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SHIFT TRANSFORM APPROACH TO THE TWO-SIDED BALLOT THEOREM

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Abstract

In this paper, we present a recursive formula for the two-sided Ballot Theorem using left and right shift transformations. We show that the x^{th} entry of the image of the $(d+1)$-dimensional unit vector under the sum of the left and right shift operators is the number of walks in the lattice interval $[0, d]$ that starts at the origin and ends at the location x. This approach enables us to write a recursive formula for the number of possible n-walks between two obstacles that stop at a predetermined location.

Key Words: Ballot Theorem, Random Walks, Self-avoiding Walks, Reflection Method.

Özet

Bu makalede, sola ve sağa kaydırma dönüşümlerini kullanarak iki yönlü Oy Pusulası Teoremi için bir rekürans formülü vereceğiz. Özellikle $(d+1)$-boyutlu birim vektörün sola ve sağa kaydırma operatörlerinin toplamı altındaki görüntüsünün x-inci girdisinin, $[0, d]$ tamsayı latis aralığında gerçekleşen ve orijinde başlayıp x lokasyonunda biten yürüyüşlerin sayısına eşit olduğunu göstereceğiz. Bu yaklaşım, önceden belirlenmiş yerlerde bulunan iki engel arasındaki olası n-yürüyüşlerinin sayısı için bir rekürans formülü yazmamıza olanak sağlayacaktır.

Anahtar Kelimeler: Oy Pusulası Teoremi, Rastgele Yürüyüşler, Kendinden Kaçınan Yürüyüşler, Yansıma Yöntemi.

1. Introduction

Our goal in this paper is to give a recursive formula for the two-sided Ballot Theorem. The classical Ballot Theorem, first introduced by Bertrand [4], presents a formula for the number of possible vote castings where one candidate always maintains k times more votes than the other candidate. It is stated as follows.

Theorem 1.1. *Suppose that in an election, candidate A receives a votes and candidate B receives b votes, where* $a \geq kb$ *for some positive integer* k*. Then there are*

$$\frac{a-kb}{a+b}\binom{a+b}{a}$$

number of ways the ballots can be ordered so that candidate A maintains more than k times as many votes as B throughout the counting of the ballots.

Barbier [3] proposed a solution to the Ballot Theorem for $k \geq 2$ without proof. Soon after, André [2] gave a combinatorial proof for the case $k = 1$, following the proof by Aeppli [1] when $k \geq 1$. André's approach was based on counting the number of bad ballot permutations and removing them from the number of all possible permutations. Takács [11] detailed an outline of the history and its evolution of the Ballot Theorem as well as various proofs together with the original proofs by André and Aeppli. Renault [9] outlined four different proofs of the Ballot Theorem and discussed how it had influenced other mathematicians and lead the way to amazing discoveries such as MacMahon's [7] "reflection method."

The wo-sided Ballot Theorem is a generalization of the classical Ballot Theorem. It restricts the difference between the number of vote castings to stay within a specified range. One version of the two-sided Ballot Theorem is given in [6] as follows.

Theorem 1.2. *Let p and q be two positive integers. Let n and x be positive integers, such that* $-a < x < b$*. Then the number of* $n-$*paths from* 0 *to* x *such that the path never touches or crosses level* $-a$ *or level b, is given follows:*

$$k = \binom{n}{\frac{n+x}{2}} - \binom{n}{\frac{n+x+2a}{2}} - \binom{n}{\frac{n+x-2b}{2}} + \binom{n}{\frac{n+x+2a+2b}{2}} + \binom{n}{\frac{n+x-2a-2b}{2}} - \binom{n}{\frac{n+x+4a+2b}{2}}$$
$$- \binom{n}{\frac{n+x-2a-4b}{2}} + \binom{n}{\frac{n+x+4a+4b}{2}} + \binom{n}{\frac{n+x-4a-4b}{2}} - \binom{n}{\frac{n+x+8a+4b}{2}} - \binom{n}{\frac{n+x-4a-8b}{2}} + \cdots$$

