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Research Article

Morgan-Voyce Polynomial Approach for Solution of High-Order Linear Differential-Difference Equations with Residual Error Estimation

Bengü TÜRKYILMAZ[a,*], Burcu GÜRBÜZ[b], Mehmet SEZER[b]

[a] *Alaşehir Vocational School, Manisa Celal Bayar University, Alaşehir, Manisa, TURKEY*
[b] *Department of Mathematics, Faculty of Science& Art, Manisa Celal Bayar University, Manisa, TURKEY*
** Corresponding author's e-mail address: benguturkyilmaz@hotmail.com*

ABSTRACT

The main aim of this study is to apply the Morgan-Voyce polynomials for the solution of high-order linear differential difference equations with functional arguments under initial boundary conditions. The technique we have used is essentially based on the truncated Morgan-Voyce series and its matrix representations along with collocation points. Also, by using the Mean-Value Theorem and residual function, an efficient error estimation technique is proposed and some illustrative examples are presented to demonstrate the validity and applicability of the method.

Keywords: *Morgan- Voyce polynomials, Differential-difference equations, Collocation method, Matrix method, Error estimation, Residual function.*

Yüksek Mertebeden Lineer Diferansiyel Fark Denklemlerinin Hata Tahmini ile Morgan-Voyce Yaklaşımı

ÖZET

Bu çalışmanın amacı, yüksek mertebeden fonkisyonel argümanlı lineer diferansiyel fark denklemlerinin başlangıç sınır koşulları altında Morgan- Voyce polinomlarına bağlı çözümlerini araştırmaktır. Kullanılan metot esas olarak kesilmiş Morgan- Voyce serilerinin ve sıralama noktalarına bağlı matris gösterimlerine dayalıdır. Ayrıca, Ortalama Değer Teoremi ve rezidüel fonksiyonu ile etkin bir hata tahmini yöntemi verilmektedir. Bazı örneklerle metotun geçerlilik ve uygulanabilirliği gösterilmektedir.

Anahtar Kelimeler: *Morgan- Voyce polinomları, Diferansiyel-fark denklemleri, Kolokasyon metotu, Matris metotu, Hata tahmini, Rezidüel fonksiyon.*

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I. Introduction

Differential-difference equations $[1-10]$; which are a class of functional differential equations, have been treated as models of some physical phonemena. When a mathematical model is developed for a physical system, it is usually assumed that all of the variables, such as space and time, are continuous. This assumption leads to a realistic and justified approximation of the real variables of the system. However, for some of the physical systems, these continuous variable assumptions can not be made. Then differential-difference equations have played an important role in modelling problems that appear in various branches of science; such as mechanical engineering, condensed matter, biophysics, mathematical statistics and control theory. In recent years, the studies of differential-difference equations are developed very rapidly and intensively. It is well known that linear differential-difference equations have been considered by many authors,and have been used in the applications of difference models to problems in biology, physics and engineering.

Recently, a number of different methods associated with the solution of higher-order differential-difference equations, which are the inverse scatterining method $[1]$, Hirota's bilinear form method $[2]$, Tanh-method $[3]$, Jacobian elliptic function method $[4]$, Numerical techniques $[5,6]$, Taylor polynomial methods $[11,12]$ and Chebyshev methods $[13,15]$, have been given.

In this study, the basic ideas of the mentioned studies are developed to obtain the approximate solutions of high-order linear differential-difference equation with functional arguments (advanced, neutral or delayed) and variable coefficients in the form

$$\sum_{k=0}^{m} f_k(x)y^{(k)}(x)+\sum_{j=0}^{J} p_j(x)y^{(j)}(\alpha x+\beta)=g(x), J\le m \tag{1}$$

under the mixed conditions

$$\sum_{k=0}^{m-1}(a_{lk}y^{(k)}(a)+b_{lk}y^{(k)}(b))=\lambda_l, l=0,1,...,m-1. \tag{2}$$

Here $f_k(x), P_j(x)$ and $g(x)$ are known functions defined on the interval $a\le x\le b$;

$\alpha,\beta,a_{lk},b_{lk}$ and λ_l are appropriate constants; $y(x)$ is an unknown function to be determined. The aim of this study is to get the solution of the problem $(1)-(2)$ as the truncated Morgan-Voyce series defined by

$$y(x)\cong y_N(x)=\sum_{n=0}^{N} a_n B_n(x), N\ge m, a\le x\le b \tag{3}$$

where $a_n, n=0,1,...,N$, are unknown coefficients; $B_n(x), n=0,1,...,N$, denote the Morgan-Voyce polynomials defined by $[16-20]$

$$B_n(x)=\sum_{j=0}^{n}\binom{n+j+1}{n-j}x^j \qquad (4)$$

or recursively

$$B_n(x)=(x+2)B_{n-1}(x)-B_{n-2}(x), n\geq 2$$

with $B_0(x)=1$ and $B_1(x)=x+2$.

