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LOCAL T_0 AND T_1 QUANTALE-VALUED PREORDERED SPACES

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Abstract. In this paper, we characterize explicitly the separation properties T_0 and T_1 at a point p in the topological category of quantale-valued preordered spaces and investigate how these characterizations are related. Moreover, we prove that local T_0 and T_1 quantale-valued preordered spaces are hereditary and productive.

1. Introduction

Classical separation axioms of topology have been extended to topological category by several authors. Baran [2], in 1991, introduced these axioms in a set-based topological category in terms of initial, final structures and discreteness. He defined separation properties first locally, i.e., at a point p [4], then they are expanded to point-free concepts. Using local lower separation axioms, Baran [2, 3] introduced the notion of (strongly) closedness in set-based topological categories that creates closure operators in sense of Dikranjan and Giuli [16] in some well-known topological categories **Conv** (the category of convergence spaces and filter convergence maps) [11], **Lim** (the category of limit spaces and filter convergence maps) [9], **Prord** (the category of preordered sets and monotone maps) [12] and **SUConv** (the category of semiuniform convergence spaces and uniformly continuous maps) [14]. The other significant use of these concepts to define the notions of Hausdorffness [5], compactness, perfectness [9], connectedness [10], regular, completely regular, normal objects [7, 8] in set-based topological categories.

A topological space defines a preorder (reflexive and transitive) relation, and a topology can be obtained from a preorder relation on a set [17, 20]. This indicates

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a connection between topology and order. Domain theory that was introduced by Dana Scott in the 1960s, is a branch of order theory which studies special kinds of partially ordered sets generally named as domains. In computer science, this field is used to establish denotational semantics, particularly for functional programming languages [18, 29]. Domain theory is closely related to topology and formalizes the intuitive principles of convergence and approximation in a general way.

With the advancement of lattice theory, distinct mathematical frameworks have been studied with lattice structures including lattice-valued topology [15], quantale-valued approach space [23, 24, 28], quantale-valued metric space [25], lattice-valued convergence space [22] and lattice-valued preordered space [15]. This motivates us to study local T_0 and T_1 separation axioms in quantale-valued preordered spaces.

The main purpose of this paper is to give an explicit characterization for the local T_0 and T_1 separation axioms in the category of quantale-valued preordered spaces as well as to examine the relationship between them and to investigate their some invariance properties.

2. Preliminaries

Recall [24] that a partially ordered set $(L, \leq)$ is called a complete lattice if all subsets of L have both infimum ($\bigwedge$) and supremum ($\bigvee$). For any complete lattice, the bottom element and top element is denoted by $\perp$ and $\top$, respectively.

Definition 1. *Let $(L, \leq)$ be a complete lattice. We identify*

(1) $\alpha \lhd \beta$ *(the well-below relation) if $\forall X \subseteq L$ such that $\beta \leq \bigvee X$ there exists $\delta \in X$ such that $\alpha \leq \delta$.*
(2) $\alpha \prec \beta$ *(the well-above relation) if $\forall X \subseteq L$ such that $\bigwedge X \leq \alpha$ there exists $\delta \in X$ such that $\delta \leq \beta$.*

Definition 2. *A complete lattice $(L, \leq)$ is called a completely distributive iff for any $\alpha \in L$, $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$.*

Definition 3. *The triple $(L, \leq, *)$ is called a quantale if the following conditions are satisfied.*

(1) *$(L, \leq)$ is a complete lattice.*
(2) *$(L, *)$ is a semi group.*
(3) *$(\bigvee_{i\in I} \alpha_i)*\beta = \bigvee_{i\in I}(\alpha_i*\beta)$ and $\beta*(\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I}(\beta*\alpha_i)$ for all $\alpha_i, \beta \in L$,*

Note that if $(L, *)$ is a commutative semi group, then the quantale $(L, \leq, *)$ is named as commutative and if for all $\alpha \in L$, $\alpha * \top = \top * \alpha = \alpha$, then it is called integral.

We denote a quantale by $\mathsf{L} = (L, \leq, *)$ if it is integral and commutative, where $(L, \leq)$ is completely distributive.