2. Shift Transformations

Let $\mathbb{Z}$ be the integer lattice. For two integers a and b with $a \leq b$, let $[a, b]$ be an interval in the lattice, i.e.,

$$[a, b] = \{a, a + 1, a + 2, \dots , b - 2, b - 1, b\}.$$

For $n, d \in \mathbb{N}$, let $x \in [0, d]$ and define $\omega_n(x, d) = \omega_n(x)$ as the number of walks starting from 0, traveling between 0 and d, and stopping at some $x \in \{0,1, \dots , d\}$ with n many steps. For example,

$$\omega_8(2,5) = 28,$$

$$\omega_8(4,9) = 20,$$

$$\omega_{12}(4,9) = 275.$$

Let $e_i \in \mathbb{Z}^{d+1}$ be the i^{th} unit vector. Define the operators L and R as left shift and right shift operators on $(d$+1)-dimensional integer lattice, respectively. If $v = (v_1, v_2, \dots , v_{d+1}) \in \mathbb{Z}^{d+1}$, then the operators $L: \mathbb{Z}^{d+1} \to \mathbb{Z}^{d+1}$ and $R: \mathbb{Z}^{d+1} \to \mathbb{Z}^{d+1}$ are given by

$$L(v) = (v_2, v_3, \dots , v_{d+1}, 0),$$

and

$$R(v) = (0, v_1, v_2, \dots , v_d).$$

In the following, we will count the number of walks starting at 0, traveling between the integers 0 and d, and ending at an integer x. To this purpose, we introduce the sum $S = L + R$ of left and right shift transformations. When S is applied to the unit vector e_1, we obtain the n-step walks within the integer block $[0, d]$ starting at the site 0 and ending at the site $x \in \{0,1, \dots , d\}$. For example, $Se_1 = Le_1 + Re_1 = (0,1,0, \dots ,0)$ indicates that there is one single 1-step walk that starts at the location 0 and ends at the location 1. Likewise, $S^2e_1 = L^2e_1 + LRe_1 + RLe_1 + R^2e_1 = (1,0,1,0, \dots ,0)$ indicates that there is one 2-step walk that starts at the location 0 and ends at the location 0; there is one 2-step walk that starts at the location 0 and ends at the location 2. These are the only 2-step walks starting from the location 0.

Example 2.1 Let us use a table to calculate $\omega_{12}(4,9)$. In the following, the first column lists the number of steps, and the rest of the columns lists the locations in the interval.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	1	0	0	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0	0
3	0	2	0	1	0	0	0	0	0	0
4	2	0	3	0	1	0	0	0	0	0

5	0	5	0	4	0	1	0	0	0	0
6	5	0	9	0	5	0	1	0	0	0
7	0	14	0	14	0	6	0	1	0	0
8	14	0	28	0	20	0	7	0	1	0
9	0	42	0	48	0	27	0	8	0	1
10	42	0	90	0	75	0	35	0	9	0
11	0	132	0	165	0	110	0	44	0	9
12	132	0	297	0	275	0	154	0	53	0

The last row is $S^{12}e_1 = (132, 0, 297, 0, 275, 0, 154, 0, 53, 0)$, and $S^{12}e_1[5] = \omega_{12}(4,9) = 275$.

Now we are ready to state our main result.

Theorem 2.2. *Let $\omega_n(x)$ be defined as above. Then we have*

$$\omega_n(x) = \binom{n}{\frac{n-x}{2}} - \sum_{j=1}^{n} \left[\omega_{j-1}(0) \binom{n-j}{\frac{n-x}{2} - \frac{j+1}{2}} + \omega_{j-1}(d) \binom{n-j}{\frac{n-x}{2} - \frac{j-d-1}{2}} \right]$$

where $\binom{p}{q} = 0$ if $p, q \notin \mathbb{N}$ or if $q > p$. Moreover, the combinations equal zero if the quotients and differences are not nonnegative integers.

