is isomorphic to the Hirzebruch surface with the same name, h is the only section of its ruling with negative self-intersection and u is induced by the linear system $|h+2f|$. We have $h^2 = -2$ and $\omega_{F_2} \cong \mathscr{O}_{F_2}(-2h-4f)$. Let $C' \subset F_2$ denote the strict transform of C, with $C' \in |ah+bf$ and $b \geq 2a$. Since C is smooth, u sends isomorphically C' to C. Hence $11 = b$ and $b \in \{2a, 2a+1\}$. Since $h^0(\mathscr{O}_{F_2}(4h+8f-C)) = 0$, any W containing C contains A, a contradiction. □

Lemma 7.3. *Fix $C \in M'_{d,g}(M)$, $d \leq 13$, and let H be a general hyperplane of M. We have $h^1(H,\mathscr{I}_{H\cap C,H}(4)) = 0$ and $h^1(H,\mathscr{I}_{H\cap C,H}(3)) \leq \max\{0, d-10\}$.*

Proof. Any $S \subseteq C \cap H$ with $\#(S) \leq 10$ (resp. $\#(S) \leq 13$) is in linearly general position in M and hence $h^(M, \mathscr{I}_{S,M}(3)) = 0$ (resp. $h^1(M, \mathscr{I}_{C,M}(4)) = 0$ by [22, Theorem 3.2]. □

Lemma 7.4. *Let $N \subset M$ be a hyperplane and let $Z \subset N$ be a degree $d \leq 11$ zero-dimensional scheme spanning N. If there are neither a line $R \subset N$ with $\deg(R \cap Z) \geq 6$ nor a plane conic $D \subset N$ with $\deg(D \cap Z) = 10$, then $h^1(N, \mathscr{I}_{Z,N}(4)) = 0$.*

Proof. Let $U \subset N$ be a plane of N with maximal $a := \deg(Z \cap N)$. Since Z spans N, we have $a \geq 3$. Assume for the moment $a = 3$, i.e. assume that Z is in linearly general position. Since $d \leq 13$, we have $h^1(N, \mathscr{I}_{Z,MN}(4)) = 0$ [22, Theorem 3.2]. Hence we may assume $a \geq 4$.
First assume $h^1(U, \mathscr{I}_{Z \cap U,U}(4)) > 0$. Since Z spans N, we have $a \leq d - 1 \leq 10$. Use [24, Corollaire 2 or Remarques (i) at page 116].
Now assume $h^1(N, \mathscr{I}_{Z \cap N}(4)) = 0$. The residual exact sequence of U in N gives $h^1(N, \mathscr{I}_{\mathrm{Res}_U(Z)}(3)) > 0$. Since $\deg(\mathrm{Res}_U(Z)) = d - a \leq 7$, [23, Lemma 34] gives the existence of a line $L \subset N$ such that $\deg(L \cap Z) \geq 5$. Then we continue as in step (a) of the proof of Lemma 6.2. the residual exact sequence of M gives $h^1(M, \mathscr{I}_{\mathrm{Res}_N(Z),M}(3)) > 0$. Since $d - a \leq 7$, then there is a line $L \subset M$ such that $\deg(\mathrm{Res}_N(Z)) \geq 5$ [23, Lemma 34]. By assumption we have $\deg(L \cap Z) = 5$. Since $\deg(Z \cap L) \geq 5$, the maximality property of a gives $a \geq 7$. Since $d - a \geq 5$, we get $d \geq 12$, a contradiction. □

Lemma 7.5. *A general $W \in \mathbb{W}$ contains no $C \in M'_{d,g}(M)$ such that there a plane conic D with $\deg(D \cap C) \geq 10$ (if D is singular also assume that $\deg(L \cap C) \leq 5$ for each line $L \subset D$).*