A quantale $\mathsf{L} = (L, \leq, *)$ is named as a value quantale if $(L, \leq)$ is completely distributive lattice such that $\forall \alpha, \beta \lhd \top$, $\alpha \vee \beta \lhd \top$ [19].

Definition 4. *[25, 30] Let $X \neq \emptyset$ be a set. A map $\mathsf{R} : X \times X \to \mathsf{L} = (L, \leq, *)$ is called an L-preorder relation on X and the pair (X, R) is called an L-preordered space if it satisfies*

(1) *reflexivity, i.e., for all $x \in X$, $\mathsf{R}(x, x) = \top$,*
(2) *transitivity, i.e., for all $x, y, z \in X$, $\mathsf{R}(x, y) * \mathsf{R}(y, z) \leq \mathsf{R}(x, z)$.*

Note that an L-preordered space (X, R) is named as an L-equivalence space (X, R) if for all $x, y \in X$, $\mathsf{R}(x, y) = \mathsf{R}(y, x)$ (symmetry). Also, (X, R) is called separated L-preordered space if $x = y$ whenever $\mathsf{R}(x, y) = \top$.

A map $f : (X, \mathsf{R}_X) \to (Y, \mathsf{R}_Y)$ is called an L-order preserving map if for all $x_1, x_2 \in X$, $\mathsf{R}_X(x_1, x_2) \leq \mathsf{R}_Y(f(x_1), f(x_2))$.

Let L-**Prord** denotes the category whose objects are L-preordered spaces and morphisms are L-order preserving mappings.

Example 1. (i) *For $\mathsf{L} = 2 = (\{0, 1\}, \leq, \wedge)$, **2-Prord** $\cong$ **Prord**, where **Prord** is the category of preordered sets and order preserving maps.*

(ii) *For $\mathsf{L} = ([0, \infty], \geq, +)$ (Lawvere's quantale), $[0, \infty]$-**Prord** $\cong$ ∞**qMet**, where ∞**qMet** is the category of extended quasi metric spaces and non-expansive maps.*

(iii) *For $\mathsf{L} = (\triangle^+, \leq, *)$ (distance distribution functions quantale defined in [24]), then $\triangle^+$-**Prord** $\cong$ **ProbqMet**, where **ProbqMet** is the category of probabilistic quasi metric spaces and non-expansive maps [19].*

Note that in some literature, L-preordered space is often called a continuity space if L is a value quantale (see [19]), an L-metric space (see [25]) and an L-category (see [21]).

Recall [1], let E be a category, **Set** be the category of sets and functions and $\mathsf{U} : \mathsf{E} \to$ **Set** be a functor. U is called topological or E is called topological category on **Set** if

(i) U is amnestic and faithful (i.e., concrete),
(ii) U consists of small fibers,
(iii) Every U-source has a unique initial lift.

In addition, a topological functor is said to be normalized if constant objects, i.e., subterminals, have a unique structure.

Note that the forgetful functor $\mathsf{U} : \mathsf{L}$-**Prord** $\to$ **Set** is topological (see [21]) and it is also normalized.

Lemma 1. *[21] Let (X_i, R_i) be a collection of L-preordered spaces. A source $(f_i : (X, \mathsf{R}) \to (X_i, \mathsf{R}_i))_{i \in I}$ is initial in L-**Prord** iff $\forall a, b \in X$,*

$$\mathsf{R}(a, b) \quad = \quad \bigwedge_{i \in I} \mathsf{R}_i(f_i(a), f_i(b)).$$

Lemma 2. *[21] Let X be a non-empty set and (X, R) be an L-preordered space. For all $a, b \in X$,*

(i) *The discrete* L*-preorder structure on* X *is stated by*

$$\mathsf{R}_{dis}(a,b) = \begin{cases} \top, & a = b, \\ \bot, & a \neq b. \end{cases}$$

(ii) *The indiscrete* L*-preorder structure on* X *is stated by*

$$\mathsf{R}_{ind}(a,b) \quad = \quad \top.$$

3. Local T_0 and T_1 Objects

Let X be a set, $p \in X$ be a point and $X \vee_p X$ be the wedge product of X at p [2], i.e., two separate copies of X identified at p.