Proof. Let $S = R + L$, the sum of the left shift and right shift operators. It can be seen that

$$S^n e_1[x+1] = \omega_n(x).$$

Now consider the sum $S^n = (R + L)^n$. Note that the operators R and L are not commutative, i.e., $RL \neq LR$. Therefore, the Binomial formula cannot be applied. However, it can be used to count the number of certain terms. With an abuse of notation, let us denote by $L^k R^{n-k}$ the set of all words of length n obtained from the alphabet $\{R, L\}$ that contains k left shifts, and $n-k$ right shifts. For example,

$$L^2 R = \{LLR, LRL, RLL\}.$$

Then it can be seen by the Binomial formula that

$$|L^k R^{n-k}| = \binom{n}{k}.$$

Let us denote by $\binom{n}{k} L^k R^{n-k}$ the sum of all words in the set $L^k R^{n-k}$. For example,

$$\binom{3}{2} L^2 R = LLR + LRL + RLL.$$

Since $L^s e_1 = 0$ and $R^q e_1 = 0$ whenever $s \geq 1$ and $q \geq d+1$, some operators in $L^k R^{n-k}$ evaluated at e_1 are 0. However, the rest are nonzero vectors. Indeed, if $T \in L^k R^{n-k}$ and Te_1 is nonzero, then

$$Te_1 = e_{n-2k}.$$

If we would like to find the $(x+1)^{\text{th}}$ entry of $S^n e_1 \in \mathbb{Z}^{d+1}$ for $x \in \{0,1,\dots,d\}$, it is not very difficult to see that the words with x more Rs than Ls have some contributions, and the rest will play no role at all. We know that n and x must have the same parity and that $n \geq d$, in general. To land on the site x, the equation $(n-k)-k = x$ must hold. Solving it for k, we get $k = (n-x)/2$. In other words, we need transformations only from the set $L^{(n-x)/2}R^{(n+x)/2}$. Not that this condition is necessary but not enough for a contribution. When these words are applied to e_1, we either get e_{x+1} or the zero vector. The sum of e_{x+1}'s will then be the $(x+1)^{\text{th}}$ entry of $S^n e_1$. We now will use some exclusion argument to find this value.

Note that $Te_1 = 0$ for any operator $T \in L^{(n-x)/2}R^{(n+x)/2}$ starting with L. Hence, it does not contribute to $S^n e_1[x+1]$. This is the case when the walker is kicked out at the first step. There are $\binom{n-1}{k-1}$ operators of this kind in $L^{(n-x)/2}R^{(n+x)/2}$. As a result,

$$\binom{n}{k} - \binom{n-1}{k-1}$$

of the operators start with R where $k = (n-x)/2$. Likewise, for any operator T that starts with R^{d+1}, we get $Te_1 = 0$. There are $\binom{n-d-1}{k}$ many operators of this kind. This corresponds to the case when the walker is kicked out from right the first time. By excluding these operators, we obtain

$$\omega_n(x) \leq \binom{n}{\frac{n-x}{2}} - \binom{n-1}{\frac{n-x}{2}-1} - \binom{n-d-1}{\frac{n-x}{2}}.$$

However, these are not the only ones that lead to no contribution. Pathological cases arise when the walker is kicked out from the left or right. Suppose that the walker is kicked out from left at the j^{th} step where $j \in \{1,2,\dots,n\}$. This means that it managed to survive the first $(j-1)$-steps. There are $\omega_{j-1}(0)$ many $(j-1)$-walks. If i is the number of Ls in these first j steps, then we must have $j-i$ many Rs, and $i-(j-i)=1$, i.e., $i = (j+1)/2$. Note that there are $n-j$ steps left and $\frac{n-x}{2} - \frac{j+1}{2}$ of them are left moves. Hence, there are

$$\omega_{j-1}(0) \binom{n-j}{\frac{n-x}{2} - \frac{j+1}{2}}$$

many combinations that lead to no contributions. Similarly, suppose that the walker is kicked out from right at the j^{th} step. There $\omega_{j-1}(d)$ many $(j-1)$-walks where the walker is at the right end point. If i is the number of left moves in these j steps, then we must have $(j-i)-i=d+1$, i.e., $i=\frac{j-d-1}{2}$. As a result, there are

$$\omega_{j-1}(d)\binom{n-j}{\frac{n-x}{2}-\frac{j-d-1}{2}}$$

many combinations that lead to no contribution. The last assertion leads us to the formula.

3. Conclusions

In this paper, we provided a different proof of the two-sided Ballot Theorem. Our approach used the left and right shift transformations to count the number of walks in the lattice interval $[0,d]$ that starts from 0, traveling between 0 and d and landing in the location $x=0,1,\dots,d$. This approach can be generalized to the case when the starting point is any location other than 0 in the interval $[0,d]$. However, the formulae should be adjusted accordingly.

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Conflicts of interest

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