On the other hand, by using (3) or (4), the first four Morgan-Voyce polynomials are given by

$B_0(x)=1, B_1(x)=x+2, B_2(x)=x^2+4x+3, B_3(x)=x^3+6x^2+10x+4,\ldots$ and the Morgan-Voyce polynomials $B_n(x)$ are solutions of the following differential equation:

$$x(x+4)B_n''(x)+3(x+2)B_n'(x)-n(n+2)B_n(x)=0$$

II. METHOD

A. FUNDAMENTAL MATRIX RELATIONS

We first consider the solution $y(x)$ of Eq.(1) defined by the truncated Morgan-Voyce series (3) ,which is given in the form

$$y(x)\cong y_N(x)=\sum_{n=0}^{N}a_nB_n(x), a\leq x\leq b.$$

Then we can convert the finite series (3) to the matrix form as, for $n=0,1,\ldots,N$,

$$y(x)\cong y_N(x)=\mathbf{B}(x)\mathbf{A} \qquad (5)$$

so that

$$\mathbf{B}(x)=\begin{bmatrix}B_0(x) & B_1(x) & \ldots & B_N(x)\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}a_0 & a_1 & \ldots & a_N\end{bmatrix}^T.$$

On the other hand, by using the relation (4), the matrix $B(x)$ is obtained as

$$\mathbf{B}(x)=\mathbf{X}(x)\mathbf{R}^{\boldsymbol{T}} \qquad (6)$$

where

$$\mathbf{X}(x)=[x^0 \;\; x^1 \;\; x^2 \;\; \ldots \;\; x^N]$$

$$\mathbf{R}=\begin{bmatrix}\binom{1}{0} & 0 & 0 & \cdots & 0\\ \binom{2}{1} & \binom{3}{0} & 0 & \cdots & 0\\ \binom{3}{2} & \binom{4}{1} & \binom{5}{0} & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ \binom{N+1}{N} & \binom{N+2}{N-1} & \binom{N+3}{N-2} & \cdots & \binom{2N+1}{0}\end{bmatrix}$$

Also, it is clearly seen from (6) that the relation between the matrix $\mathbf{B}(x)$ and its derivative $\mathbf{B}^{'}(x)$ is

$$\mathbf{B}'(x)=\mathbf{X}'(x)\mathbf{R}^T = X(x)T^T R^T$$

and that repeating the process

$$\mathbf{B}^{(k)}(x)=\mathbf{X}(x)(\mathbf{T}^T)^k\mathbf{R}^T, k=0,1,2,...,m \tag{7}$$

where

$$\mathbf{T}=\begin{bmatrix}0 & 0 & 0 & \cdots & 0 & 0 & 0\\ 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 2 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & (N-1) & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & N & 0\end{bmatrix}, \quad T^0=diag\begin{bmatrix}1 & 1 & 1 & \cdots & 1\end{bmatrix} \text{ (unite matrix).}$$

From the matrix relations (5), (6) and (7), it follows that

$$y(x)\cong y_N(x)=\mathbf{B}(x)\mathbf{A} = X(x)R^T A$$

and

$$y^{(k)}(x)\cong y_N^{(k)}(x)=\mathbf{B}^{(k)}(x)\mathbf{A} = \mathbf{X}^{(k)}(x)\mathbf{R}^T\mathbf{A}$$

$$=\mathbf{X}(x)(\mathbf{T}^T)^k\mathbf{R}^T\mathbf{A},\ k=0,1,\dots. \tag{8}$$

By substituting $x\to\alpha x+\beta$ into the relation (8), we get, $j=0,1,...,$

$$y^{(j)}(\alpha x+\beta)=\mathbf{X}(\alpha x+\beta)(\mathbf{T}^T)^j\mathbf{R}^T\mathbf{A}$$