In the wedge $X \vee_p X$, a point x is represented as x_k if it lies in the k-th component for $k = 1, 2$.

Definition 5. *[2] Let* $X \vee_p X$ *be the wedge product at* p *and* X^2 *be the cartesian product of* X*.*

(1) $\mathsf{A}_p : X \vee_p X \to X^2$ *(the principal p-axis mapping) is given by*

$$\mathsf{A}_p(x_1) = (x,p) \text{ and } \mathsf{A}_p(x_2) = (p,x).$$

(2) $\mathsf{S}_p : X \vee_p X \to X^2$ *(the skewed p-axis mapping) is given by*

$$\mathsf{S}_p(x_1) = (x,x) \text{ and } \mathsf{S}_p(x_2) = (p,x).$$

(3) $\nabla_p : X \vee_p X \to X$ *(the fold mapping at p) is given by*

$$\nabla_p(x_1) = \nabla_p(x_2) = x.$$

Definition 6. *Let* (X,τ) *be topological space and* $p \in X$*. For each point* $x \neq p$*, there exists an open set* A *such that* $p \in A$*,* $x \notin A$ *or (resp. and) there exists an open set* B *such that* $x \in B$*,* $p \notin B$*, then* (X,τ) *is said to be* T_0 *(resp.* T_1*) at* p *[2, 6].*

Theorem 1. *Let* (X,τ) *be topological space and* $p \in X$*. Then* (X,τ) *is* T_0 *(resp.* T_1*) at* p *iff the initial topology induced by* $\{\mathsf{A}_p$ *(resp.* $\mathsf{S}_p) : X \vee_p X \to (X^2,\tau_*)$ *and* $\nabla_p : X \vee_p X \to (X, P(X))\}$ *is discrete, where* τ_* *is the product topology on* X^2*.*

Proof. The proofs are given in [6]. □

Definition 7. *[2] Let* $\mathsf{U} : \mathsf{E} \to$ **Set** *be topological functor,* $X \in Ob(\mathsf{E})$ *with* $\mathsf{U}(X) = B$ *and* $p \in B$*.*

(i) X *is* $\overline{T_0}$ *at* p *provided that the initial lift of the* U*-source* $\{\mathsf{A}_p : B \vee_p B \to \mathsf{U}(X^2) = B^2$ *and* $\nabla_p : B \vee_p B \to \mathsf{UD}(B) = B\}$ *is discrete, where* D *is the discrete functor that is a left adjoint to* U*.*

(ii) X *is* T_1 *at* p *provided that the initial lift of the* U*-source* $\{\mathsf{S}_p : B \vee_p B \to \mathsf{U}(X^2) = B^2$ *and* $\nabla_p : B \vee_p B \to \mathsf{UD}(B) = B\}$ *is discrete.*

Remark 1. (1) *Separation axioms $\overline{T}_0$ at p and T_1 at p are used to identify the notions of (strong) closedness in arbitrary set-based topological categories [2, 3].*

(2) *In **Top** (the category of topological spaces and continuous mappings), by Theorem 1, $\overline{T}_0$ at p and T_1 at p reduce to Definition 6 [2].*

(3) *A topological space X is T_i, $i = 0, 1$ if and only if X is T_i, $i = 0, 1$, at p for all points p in X ([6], Theorem 1.5(5)).*

(4) *Let $\mathsf{U} : \mathsf{E} \to \mathbf{Set}$ be a topological functor, X an object in E and $p \in \mathsf{U}(X)$ be a retract of X, i.e., the initial lift $h : \overline{1} \to X$ of the U-source $p : 1 \to \mathsf{U}(X)$ is a retract, where 1 is the terminal object in **Set**. Then if X is $\overline{T_0}$ (resp. T_1), then X is $\overline{T_0}$ at p (resp. T_1 at p) but the converse of implication is not true, in general ([4], Theorem 2.6).*

(5) *Specially, if $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ is normalized, then $\overline{T_0}$ and T_1 imply $\overline{T_0}$ at p and T_1 at p, respectively. ([4], Corollary 2.7).*

Theorem 2. *An L-preordered space (X, R) is $\overline{T_0}$ at p iff $\mathsf{R}(x, p) \wedge \mathsf{R}(p, x) = \bot$ for all $x \in X$ distinct from p.*

