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Numerical Approximation of an Optimal Control Problem for Quasi Optics Equation

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Abstract

In this paper, difference method is applied to the optimal control problem arising in non-linear optics. Firstly, the difference scheme is established for the problem. Then stability of the difference scheme is given and the error analysis for this scheme is evaluated. Finally, the covergence according to the functional of the difference approximation is proved.

1.Introduction

Optimal control problems are often not linear and, therefore, have no analytical solution. As a result, it is necessary to use numerical methods for solving optimal control problems. The methods used for these solutions are divided into two: direct methods and indirect methods. In indirect methods, calculus of variation used to determine the optimal condition of the first order of the original optimal control problem. Indirect methods lead to a boundary value problem to determine the optimal trajectories. The lowest cost is selected in locally-optimized solutions. the disadvantage of the indirect method is that it is extremely difficult the solution of boundary value problems. In the direct method the optimal control problem is discretized converted to a non-linear optimization problem. After the non-linear optimization problem is solved by well known techniques. Solving nonlinear optimization problem is easier than solving boundary value problems [ANIL V. RAO].

The optimal control problem for the Schrödinger equation is one of the major interests of modern optimal control theory. The equation of Quasi optics is a special form of Schrödinger equation with complex potential. Potentials of this equation consists of refraction and absorption coefficients and these coefficients are often taken as control functions [KOÇAK, Y., ÇELİK, E., (2012)].

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Also the initial position of the system, usually taken as a control [KOÇAK, Y., ÇELİK, E., (2012), KOÇAK, Y., ÇELİK, E., YILDIRIM AKSOY, N., (2015)]. Such problems of modern physics, nonlinear optics and quantum mechanics arises in various branches [POTAPOV, M.N. AND RAZGULİN, A.V. (1990), YAGUBOV, G.Y. (1994), TOYOĞLU F., AND YAGUB, Y., (2015)].
Overall, the finite difference approach is used for the creation of numerical methods to solve optimal control problems. The finite difference method of solution of a system with optimal control problems governed by the Schrödinger equation were addressed in the studies [YAGUBOV, G.Y. AND MUSAYEVA, M.A. (1994), YILDIRIM, N., YAGUBOV, G.Y. AND YILDIZ B. (2012), TOYOĞLU F., AND YAGUB, Y., (2015)].

2. Formulation of the Problem

The following optimal control problem we consider in this paper

$$Minimize\left\{J(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)}\right\} \quad (1)$$

in the set

$$V \equiv \left\{v = (v_0, v_1), v_m \in L_2(0, L), \|v_m\|_{L_2(0,L)} \le b_m, v_1(z) \ge 0, \forall z \in (0, L), m = 0,1\right\}$$

subject to a systems of stationary equation of quasi optics

$$i\frac{\partial\psi_k}{\partial z} + a_0\frac{\partial^2\psi_k}{\partial x^2} - a(x)\psi_k + v_0(z)\psi_k + iv_1(z)\psi_k = f_k(x, z)$$

$$(x, z) \in \Omega, \mathrm{k} = 1,2, \quad (2)$$

with the conditions

$$\psi_k(x, 0) = \varphi_k(x), x \in (0, l), k = 1,2 \quad (3)$$

$$\psi_k(0, z) = \psi_k(l, z) = 0, z \in (0, L). \quad (4)$$

$$\frac{\partial\psi_2(0,z)}{\partial x} = \frac{\partial\psi_2(l,z)}{\partial x} = 0, z \in (0, L). \quad (5)$$

where $\psi_k = \psi_k(x, z)$ is a wave function,

$$\Omega = (0, \mathrm{l}) \times (0, \mathrm{l}), \mathrm{i} = \sqrt{-1}, a_0 > 0, l > 0, L > 0, b_m > 0 \ (m = 0,1)$$

are given numbers, $a(x)$ is a measurable bounded function that satisfies the following conditions:

$$0 < \mu_0 \le a(x) \le \mu_1, \left|\frac{da(x)}{dx}\right| \le \mu_2, \left|\frac{d^2a(x)}{dx^2}\right| \le \mu_3,$$

$$\forall x \in (0,l), \mu_m = constant > 0.$$

$\varphi_k(x)$ and $f_k(x,z)$ are given functions that satisfy the condition

$$\varphi_1 \in \overset{o}{W_2^2}(0,l), \varphi_2 \in W_2^2(0,l), \frac{d\varphi_2(0)}{dx} = \frac{d\varphi_2(l)}{dx} = 0 \quad (6)$$

$$f_1 \in \overset{o}{W_2^{2,0}}(\Omega), f_2 \in W_2^{2,0}(\Omega), \frac{\partial f(0,z)}{\partial x} = \frac{\partial f(l,z)}{\partial x} = 0 \quad (7)$$

The spaces $W_l^{k,m}(\Omega)$ are Sobolev spaces defined as in LADYZENSKAJA et al. (1968).