Proof. The pencil of hyperplanes of M containing the plane U spanned by D shows that $d = 11$, $\deg(D \cap C) = 10$, and $g = 0$. First assume that D is a double line. Fix $W \in \mathbb{W}$ with $W \supset C$. Set $L := D_{\mathrm{red}}$. Since $\deg(L \cap C)$, we have $L \subset W$ for any $W \in \mathbb{W}$ with $W \supset C$. Let $\mathrm{Res}_L(C \cap D)$ be the residual scheme with respect to the divisor L of U. Since $\deg(C \cap L) \geq \deg(C \cap D)/2$, our assumptions give $\deg(L \cap C) = 5$ and hence $\deg(\mathrm{Res}_L(C \cap D)) = 5$. Since $C \cap D \subset D$, we have $\mathrm{Res}_L(C \cap D) \subset L$. Since $D \nsubseteq W$ (Lemma 4.9), we have $W \cap U = L \cup T$ with T a plane cubic not containing L. Hence $\deg(L \cap T)$ =3. Since $\mathrm{Res}_L(C \cap D)$ is contained both in L and in T, we get a contradiction.
Now assume $D = R \cup L$ with R, L lines and $L \neq R$. Since $\deg(L \cap C) \leq 5$ and $\deg(R \cap C) \leq 5$ by assumption, we have $\deg(R \cap C) = \deg(R \cup L) = 5$. Hence $D \subset W$, contradicting Lemma 4.9.
Now assume that D is smooth. Since $g = 0$ for each $Z \subset D$ with $\deg(D) = 10$, we have $h^1(N_{C,M}(-Z)) = 0$ and so $h^0(N_{C,M}) = 45 - 30$. Since D has ∞^{10} degree 10 subschemes, M has ∞^6 planes, each plane has ∞^5 conics and $\mathbb{P}^5$ has ∞^5, hyperplanes, each irreducible component of the set of all (C, D, M) with D a smooth conic and $C\imath M'_{11,0}(M)$ has dimension at most 41, i.e. codimension at least 17 in $M_{11,0}$. Hence to avoid these curves we may assume $h^1(\mathscr{I}_C(4)) \geq 16$. Lemma 7.3 gives $h^1(M, \mathscr{I}_C(2)) \geq 15$. Hence $h^0(M, \mathscr{I}_C(2)) \geq 7$, contradicting Lemma 7.1. □

Lemma 7.6. *A general $W \in \mathbb{W}$ contains no $C \in M'_{d,g}(M)$, $d \leq 11$, for some hyperplane M such that there is no line $R \subset M$ with $\deg(R \cap C) \geq 6$.*

Proof. By Lemma 7.5 we may assume that there is no conic D with $\deg(D \cap C) \geq 10$. Since $d \leq 11$, Lemmas 4.10 and 7.4 give $h^1(M, \mathscr{I}_{C,M}(3)) \geq 4 + h^1(\mathscr{I}_{C \cap M,M}(3)) \geq d - g$. Assume for the moment that either $d \leq 10$ or $d = 11$ and $h^1(H, \mathscr{I}_{C \cap H,H}(3)) = 0$ for a general hyperplane H of M. Lemma 7.3 gives $h^1(M, \mathscr{I}_{C,M}(2)) \geq d - g$ and so $h^0(M, \mathscr{I}_C(2)) \geq 15 + d - g - 2d - 1 + g = 14 - d$. Hence if $d \leq 10$ Lemma 7.1 concludes the proof. If $d = 11$ and $h^1(H, \mathscr{I}_{C \cap H,H}(3)) = 1$, we get $h^0(M, \mathscr{I}_C(2)) \geq 2$. Assume $h^0(\mathscr{I}_C(2)) = 2$ and let K be the intersection of two general elements of $|\mathscr{I}_{C,M}(2)|$ and call $A \subseteq K_{\mathrm{red}}$ any irreducible component containing C. Since $h^1(M, \mathscr{I}_{C,M}(3)) \geq 11 - g$, we have $h^0(M, \mathscr{I}_C(3)) \geq 45 - 2d > 10$. Hence the map $H^0(M, \mathscr{I}_{C,M}(2)) \otimes H^0(\mathscr{O}_M(1)) \to H^0(M, \mathscr{I}_{C,M}(3))$ is not surjective. Take $U \in |\mathscr{I}_{C,M}(3)|$ not containing K. Since $\deg(C) > 9$, we first get $A = K$, and then (since $d = 11$), that the complete intersection $K \cap U$ links C to a line. Hence C is arithmetically Cohen-Macaulay, contradicting the assumption $h^1(M, \mathscr{I}_{C,M}(4)) > 0$. □