Proof. Assume (X, R) is $\overline{T_0}$ at p and $x \in X$ with $x \neq p$. Let R_{dis} be the discrete L-preorder relation on X and for $i = 1, 2$, $\pi_i : X^2 \to X$ be the projection maps. For $x_1, x_2 \in X \vee_p X$,

$$\begin{aligned}
\mathsf{R}(\pi_1 \mathsf{A}_p(x_1), \pi_1 \mathsf{A}_p(x_2)) &= \mathsf{R}(\pi_1(x, p), \pi_1(p, x)) = \mathsf{R}(x, p)\\
\mathsf{R}(\pi_2 \mathsf{A}_p(x_1), \pi_2 \mathsf{A}_p(x_2)) &= \mathsf{R}(\pi_2(x, p), \pi_2(p, x)) = \mathsf{R}(p, x)\\
\mathsf{R}_{dis}(\nabla_p(x_1), \nabla_p(x_2)) &= \mathsf{R}_{dis}(x, x) = \top
\end{aligned}$$

Since (A, R) is $\overline{T_0}$ and $x_1 \neq x_2$, by Definition 7 and Lemmas 1, 2,

$$\begin{aligned}
\bot &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{A}_p(x_1), \pi_1 \mathsf{A}_p(x_2)), \mathsf{R}(\pi_2 \mathsf{A}_p(x_1), \pi_2 \mathsf{A}_p(x_2)), \mathsf{R}_{dis}(\nabla_p(x_1), \nabla_p(x_2))\}\\
&= \bigwedge \{\mathsf{R}(x, p), \mathsf{R}(p, x), \top\}\\
&= \mathsf{R}(x, p) \wedge \mathsf{R}(p, x)
\end{aligned}$$

Hence, we have $\mathsf{R}(x, p) \wedge \mathsf{R}(p, x) = \bot$.

Conversely, let R' be the initial L-preorder relation on $X \vee_p X$ induced by $\mathsf{A}_p : X \vee_p X \to \mathsf{U}(X^2, \mathsf{R}^2) = X^2$ and $\nabla_p : X \vee_p X \to \mathsf{U}(X, \mathsf{R}_{dis}) = X$, where R^2 is the product structure on X^2 induced by the projection maps π_1 and π_2.

Assume that the condition holds, i.e., for all $x \in X$ distinct from p, $\mathsf{R}(x, p) \wedge \mathsf{R}(p, x) = \bot$. Let v and w be any points in the wedge.

(1) If $v = w$, then $\mathsf{R}'(v, w) = \top$.

(2) If $v \neq w$ and $\nabla_p v \neq \nabla_p w$, then $\mathsf{R}_{dis}(\nabla_p v, \nabla_p w) = \bot$. By Lemma 1,

$$\begin{aligned}
\mathsf{R}'(v, w) &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{A}_p v, \pi_1 \mathsf{A}_p w), \mathsf{R}(\pi_2 \mathsf{A}_p v, \pi_2 \mathsf{A}_p w), \mathsf{R}_{dis}(\nabla_p v, \nabla_p w)\}\\
&= \bot
\end{aligned}$$

(3) Suppose $v \neq w$ and $\nabla_p v = \nabla_p w$. It follows that $\nabla_p v = x = \nabla_p w$ for some points $x \in X$ with $x \neq p$. We must have $v = x_1$ and $w = x_2$ or $v = x_2$ and $w = x_1$ since $v \neq w$.

(a) If $v = x_1$ and $w = x_2$, then

$$\begin{aligned} \mathsf{R}(\pi_1 \mathsf{A}_p v, \pi_1 \mathsf{A}_p w) &= \mathsf{R}(x,p) \\ \mathsf{R}(\pi_2 \mathsf{A}_p v, \pi_2 \mathsf{A}_p w) &= \mathsf{R}(p,x) \\ \mathsf{R}_{dis}(\nabla_p v, \nabla_p w) &= \mathsf{R}_{dis}(x,x) = \top \end{aligned}$$

and it follows that

$$\begin{aligned} \mathsf{R}'(v,w) &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{A}_p v, \pi_1 \mathsf{A}_p w), \mathsf{R}(\pi_2 \mathsf{A}_p v, \pi_2 \mathsf{A}_p w), \mathsf{R}_{dis}(\nabla_p v, \nabla_p w)\} \\ &= \bigwedge \{\mathsf{R}(x,p), \mathsf{R}(p,x), \top\} \\ &= \mathsf{R}(x,p) \wedge \mathsf{R}(p,x) \end{aligned}$$

By the assumption $\mathsf{R}(x,p) \wedge \mathsf{R}(p,x) = \bot$, we get $\mathsf{R}'(v,w) = \bot$.