In study [IBRAHIMOV, N.S. (2010)], it was shown that the problem (1) to (4) has unique solution for each $v \in V$ and the following estimation is valid for this solution:

$$\|\psi_1\|_{\overset{o}{W_2^{2,0}}(\Omega)} \le c_1 \left(\|\varphi_1\|_{\overset{o}{W_2^{2,0}}(0,l)} + \|f_1\|_{\overset{o}{W_2^{2,0}}(\Omega)} \right) \quad (8)$$

$$\|\psi_2\|_{W_2^{2,1}(\Omega)} \le c_2 \left(\|\varphi_2\|_{W_2^2(0,l)} + \|f_2\|_{W_2^{2,0}(0,l)} \right) \quad (9)$$

for each $z \in (0,L)$.

Now, we shall discretize the optimal control problem (1) to (5) as in the study [KOÇAK, Y., ÇELİK, E., YILDIRIM AKSOY, N., (2015)]. For this purpose, let us transform the region Ω into the following scheme

$$\left\{(x_j,z_k)_n\right\}, n = 1,2,\dots, x_j = jh - \frac{h}{2}, j = \overline{1,M_{n-1}}, z_k = k\tau, k = \overline{1,N_n}$$

$$h = h_n = {}^{l}/_{M_n - 1}, \tau = \tau_n = {}^{\tau}/_{N_n}, M = M_n, N = N_n.$$

and let us make the following assignments

$$\delta_{\bar{x}}\phi_{jk} = \frac{\phi_{jk} - \phi_{jk-1}}{h}, \qquad \delta_{\bar{z}}\phi_{jk} = \frac{\phi_{jk} - \phi_{jk-1}}{\tau}$$

$$\delta_x\phi_{jk} = \frac{\phi_{j+1k} - \phi_{jk}}{h}, \qquad \delta_{x\bar{x}}\phi_{jk} = \frac{\phi_{j+1k} - 2\phi_{jk} - \phi_{jk-1}}{h^2}$$

For arbitrary natural number, $n \ge 1$, let us consider the minimizing problem of the function

$$I_n([v]_n) = h\sum_{j=1}^{M-1}\left|\phi_{jN}^1 - \phi_{jN}^2\right|^2 \quad (10)$$

in the set

$$V \equiv \{[v]_n : [v]_n = ([v_0]_n, [v_1]_n), v_{1k} \ge 0, k = \overline{1,N},$$

$$\left.[v_p] = (v_{p1}, v_{p2}, \dots, v_{pN}), \left(h\sum_{k=1}^{N}|v_{pk}|^2\right)^{1/2} \le b_p, p = 0,1, k = \overline{1,N}\right\}$$

under the conditions

$$i\delta_{\bar{z}}\phi_{jk}^p + a_0\delta_{x\bar{x}}\phi_{jk}^p - a_j\phi_{jk}^p + v_{0k}\phi_{jk}^p + iv_{1k}\phi_{jk}^p = f_{jk}^p, j = \overline{1,M-1}, k = \overline{1,N}, \quad (11)$$

$$\phi_{j0}^p = \varphi_j^p, j = \overline{0,M}, p = 1,2 \quad (12)$$

$$\phi_{0k}^1 = \phi_{Mk}^1 = 0, k = \overline{1,N}, \quad (13)$$

$$\delta_{\bar{x}}\phi_{1k}^2 = \delta_{\bar{x}}\phi_{Mk}^2 = 0, k = \overline{1,N}, \quad (14)$$

where the scheme functions $a_j, \varphi_j^p, f_{jk}^p, p = 1,2$ are defined by

$$a_j = \frac{1}{h}\int_{x_j - h/2}^{x_j + h/2} a(x)dx, j = \overline{1,M-1} \quad (15)$$

$$\varphi_j^p = \frac{1}{h}\int_{x_j - h/2}^{x_j + h/2} \varphi_p(x)dx, p = 1,2, j = \overline{1,M-1} \quad (16)$$

$$\varphi_0^1 = \varphi_M^1 = 0, \varphi_0^2 = \varphi_1^2, \varphi_M^2 = \varphi_{M-1}^2$$

$$f_{jk}^p = \frac{1}{\tau h}\int_{z_{k-1}}^{z_k}\int_{x_j - h/2}^{x_j + h/2} f_p(x,z)dxdx, p = 1,2, j = \overline{1,M-1}, k = \overline{1,N}. \quad (17)$$

As we have seen discrete problem (10)-(14) is the same as problem (1)-(5). So we can say the problem (10)-(14) has at least solution.