Lemma 7.7. *A general $W \in \mathbb{W}$ contains no curve C with $C \in M'_{d,g}(M)$ for some hyperplane and with a line R such that $\deg(R \cap C) \geq 6$.*

Proof. Note that if W, C, R are as in the statement of the lemma with $C \subset W$, then $R \subset W$ (Bezout). Let $\mathscr{G}$ be the set of all quadruples (W, H, L, C) with $W \in \mathbb{W}'$, M a hyperplane, $L \subset W \cap M$ a line, $C \in M'_{d,g}(M)$ and $\deg(L \cap C) \geq 6$. Fix M, a line $L \subset M$ and $Z \subset R$ with $\deg(Z) = 6$. First assume $d \geq 2g - 1 + 6$. Lemma 6.6 gives $h^1(M, N_{C,M}(-Z)) = 0$, i.e. $h^0(N_{C,M}(-Z)) = 5d + 1 - g - 18$. Since L has ∞^6 degree 6 zero-dimensional schemes, M has ∞^6 lines and $\mathbb{P}^5$ has ∞^5 hyperplanes, and each $W \in \mathbb{W}'$ contains only finitely many lines, we get that each irreducible component of $\mathscr{G}$ has dimension at most $5d - g$. Hence to prove the lemma it is sufficient to exclude the curves $C \in M'_{d,g}(M)$ with $h^1(\mathscr{I}_C(4)) \geq d - g + 2$. Lemma 7.3 gives $h^1(M, \mathscr{I}_{C,M}(3)) \geq d - g + 2$. Hence $h^1(M, \mathscr{I}_{C,M}(2)) \geq d - g + 1$ (Lemma 7.3) and so $h^0(M, \mathscr{I}_{C,M}(2)) \geq 15 - d \geq 4$, contradicting Lemma 7.1. Now assume $d \leq 2g + 4$. Since $d \geq 7$ and $g = 0$ if $d = 7$, then $(d, g) \in \{(8,2), (8,3), (9,3), (10,3)\}$. Assume $d = 8$. The net of all hyperplanes of M containing R induces a g^2_2 on C and hence $g = 0$, a contradiction. Now assume $(d, g) \in \{(9,3), (10,3)\}$. We take $Z' \subset R$ with $\deg(Z') = 4$. Since $d \geq 2g - 1 + \deg(Z')$, as above we get that we may assume $h^1(\mathscr{I}_C(4)) \geq d - g$. Since $d \leq 10$, we have $h^1(M, \mathscr{I}_{C,M}(2)) \geq h^1(M, \mathscr{I}_{C,M}(3)) \geq h^1(M, \mathscr{I}_{C,M}(4))$ (Lemma 7.3) and hence $h^0(M, \mathscr{I}_{C,M}(2)) \geq 14 - d \geq 4$, contradicting Lemma 7.1. □

End of the proof of Theorem 1.1: The last lemma concludes the proof of Theorem 1.1 for all $C \in M_{d,g}(4)$. Since in section 6 we checked all $C \in M_{d,g}(5)$, in Remark 4.3 all $C \in M_{d,g}(1)$, in Remark 4.4 all $C \in M_{d,g}(2)$ and in Lemma 4.6 all $C \in M_{d,g}(3)$, we have completed the proof of Theorem 1.1. □

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