(b) Similarly, if $v = x_2$ and $w = x_1$, then $\mathsf{R}'(v,w) = \bot$.

Consequently, for all v, w in the wedge $X \vee_p X$, we obtain

$$\mathsf{R}'(v,w) = \begin{cases} \top, & v = w \\ \bot, & v \neq w \end{cases}$$

By Lemma 2, R' is the discrete L-preorder relation on the wedge. Hence, by Definition 7, (X, R) is $\overline{T_0}$ at p. □

Theorem 3. *An* L*-preordered space* (X, R) *is* T_1 *at* p *iff* $\mathsf{R}(x,p) = \bot = \mathsf{R}(p,x)$ *for all* $x \in X$ *distinct from* p.

Proof. Assume that (X, R) is T_1 at p and $x \in X$ with $x \neq p$. Let $v = x_1$, $w = x_2 \in X \vee_p X$. Note that,

$$\begin{aligned} \mathsf{R}(\pi_1 \mathsf{S}_p v, \pi_1 \mathsf{S}_p w) &= \mathsf{R}(\pi_1(x,x), \pi_1(p,x)) = \mathsf{R}(x,p) \\ \mathsf{R}(\pi_2 \mathsf{S}_p v, \pi_2 \mathsf{S}_p w) &= \mathsf{R}(\pi_2(x,x), \pi_2(p,x)) = \mathsf{R}(x,x) = \top \\ \mathsf{R}_{dis}(\nabla_p v, \nabla_p w) &= \mathsf{R}_{dis}(x,x) = \top \end{aligned}$$

where R_{dis} is the discrete L-preorder relation on X and for each $i = 1, 2$, $\pi_i : X^2 \to X$ is the projection map. Since $v \neq w$ and (X, R) is T_1 at p, by Definition 7 and Lemmas 1, 2,

$$\begin{aligned} \bot &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{S}_p v, \pi_1 \mathsf{S}_p w), \mathsf{R}(\pi_2 \mathsf{S}_p v, \pi_2 \mathsf{S}_p w), \mathsf{R}_{dis}(\nabla_p v, \nabla_p w)\} \\ &= \bigwedge \{\mathsf{R}(x,p), \top\} \\ &= \mathsf{R}(x,p) \end{aligned}$$

Similarly, if $v = x_2$, $w = x_1 \in X \vee_p X$, then by Lemma 1, we have

$$\bot = \bigwedge \{\mathsf{R}(p,x), \top\} = \mathsf{R}(p,x)$$

Conversely, let R' be the initial L-preorder relation on $X \vee_p X$ induced by $\mathsf{S}_p : X \vee_p X \to \mathsf{U}(X^2, \mathsf{R}^2) = X^2$ and $\nabla_p : X \vee_p X \to \mathsf{U}(X, \mathsf{R}_{dis}) = X$, where R^2 is the product structure on X^2 induced by the projection maps π_1 and π_2.

Assume that for all $x \in X$ distinct from p, $\mathsf{R}(x,p) = \bot = \mathsf{R}(p,x)$. Let v and w be any points in the wedge.

(1) If $v = w$, then $\mathsf{R}'(v,w) = \top$.

(2) If $v \neq w$ and $\nabla_p v \neq \nabla_p w$, then $\mathsf{R}_{dis}(\nabla_p v, \nabla_p w) = \bot$ since R_{dis} is the discrete structure. By Lemma 1,

$$\begin{aligned}\mathsf{R}'(v,w) &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{S}_p v, \pi_1 \mathsf{S}_p w), \mathsf{R}(\pi_2 \mathsf{S}_p v, \pi_2 \mathsf{S}_p w), \mathsf{R}_{dis}(\nabla_p v, \nabla_p w)\} \\ &= \bot\end{aligned}$$

(3) Suppose $v \neq w$ and $\nabla_p v = \nabla_p w$. It follows that we must have $v = x_1$ and $w = x_2$ or $v = x_2$ and $w = x_1$.