Using the study [11], we can write Theorem 1 for the stability of difference scheme.

Theorem 1. For each $[v]_n \in V_n$, the solution of the difference scheme (10)-(14) satisfies the following estimation.

$$h\sum_{j=1}^{M-1}\left|\phi_{jk}^p\right|^2 \le c_3\left(h\sum_{j=1}^{M-1}\left|\varphi_j^p\right|^2 + \tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|f_{jk}^p\right|^2\right), m = 1,2,\dots,N, p = 1,2. \quad (18)$$

where $c_3 > 0$ is a constant that does not depend on τ and h.

3. An Estimation for the Error of the Difference Schemes

In this section, we will evaluate the error of the difference scheme (10)-(14). For this purpose, let us consider the following system.

$$i\delta_{\bar{z}}Z_{jk}^p + a_0\delta_{x\bar{x}}Z_{jk}^p - a_jZ_{jk}^p + v_{0k}Z_{jk}^p + iv_{1k}Z_{jk}^p = F_{jk}^p, j = \overline{1,M-1}, k = \overline{1,N}, \quad (19)$$

$$Z_{j0}^p = 0, j = \overline{0,M}, p = 1,2 \quad (20)$$

$$Z_{0k}^1 = Z_{Mk}^1 = 0, k = \overline{1,N}, \quad (21)$$

$$\delta_{\bar{x}}Z_{1k}^2 = \delta_{\bar{x}}Z_{Mk}^2 = 0, k = \overline{1,N}, \quad (22)$$

where $[Z^p]_n = \{Z_{jk}^p\} = \{\phi_{jk}^p\} - \{\psi_{jk}^p\}, p = 1,2$ is the solution of the system (10)-(14), $\{\psi_{jk}^p\}$ is defined by

$$\psi_{jk}^p = \frac{1}{\tau h}\int_{z_{k-1}}^{z_k}\int_{x_j - h/2}^{x_j + h/2} \psi_p(x,z)dxdx, j = \overline{1,M-1}, k = \overline{1,N}. \quad (23)$$

and the scheme function F_{jk}^p is defined by

$$F_{jk}^{p} = \frac{1}{\tau h}\int_{z_{k-1}}^{z_k}\int_{x_j-h/2}^{x_j+h/2}\left(i\frac{\partial\psi_k}{\partial z} + a_0\frac{\partial^2\psi_k}{\partial x^2} - a(x)\psi_k + v_0(z)\psi_k + iv_1(z)\psi_k\right)dxdx$$
$$-i\delta_{\bar z}\psi_{jk}^{p} + a_0\delta_{x\bar x}\psi_{jk}^{p} - a_j\psi_{jk}^{p} + v_{0k}\psi_{jk}^{p} + iv_{1k}\psi_{jk}^{p}, j = \overline{1, M-1}, k = \overline{1, N}, p = 1,2. \quad (24)$$

Also, let us define the operator Q_n such that

$$Q_n : V \to V_n, Q_n(v) = [w]_n = ([w_0]_n, [w_1]_n)$$
$$w_{pk} = \frac{1}{\tau}\int_{z_{k-1}}^{z_k} v_p(z)dz,\ k = \overline{1,N}, p = 1,2 \quad (25)$$

Now, we can write the following theorem that expresses the error of the finite difference approximations:

Theorem 2. Suppose that the step τ and h satisfies the condition $c_4 \le \frac{\tau}{h} \le c_5$ and ψ_p satisfy following inequality:

$$\underset{z\in[0,L]}{\mathrm{vraimax}} \left\|\frac{\partial\psi_p(.,z)}{\partial z}\right\|_{L_2(0,l)} \le c_6.$$

Then, the estimation is valid:

$$h\sum_{j=1}^{M-1}\left|Z_{jk}^{p}\right|^2 \le c_6(\beta_{\tau h} + \|Q_n(v) - [v]_2\|^2), m = \overline{1,N}, p = 1,2. \quad (26)$$

where $c_6^p > 0$ is a constant independent from τ and h, $\beta_{\tau h} > 0, \beta_{\tau h} \to 0$ for $\tau \to 0$ and $h \to 0$. $\beta_{\tau h} > 0$, for $\tau \to 0$ and $h \to 0, \beta_{\tau h} \to 0$. Here $\|Q_n(v) - [v]_2\|^2$ is defined by following equality

$$\|Q_n(v) - [v]_2\|^2 = \tau\sum_{k=1}^{N}(|w_{0k} - v_{0k}|^2 + |w_{1k} - v_{1k}|^2).$$

Proof: The proof of Theorem 2 can be obtain by similar process given in [8,9].