$$=\mathbf{X}(x)\mathbf{D}(\alpha,\beta)(\mathbf{T}^T)^j\mathbf{R}^T\mathbf{A} \tag{9}$$

so that, for $\alpha \neq 0$,

$$\mathbf{D}(\alpha,\beta)=\begin{bmatrix} \binom{0}{0}\alpha^0\beta^0 & \binom{1}{0}\alpha^0\beta^1 & \binom{2}{0}\alpha^0\beta^2 & \cdots & \binom{N}{0}\alpha^0\beta^N \\ 0 & \binom{1}{1}\alpha^1\beta^0 & \binom{2}{1}\alpha^1\beta^1 & \cdots & \binom{N}{1}\alpha^1\beta^{N-1} \\ 0 & 0 & \binom{2}{2}\alpha^2\beta^0 & \cdots & \binom{N}{2}\alpha^2\beta^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}\alpha^N\beta^0 \end{bmatrix}$$

B. MORGAN- VOYCE COLLOCATION METHOD

For constructing the fundamental matrix equation , we first consider the collocation points defined by; for $i = 0,1,\ldots,N$,

$$x_i = a + \frac{b-a}{N}i,\ \text{(Standard)}$$

and

$$x_i = \frac{b+a}{2} - \frac{b-a}{2}\cos\left(\frac{\pi i}{N}\right),\ \text{(Chebyshev-Lobatto)} \tag{10}$$

Then, by using the collocation points (10) into (1), we have the system of the equations

$$\sum_{k=0}^{m-1} f_k(x_i)y^{(k)}(x_i) + \sum_{j=0}^{J} p_j(x_i)y^{(j)}(\alpha x_i + \beta) = g(x_i)$$

or briefly the corresponding matrix equation

$$\sum_{k=0}^{m} F_k Y^{(k)} + \sum_{j=0}^{J} P_j Y^{(j)}(\alpha,\beta) = G \tag{11}$$

where

$$F_k = diag\left[f_k(x_0) \quad f_k(x_1) \quad \cdots \quad f_k(x_N)\right],\ P_j = diag\left[p_j(x_0) \quad p_j(x_1) \quad \cdots \quad p_j(x_N)\right]$$

$$\mathbf{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix}, \mathbf{Y}^{(j)}(\alpha,\beta) = \begin{bmatrix} y^{(j)}(\alpha x_0 + \beta) \\ y^{(j)}(\alpha x_1 + \beta) \\ \vdots \\ y^{(j)}(\alpha x_N + \beta) \end{bmatrix}, \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}$$

On the other hand,by substituting the collocation points (10) into (8) and (9), we obtain the matrix relations

$$\mathbf{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix} = \begin{bmatrix} \mathbf{X}(x_0)(\mathbf{T}^T)^k \mathbf{R}^T \mathbf{A} \\ \mathbf{X}(x_1)(\mathbf{T}^T)^k \mathbf{R}^T \mathbf{A} \\ \vdots \\ \mathbf{X}(x_N)(\mathbf{T}^T)^k \mathbf{R}^T \mathbf{A} \end{bmatrix} = \mathbf{X}(\mathbf{T}^T)^k \mathbf{R}^T \mathbf{A}$$

$$\mathbf{Y}^{(j)}(\alpha,\beta) = \begin{bmatrix} y^{(j)}(x_0) \\ y^{(j)}(x_1) \\ \vdots \\ y^{(j)}(x_N) \end{bmatrix} = \begin{bmatrix} \mathbf{X}(\alpha x_0+\beta)\mathbf{D}(\alpha,\beta)(\mathbf{T}^T)^j \mathbf{R}^T \mathbf{A} \\ \mathbf{X}(\alpha x_1+\beta)\mathbf{D}(\alpha,\beta)(\mathbf{T}^T)^j \mathbf{R}^T \mathbf{A} \\ \vdots \\ \mathbf{X}(\alpha x_N+\beta)\mathbf{D}(\alpha,\beta)(\mathbf{T}^T)^j \mathbf{R}^T \mathbf{A} \end{bmatrix} = \mathbf{X}\mathbf{D}(\alpha,\beta)(\mathbf{T}^T)^j \mathbf{R}^T \mathbf{A}$$

so that

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \cdots & x_0{}^N \\ 1 & x_1 & \cdots & x_1{}^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N{}^N \end{bmatrix}.$$