If $v = x_1$ and $w = x_2$, then by Lemma 1,

$$\mathsf{R}'(v,w) = \bigwedge \{\mathsf{R}(x,p), \top\} = \mathsf{R}(x,p)$$

By the assumption $\mathsf{R}(x,p) = \bot = \mathsf{R}(p,x)$, we get $\mathsf{R}'(v,w) = \bot$.

Similarly, we obtain $\mathsf{R}'(v,w) = \bot$ for $v = x_2$ and $w = x_1$.

Hence, for all $v, w \in X \vee_p X$, we have

$$\mathsf{R}'(v,w) = \begin{cases} \top, & v = w \\ \bot, & v \neq w \end{cases}$$

By Lemma 2, it follows that R' is the discrete L-preorder relation on the wedge. Consequently, by Definition 7, (X, R) is T_1 at p. □

Example 2. *Let $*$ be a binary operation identified as $\forall \alpha, \beta \in [0,1]$, $\alpha * \beta = (\alpha - 1 + \beta) \vee 0$ and $\mathsf{L} = ([0,1], \leq, *)$ be a triangular norm (Lukasiewicz t-norm) [26], where the bottom and top elements are $\bot = 0$ and $\top = 1$, respectively. Let $X = \{a,b,c\}$ and an L-preorder relation $\mathsf{R} : X \times X \to \mathsf{L}$ defined by*

$$\mathsf{R}(v,w) = \begin{cases} \top, & v = w \\ \frac{1}{2}, & (v,w) = (a,c) \\ \bot, & \textit{otherwise.} \end{cases}$$

Clearly, (X, R) is an L-preordered space. By Theorem 2, (X, R) is $\overline{T}_0$ at p for all $p \in X$, and by Theorem 3, (X, R) is T_1 at b but it is neither T_1 at a nor at c.

Remark 2. (1) *By Theorems 2 and 3, if an L-preordered space (X, R) is T_1 at p, then it is $\overline{T}_0$ at p. But in general, the converse is not true (see previous Example).*

(2) *In an arbitrary set-based topological category, $\overline{T_0}$ at p and T_1 at p objects may be equivalent, for example, in* **Prox** *(the category of proximity spaces and p-maps) [27],* **CP** *(the category of pairs and pair preserving maps) [3],*

Born *(the category of bornological spaces and bounded maps) [3],* **SULim** *(the category of semiuniform limit spaces and uniformly continuous maps) [13], Remark 3.6.*

4. Hereditary and Productive Properties

Definition 8. *Let (X, R) be an* L*-preordered space and $A \subset X$. A subspace (A, R_A) is defined by $\mathsf{R}_A(x, y) = \mathsf{R}(x, y)$ for all $x, y \in A$, where R_A is the initial* L*-preorder structure on A induced by the inclusion map $i : A \to X$.*

Theorem 4. *Let (X, R) be an* L*-preordered space, $A \subset X$ and $p \in A$.*

(i) *If (X, R) is $\overline{T}_0$ at p, then (A, R_A) is $\overline{T}_0$ at p.*
(ii) *If (X, R) is T_1 at p, then (A, R_A) is T_1 at p.*

Proof. (i) Suppose that $p \in A$ and (X, R) is $\overline{T}_0$ at p. By Theorem 2, $\mathsf{R}(x, p) \wedge \mathsf{R}(p, x) = \bot$ for $x \in A \subset X$ with $x \neq p$. By Definition 8, we have $\mathsf{R}_A(x, p) = \mathsf{R}(x, p)$ and $\mathsf{R}_A(p, x) = \mathsf{R}(p, x)$ for $x, p \in A \subset X$. It follows that $\mathsf{R}_A(x, p) \wedge \mathsf{R}_A(p, x) = \bot$. Hence, by Theorem 2, the subspace (A, R_A) is also $\overline{T}_0$ at p.