4. The convergence of the difference approximations

In this section, we will investigate the convergence of the difference approximations according to functional.

Theorem 3. Suppose that the conditions of Theorem 2 hold. Then, the inequality

$$|J(v) - I_n([v]_n| \le c_7\left(\sqrt{\beta_{\tau h}} + \|Q_n\|(v) - [v]_n\right) \quad (27)$$

is valid for $\forall v \in V$ and $\forall [v]_n \in V_n$.

Here the number of $c_7 > 0$ is independent from τ and h.

Proof: We consider the difference $J(v) - I_n([v]_n)$. We can write the following equation using (1) and (10):

$$J(v) - I_n([v]_n) = \int_{\Omega} |\psi_1(x,z) - \psi_2(x,z)|^2 dxdz - h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\left|\phi_{jk}^1 - \phi_{jk}^2\right|^2$$
$$= \sum_{k=1}^{N}\sum_{j=1}^{M-1}\int_{z_{k-1}}^{z_k}\int_{x_{j-\frac{h}{2}}}^{x_{j+\frac{h}{2}}}\left(\left(|\psi_1(x,z) - \psi_2(x,z)| + \left|\phi_{jk}^1 - \phi_{jk}^2\right|\right)\times\right.$$
$$= \left.\left(|\psi_1(x,z) - \psi_2(x,z)| + \left|\phi_{jk}^1 - \phi_{jk}^2\right|\right)\right) dxdz.$$

Using the estimates (8), (9) and applying the Cauchy-Bunyakovski, we obtain the following inequality:

$$|J(v) - I_n([v]_n)|$$
$$\le c_8\left[\left(\sum_{k=1}^{N}\sum_{j=1}^{M-1}\int_{z_{k-1}}^{z_k}\int_{x_{j-\frac{h}{2}}}^{x_{j+\frac{h}{2}}}\left|\psi_1(x,z) - \phi_{jk}^1\right|^2 dxdz\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{N}\sum_{j=1}^{M-1}\int_{z_{k-1}}^{z_k}\int_{x_{j-\frac{h}{2}}}^{x_{j+\frac{h}{2}}}\left|\psi_2(x,z) - \phi_{jk}^2\right|^2 dxdz\right)^{\frac{1}{2}}\right]$$
$$= c_9[J_1 + J_2].$$

$$(J_1)^2 = \sum_{k=1}^{N}\sum_{j=1}^{M-1}\int_{z_{k-1}}^{z_k}\int_{x_{j-\frac{h}{2}}}^{x_{j+\frac{h}{2}}}\left|\psi_1(x,z) - \psi_{jk}^1 + \psi_{jk}^1 - \phi_{jk}^1\right|$$
$$\le 2\sum_{k=1}^{N}\sum_{j=1}^{M-1}\int_{z_k-1}^{z_k}\int_{x_j-h/2}^{x_j+h/2}\left|\psi_1(x,z) - \psi_{jk}^1\right|^2 + 2\tau h\sum_{k=1}^{N}\sum_{j=1}^{M-1}\int_{z_{k-1}}^{z_k}\int_{x_j-h/2}^{x_j+h/2}\left|\psi_{jk}^1 - \phi_{jk}^1\right|^2$$
$$= J_{11} + J_{12} \quad (28)$$

If we use the formula (23) we can write the following inequality:

$$J_{11} \le 4\tau^2\left\|\frac{\partial\psi_1}{\partial z}\right\|_{L_2(\Omega)}^2 + 4h^2\left\|\frac{\partial\psi_1}{\partial x}\right\|_{L_2(\Omega)}^2 \quad (29)$$

We choose $p = 1$ in (26), then we obtain

$$J_{12} \le 2c_9(\beta\tau h + \|Q_n(v) - [v]_n\|^2). \quad (30)$$

Using (29) and (30) we obtain the following inequality for the J_{11}:

$$(J_1)^2 \le c_{10}(\beta\tau h + \|Q_n(v) - [v]_n\|^2) \quad (31)$$

Here the number $c_{10} > 0$ independent from τ and h.

Similarly, we can write the following inequality for the $(J_2)^2$:

$$(J_2)^2 \le c_{11}(\beta\tau h + \|Q_n(v) - [v]_n\|^2) \qquad (32)$$

Lemma 1. Suppose that the conditions of Theorem 3 hold and the operator Q_n is defined by (23). Then $Q_n(v) \in V_n$ for $\forall v \in V$ and the following estimation

$$|J(v) - I_n(Q_n(v))| \le c_{12}\sqrt{\beta_{th}}$$

is valid, where $c_{12} > 0$ is a constant independent from τ and h.