Therefore the fundamental matrix equation (11) becomes

$$\left[\sum_{k=0}^{m} \mathbf{F}_k \mathbf{X}(\mathbf{T}^T)^k + \sum_{j=0}^{J} P_j \mathbf{X} D(\alpha,\beta)(\mathbf{T}^T)^j\right] \mathbf{R}^T \mathbf{A} = \mathbf{G}\,. \tag{12}$$

Now we can find the fundamental matrix form for the conditions (2), by using the relation (8), as follows:

$$\sum_{k=0}^{m-1} (a_{lk}\mathbf{X}(a) + b_{lk}\mathbf{X}(b))(\mathbf{T}^T)^k \mathbf{R}^T \mathbf{A} = \lambda_l,\, l = 0,1,...,m-1 \tag{13}$$

Then we can write the fundamental matrix equations (12) and (13) corresponding to Eq.(1) and the conditions (2), respectively, as follows:

$$\mathbf{W}\mathbf{A} = \mathbf{G} \text{ or } [\mathbf{W};\mathbf{G}] \tag{14}$$

and

$$\mathbf{U}_l \mathbf{A} = \lambda_l \text{ or } [\mathbf{U}_l;\lambda_l],\, l = 0,1,...,m-1 \tag{15}$$

where

$$\mathbf{W}=\left[w_{pq}\right]=\left\{\sum_{k=0}^{m}\mathbf{F}_k\mathbf{X}(\mathbf{T}^T)^k+\sum_{j=0}^{J}\mathbf{P}_j\mathbf{X}\mathbf{D}(\alpha,\beta)(\mathbf{T}^T)^j\right\}\mathbf{R}^T$$

and

$$\mathbf{U}_l=\left[u_{l0}\quad u_{l1}\quad\cdots\quad u_{lN}\right]=\sum_{k=0}^{m-1}(a_{lk}\mathbf{X}(a)+b_{lk}\mathbf{X}(b))(\mathbf{T}^T)^k\mathbf{R}^T, l=0,1,...,m-1.$$

Consequently, to obtain the solution of Eq.(1) under the conditions (2), by replacing the row matrices (15) by the last (or any) m rows of the augmented matrix (14), we have the required matrix

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] \text{ or } \tilde{\mathbf{W}}\mathbf{A}=\tilde{\mathbf{G}} \tag{16}$$

If $rank\tilde{\mathbf{W}}=rank[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=N+1$, then we can write $\mathbf{A}=(\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}$. Thus the matrix $\mathbf{A}$ (thereby the coefficients $a_0,a_1,...,a_N$) is uniquely determined. Also, Eq.(1) under the conditions (2) has a unique solution.This solution is given by the truncated Morgan-Voyce series (3).

C. ACCURACY OF SOLUTIONS AND RESIDUAL ERROR ESTIMATION

We can easily check the accuracy of the obtained solutions as follows. Since the turncated Morgan-Voyce series (3) is approximate solution of (1), when the function $y_N(x)$ and its derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is, for $x=x_r\in\left[a,b\right], r=0,1,...,N,$

$$R_N(x_r)=\sum_{k=0}^{m-1}f_k(x_r)y_N^{(k)}(x_r)+\sum_{j=0}^{J}p_j(x_r)y_N^{(j)}(\alpha x_r+\beta_r)-g(x_r)\cong 0$$

or

$R_N(x_r)\leq 10^{-k_r}$, (k_r is any positive integer).

If max $10^{-k_r}=10^{-k}$ (k is an positive integer) is prescribed, then the truncation limit N is increased until the difference $R_N(x_r)$ at each of the points becomes smaller than the prescribed 10^{-k} .

On the other hand, by means of the residual function defined by $R_N(x)$ and the mean value of the function $\left|R_N(x)\right|$ on the interval $\left[a,b\right]$, the accuracy of the solution can be controlled and the error can be estimated $\left[7-9,14-22\right]$. If $R_N(x)\rightarrow 0$ when N is sufficienty large enough, then the error decreases. Also, by using the Mean-Value Theorem, we can estimate the upper bound of the mean error $\bar{R}_N$ as follows:

$$\left|\int_a^b R_N(x)dx\right| \le \int_a^b |R_N(x)|dx \quad \text{and} \quad \int_a^b R_N(x)dx = (b-a)R_N(c), a \le c \le b$$

$$\Rightarrow \left|\int_a^b R_N(x)dx\right| = (b-a)|R_N(c)| \Rightarrow (b-a)|R_N(c)| \le \int_a^b |R_N(x)|dx \Rightarrow |R_N(c)| \le \frac{\int_a^b |R_N(x)|dx}{b-a} = \overline{R}_N$$

III. EXAMPLES

1. Let us first consider the second order linear delay difference equation with variable coefficients

$$y''(x) - \frac{3}{4}y(x) - y(x+2) = -\frac{7}{4}x^2 - 4x - 7$$

under the initial conditions

$y(0) = 0$ and $y'(0) = 0$.