(ii) Similarly, let $p \in A$ and (X, R) be T_1 at p. By Theorem 3 and Definition 8, we have $\mathsf{R}_A(x, p) = \mathsf{R}(x, p) = \bot = \mathsf{R}(p, x) = \mathsf{R}_A(p, x)$ for $x, p \in A \subset X$ with $x \neq p$. Hence, by Theorem 3, the subspace (A, R_A) is also T_1 at p.

□

Theorem 5. *Let (X_i, R_i) be an* L*-preordered space for each $i \in I$ and (X, R) be the product of the spaces $\{(X_i, \mathsf{R}_i) \colon i \in I\}$, where $X = \prod_{i \in I} X_i$ and for all $x, y \in X$, $\mathsf{R}(x, y) = \bigwedge_{i \in I} \mathsf{R}_i(\pi_i(x), \pi_i(y))$. For all $i \in I$, the* L*-preordered space (X_i, R_i) is isomorphic to a subspace of the product space (X, R).*

Proof. Suppose that (X_i, R_i) is an L-preordered space for each $i \in I$ and (X, R) is the product space. Firstly, we choose a fixed point z_j in X_j for each $j \in I$ with $j \neq i$. Let $A = \{z_1\} \times \{z_2\} \times ... \times \{z_{i-1}\} \times X_i \times \{z_{i+1}\} \times ... \subset X$. Then, (A, R_A) is a subspace of the product space (X, R), where $\mathsf{R}_A(x, y) = \mathsf{R}(x, y)$ for all $x, y \in A$. Clearly, i-th projection map $\pi_i : (A, \mathsf{R}_A) \to (X_i, \mathsf{R}_i)$ defined by for $a_i \in X_i$, $\pi_i(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...) = a_i$ is bijective. For all $(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...)$, $(z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...) \in A$, we have

$$
\begin{aligned}
& \mathsf{R}_A((z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), (z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...)) \\
= \; & \mathsf{R}((z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), (z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...)) \\
= \; & \bigwedge_{j \neq i} \{\mathsf{R}_i(a_i, b_i), \mathsf{R}_j(z_j, z_j) = \top\} \\
\leq \; & \mathsf{R}_i(a_i, b_i) \\
= \; & \mathsf{R}_i(\pi_i(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), \pi_i(z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...))
\end{aligned}
$$

and it follows that π_i is an L-order preserving map.

On the other hand, let $f_i : (X_i, \mathsf{R}_i) \to (A, \mathsf{R}_A)$ be function defined by $f_i(a_i) = (z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...)$ for $a_i \in X_i$. Then, we have

$$\begin{aligned}(\pi_i \circ f_i)(a_i) &= \pi_i(f_i(a_i)) \\ &= \pi_i(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...) \\ &= a_i \\ &= 1_{X_i}(a_i)\end{aligned}$$

and

$$\begin{aligned}(f_i \circ \pi_i)(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...) &= f_i(\pi_i(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...)) \\ &= f_i(a_i) \\ &= (z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...) \\ &= 1_A(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...)\end{aligned}$$

It follows that $f_i = (\pi_i)^{-1}$ since $\pi_i \circ f_i = 1_{X_i}$ and $f_i \circ \pi_i = 1_A$.

For all $a_i, b_i \in X_i$, we obtain

$$\begin{aligned}\mathsf{R}_i(a_i, b_i) &= \bigwedge_{j \neq i} \{\mathsf{R}_i(a_i, b_i), \mathsf{R}_j(z_j, z_j) = \top\} \\ &= \mathsf{R}((z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), (z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...)) \\ &= \mathsf{R}_A((z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), (z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...)) \\ &= \mathsf{R}_A(f_i(a_i), f_i(b_i)) \leq \mathsf{R}_A(f_i(a_i), f_i(b_i))\end{aligned}$$

and it follows that f_i is an L-order preserving map.