Proof. Let $v \in V$ is admissible control. The following formulas is written definition of Q_n:

$$Q_n(v) = ([w_0],[w_1]), [w_M] = (w_{m1}, w_{m2}, \dots, w_{mN}), \quad m = 0,1$$

$$w_{mk} = \frac{1}{\tau}\int_{z_{k-1}}^{z_k} v_m(z)dz, \qquad k = \overline{1,N}, \qquad m = 0,1.$$

$$w_{mk} = \frac{1}{\tau}\int_{z_{k-1}}^{z_k} v_m(z)dz \ge \frac{1}{\tau}\int_{z_{k-1}}^{z_k} b_0 dz = b_0,$$

$$w_{mk} = \frac{1}{\tau}\int_{z_{k-1}}^{z_k} v_m(z)dz \ge \frac{1}{\tau}\int_{z_{k-1}}^{z_k} b_1 dz = b_1$$

Thus, we obtain $b_0 \le w_{mk} \le b_1$, $k = \overline{1,N}$, and $Q_n(v) \in V_n$. Then we take $[v]_n \in V_n$ and using Theorem 3 Lemma is valid.

Now, we define the operator P_n as follows:

$$P_n([v]_n) = (P_n[v_0], P_n[v_1]) \qquad (33)$$

$$P_n([v]_m) = \tilde{v}_m(z), \qquad \tilde{v}_m(z) = v_{mk}, \qquad z_{k-1} \le z \le z_k, \ m = 0,1.$$

Lemma 2. Suppose that the conditions of Theorem 3 hold and the operator P_n is defined by (25). Then $P_n([v_n]) \in V$

$$|J(P_n([v]_n) - I_n([v]_n)| \le c_{12}\sqrt{\beta_{th}}.$$

Proof. $[v_n] \in V_n$ is discrete control. The following formulas is written definition of P_n:

$$\tilde{v}_m(z) = P_n([v]_n) = v_{mk} \ge b_0, \ z_{k-1} \le z \le z_k$$
$$\tilde{v}_m(z) = P_n([v]_n) = v_{mk} \ge b_1, \ z_{k-1} \le z \le z_k, \ k = \overline{1,N}, \ m = 0,1.$$

Thus $P_n([v]_n) \in V$. Let $\tilde{v}_m(z) = P_n([v]_n)$ instead of $v \in V$. Then, we obtain

$$|J(P_n([v]_n) - I_n([v]_n)| \le c_{13}\left(\sqrt{\beta_{th}} + \left\|Q_n\left(P_n([v]_n)\right) - I_n([v]_n)\right\|\right) \qquad (34)$$

and the following estimate:

$$\left\|Q_n\left(P_n([v]_n)\right) - I_n([v]_n)\right\|^2 = \tau\sum_{k=1}^{N}\left|\frac{1}{\tau}\int_{z_{k-1}}^{z_k} v_m(z)dz - v_{mk}\right|^2$$
$$= \tau\sum_{k=1}^{N}\left|\frac{1}{\tau}\int_{z_{k-1}}^{z_k} v_{mk}\, d_z - v_{mk}\right|^2 = \tau\sum_{j=1}^{M-1}|v_{mk} - v_{mk}|^2 = 0$$

Now, let write the convergence of the difference approximations according to functional:

Theorem 4. Suppose that the conditions of Lemma 1 and Lemma 2 hold. Also, let $v^* \in V$, $[v]_n^* \in V_n$ be solutions of the problems (1) to (5) and (10) to (14), respectively, i.e.

$$J_* = \inf_{v\in V} J(v) = J(v^*)$$
$$I_{n^*} = \inf_{[v]_n \in V} I_n([v]_n) = I_n(v_n^*)$$

Then, the solutions of the problem $(10) - (14)$ are approximate to the solution of the problem (1)-(5), i.e., $\lim_{n\to\infty} I_{n^*} = J_*$ and for the convergence according to functional the following estimation is valid:

$$|I_{n^*} - J_*| \le c_{14}\sqrt{\beta_{th}}.$$

Proof: The proof can be obtain by similar process given in [YILDIRIM, N., YAGUBOV, G.Y. AND YILDIZ, B. TOYOĞLU F., AND YAGUB, Y., (2015), KOÇAK, Y., ÇELİK, E., YİLDİRİM AKSOY, N., (2015), KOÇAK, Y., DOKUYUCU, M.A., ÇELİK, E.(2015)].

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