The exact solution of the problem is given as

$y(x) = x^2$

and we seek the solution $y(x)$ as truncated Morgan- Voyce series for $\mathrm{N} = 3$

$$y(x) = \sum_{n=0}^{2} a_n B_n(x)$$

We have the coefficients from Eq. (1)

$$f_0(x) = -\frac{3}{4},\ f_1(x) = 0,\ f_2(x) = 1,\ p_0(x) = -1, g(x) = -\frac{7}{4}x^2 - 4x - 7$$

Then, the collocation points for $\mathrm{N} = 3$; $x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3}, x_3 = \frac{1}{3}$ are obtained. And the fundamental matrix equation of the problem is defined by

$$\mathbf{W} = \left\{ \sum_{k=0}^{2} \mathbf{F}_k \mathbf{X}(\mathbf{T}^T)^k + \sum_{j=0}^{1} \mathbf{P}_j \mathbf{X}\mathbf{D}(\alpha,\beta)(\mathbf{T}^T)^j \right\} \mathbf{R}^T .$$

If these matrices are substituted in (16), it is obtained linear algebraic system. This system yields the approximate solution of the problem

$$y_N(x) = -\frac{1}{338}x^3 + x^2 + \frac{1}{100}x - \frac{1}{839}.$$

Table 1. *The comparison between approximate and exact solutions for the different N values and their absolute errors E_N.*

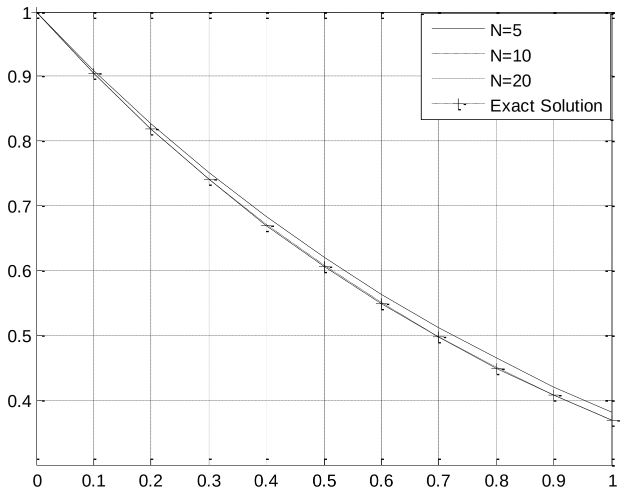
x	Exact solution	N=3	E_N	N=8	E_N
0.0	0.00	0.116E-8	1.16438E-9	0.00	0.0000
0.1	0.01	0.0100	1.38000E-9	0.01	0.0000
0.2	0.04	0.0400	1.82000E-9	0.04	0.0000
0.3	0.09	0.0900	2.46000E-9	0.09	0.0000
0.4	0.16	0.1600	3.30000E-9	0.16	0.0000
0.5	0.25	0.2500	4.30000E-9	0.25	0.0000
0.6	0.36	0.3600	5.50000E-9	0.36	0.0000
0.7	0.49	0.4900	6.80000E-9	0.49	0.0000
0.8	0.64	0.6400	8.20000E-9	0.64	0.0000
0.9	0.81	0.8100	9.60000E-9	0.81	0.0000
1.0	1.00	1.0000	1.10000E-8	1.00	0.0000

Table 1 shows the solutions of the problem for various N and their absolute error for the same N values. As we see from the Table 1, the error decreases when N value is choosen large enough. Then, we have the exact solution $y(x)=x^2$ for the $\mathrm{N}=8$. Due to this reason, we have the following example for different N values.