Consequently, L-preordered space (X_i, R_i) and the subspace (A, R_A) are isomorphic. □

Theorem 6. *Let $\{(X_i, \mathsf{R}_i) \colon i \in I\}$ be a collection of L-preordered spaces and (X, R) be the product space, where $X = \prod_{i \in I} X_i$ and $\mathsf{R}(x, y) = \bigwedge_{i \in I} \mathsf{R}_i(\pi_i(x), \pi_i(y))$ for $x, y \in X$. Let $p = (p_i)_{i \in I}$ be a point in X.*

(i) *(X, R) is $\overline{T}_0$ at p iff (X_i, R_i) is $\overline{T}_0$ at p_i for each $i \in I$.*
(ii) *(X, R) is T_1 at p iff (X_i, R_i) is T_1 at p_i for each $i \in I$.*

Proof. (i) Assume that the product space (X, R) is $\overline{T}_0$ at p. By Theorem 5, for each $i \in I$, (X_i, R_i) is isomorphic to a subspace of (X, R) and by Theorem 4, a subspace of a local $\overline{T}_0$ L-preordered space is $\overline{T}_0$ at p. Since (X, R) is $\overline{T}_0$ at p, it follows that (X_i, R_i) is $\overline{T}_0$ at p_i for each $i \in I$.

Conversely, suppose that (X_i, R_i) is $\overline{T}_0$ at p_i for each $i \in I$. Let $x = (x_i)_{i \in I}$ be a point in X with $x \neq p = (p_i)_{i \in I}$. Since $x \neq p$, there exists $i_0 \in I$ such that $x_{i_0} \neq p_{i_0}$. By the assumption L-preordered space $(X_{i_0}, \mathsf{R}_{i_0})$ is $\overline{T}_0$ at p and by Theorem 2, we have $\mathsf{R}_{i_0}(x_{i_0}, p_{i_0}) \wedge \mathsf{R}_{i_0}(p_{i_0}, x_{i_0}) = \bot$. It follows that

$$\mathsf{R}(x, p) = \bigwedge_{i \in I} \{\mathsf{R}_i(x_i, p_i)\} \leq \mathsf{R}_{i_0}(x_{i_0}, p_{i_0})$$

and

$$\mathsf{R}(p,x)=\bigwedge_{i\in I}\{\mathsf{R}_i(p_i,x_i)\}\leq \mathsf{R}_{i_0}(p_{i_0},x_{i_0})$$

Since $\mathsf{R}_{i_0}(x_{i_0},p_{i_0})\wedge \mathsf{R}_{i_0}(p_{i_0},x_{i_0})=\perp$, we get $\mathsf{R}(x,p)\wedge \mathsf{R}(p,x)=\perp$. Hence, by Theorem 2, the product space (X,R) is $\overline{T}_0$ at p.

(ii) Similarly, suppose that the product space (X,R) is T_1 at p. By the assumption and Theorems 4 and 5, we have (X_i,R_i) is T_1 at p_i for each $i\in I$.

Conversely, assume that (X_i,R_i) is T_1 at p_i for each $i\in I$. Let $x\in X$ with $x\neq p$. Then, there exists $i_0\in I$ such that $x_{i_0}\neq p_{i_0}$. By the assumption L-preordered space $(X_{i_0},\mathsf{R}_{i_0})$ is T_1 at p and by Theorem 3, we have $\mathsf{R}_{i_0}(x_{i_0},p_{i_0})=\mathsf{R}_{i_0}(p_{i_0},x_{i_0})=\perp$. It follows that

$$\begin{aligned}\mathsf{R}(x,p) &= \bigwedge\{\mathsf{R}_1(x_1,p_1),\mathsf{R}_2(x_2,p_2),...,\mathsf{R}_{i_0-1}(x_{i_0-1},p_{i_0-1}),\\ &\qquad \mathsf{R}_{i_0}(x_{i_0},p_{i_0})=\perp,\mathsf{R}_{i_0+1}(x_{i_0+1},p_{i_0+1}),...\}\\ &= \perp\end{aligned}$$

and similarly,

$$\begin{aligned}\mathsf{R}(p,x) &= \bigwedge\{\mathsf{R}_1(p_1,x_1),...,\mathsf{R}_{i_0}(p_{i_0},x_{i_0})=\perp,...\}\\ &= \perp\end{aligned}$$

Consequently, by Theorem 3, we get the product space (X,R) is T_1 at p.

□

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