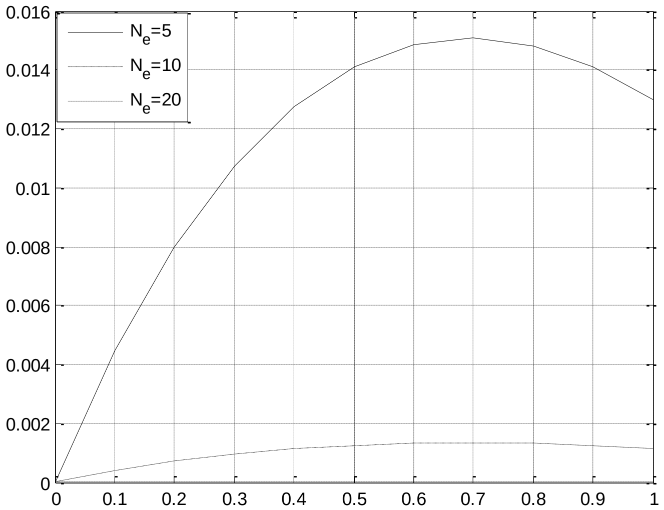
2. Let us find the Morgan- Voyce series solution of the following equation

$$y'(x)+y(x)+e^{x-1}y(x-1)=1$$

with $y(0)=1$. The exact solution of this problem is $y(x)=e^{(-x)}$. Using the same procedure in previous example for $\mathrm{N}=5,10,20$. The matrices form in Eq.(16) are computed. Hence linear algebraic system is gained. This system is approximately solved using the Maple18. The obtained results are demonstrated in



(a)



(b)

Figure 1. (a) *Numerical and exact solutions of Example 2 for the different N values.*
(b) *Error functions of the Example 2 for the various E_N.*

As can be seen in Fig. 1 (a) and (b), some differences between different N values. Familarly, we see that we approximate the exact solution by large N values. To obtain Fig. 1. (a) and (b) the equations can be written down with Matlab2013.

Table 2. The comparison between approximate and exact solution for the different N values and their absolute errors $E_{N.}$

x	Exact solution	N=5	E_N	N=10	E_N	N=20	E_N
0.0	1.00000	0.99999	4.00000E-9	1.00000	4.00000E-9	1.00000	7.10000E-18
0.1	0.90483	0.90928	4.44475E-3	0.90523	3.94605E-4	0.90483	2.66089E-10
0.2	0.81873	0.82671	7.98690E-3	0.81943	7.09007E-4	0.81873	4.78103E-10
0.3	0.74081	0.75153	1.07211E-2	0.74176	9.51724E-4	0.74081	6.41782E-10
0.4	0.67032	0.68304	1.27291E-2	0.67145	1.13006E-3	0.67032	7.62050E-09
0.5	0.60653	0.62061	1.40823E-2	0.60778	1.25034E-3	0.60653	8.43163E-09
0.6	0.54888	0.56365	1.48435E-2	0.55012	1.31806E-3	0.54888	8.88830E-09
0.7	0.49658	0.51165	1.50681E-2	0.49792	1.33806E-3	0.49658	9.02317E-09
0.8	0.44932	0.46413	1.48047E-2	0.45064	1.31468E-3	0.44932	8.86545E-08
0.9	0.40656	0.42066	1.40950E-2	0.40782	1.25184E-3	0.40656	8.44171E-08
1.0	0.36787	0.38085	1.29734E-2	0.36903	1.15322E-3	0.36787	7.77671E-08

Table 2 shows the solutions of the problem for various N and their absolute error for the same N values. Similarly, In the Table 2 the error decreases when N value is choosen large enough.

IV. CONCLUSION

In recent years, the studies of high order linear delay difference equation have attracted the attention of many mathematicians and physicists. A considerable advantage of the Morgan- Voyce method is that the approximate solutions are found very easily by using computer programs. Shorter computation time and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy. Illustrative examples are included to demonstrate the validity and applicability of the technique, and performed on the computer using a program written in Maple18. To get the best approximating solution of the equation, we take more forms from the Morgan- Voyce expansion of functions, that is, the truncation limit N must be chosen large enough. Illustrative examples with the satisfactory results are used to demonstrate the application of this method. Suggested approximations make this method very attractive and contributed to the good agreement between approximate and exact values in the numerical example.

As a result, the efficiency of the employed method is confirmed. We assured the correctness of the obtained solutions by Maple18, it provides an extra measure of confidence in the results. We predict that the Morgan- Voyce expansion method will be a promising method for investigating exact analytic solutions to linear delay difference equations. The method can also be extended to the functional equations with hybrid delay, but some modifications are required.

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