http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 954–967 (2022)
DOI:10.31801/cfsuasmas.1072728
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: February 13, 2022; Accepted: May 5, 2022

PARAMETER UNIFORM SECOND-ORDER NUMERICAL APPROXIMATION FOR THE INTEGRO-DIFFERENTIAL EQUATIONS INVOLVING BOUNDARY LAYERS

Muhammet Enes DURMAZ[1], Musa CAKIR[2] and Gabil AMIRALI[3]

[1]Department of Information Technology, Kırklareli University, Kırklareli, TURKEY
[2]Department of Mathematics, Faculty of Science, Van Yüzüncü Yıl University, Van, TURKEY
[3]Department of Mathematics, Faculty of Art and Science, Erzincan Binali Yıldırım University, Erzincan, TURKEY

Abstract. The work handles a Fredholm integro-differential equation involving boundary layers. A fitted second-order difference scheme has been created on a uniform mesh utilizing interpolating quadrature rules and exponential basis functions. The stability and convergence of the proposed discretization technique are analyzed and one example is solved to display the advantages of the presented technique.

1. Introduction

In the study, we deal with singularly perturbed Fredholm integro-differential equation (SPFIDE) in the form:

$$Lv := L_1 v + \lambda \int_0^l M(x,\zeta)v(\zeta)d\zeta = f(x), \qquad 0 < x < 1, \tag{1}$$

$$v(0) = A, \quad v(l) = B, \tag{2}$$

where $L_1 v = -\varepsilon v''(x) + a(x)v(x)$, $0 < \varepsilon \ll 1$ is a singular perturbation parameter, λ is a given parameter. The functions $a(x) \geq \alpha > 0$, $f(x)$ and $M(x,\zeta)$ are considered to be sufficiently smooth and satisfy certain regularity criteria. The solution $v(x)$ of (1)-(2) has in general boundary layers near $x = 0$ and $x = l$.

2020 *Mathematics Subject Classification.* 65R20, 65L12, 65L11, 65L20.
Keywords. Finite difference method, Fredholm integro-differential equation, singular perturbation, uniform convergence.
menesdurmaz025@gmail.com-Corresponding author; 0000-0002-6216-1032
cakirmusa@hotmail.com; 0000-0002-1979-570X
gabilamirali@yahoo.com; 0000-0001-6585-7353.

Singularly perturbed problems (SPPs) are defined by a small parameter ε multiplying the highest order derivative term. The solution to them generally involves boundary or initial layers. To quote a few, the exact solutions of SPPs and their applications may be found in [15, 18, 21]. SPPs have a wide range of applications in the fields of population dynamics, nanofluid, neurobiology, fluid dynamics, viscoelasticity, heat transfer problems, simultaneous control systems and mathematical biology etc. It is worth noting that when a small ε parameter is multiplied with the derivative, the great majority of classic numerical techniques on uniform meshes are ineffective at solving issues unless the step-size of discretization is drastically reduced. Thus, as the perturbation parameter ε goes smaller, the truncation error becomes boundless. To solve SPPs numerically, general approaches are done with the fitted finite difference method and are widely utilized [9, 12, 19, 20].

Most engineering applications and scientific disciplines have been expressed by Fredholm integro-differential equations (FIDEs). Plasma physics, biomechanics, financial mathematics, artificial neural networks, oceanopraphy, epidemic models, electromagnetic theory, fluid mechanics, biological and population dynamics processes are amongst these (see, e.g., [5, 7, 13]). For this reason, several studies have been conducted on FIDEs. Solving problems of this type is quite difficult. Therefore, we require robust and consistent numerical methods [6, 8, 14, 16, 23, 26](see, as well as the references therein).

These investigations in relation to FIDEs are just in relation to regular situations. Numerical examination of SPFIDEs has not been widespread till recently. Finite difference schemes for solving linear SPFIDEs are constructed in [1, 2]. A second order numerical tecnique for solving FIDE with boundary layer is developed in [10, 11].

The goal of this work is to propose a uniform convergence numerical technique to solve linear second-order FIDEs with boundary layers. A numerical technique that uses suitable interpolating quadrature rules and exponential basis functions is proposed on a uniform mesh. Error estimates are acquired in the discrete maximum norm with regard to the perturbation parameter. To corroborate theoretical estimates, numerical experiments are conducted and the results are presented.

The rest of the contents is organized kind of following. In Section 2, some properties of solutions (1)–(3) are presented, as well as a finite difference scheme. In Section 3, the stability and convergence analysis of this scheme are shown. In Section 4, the numerical results of an example to verify the theoretical estimates are presented. Finally, the work ends with a summary of the conclusions in Section 5.

2. Discretization Techniques

We have mentioned certain analytical bounds here, which we will use later in our error analysis.

Lemma 1. *Let* $a, f \in C^2[0,l]$, $\frac{\partial^m M}{\partial x^m} \in C[0,l]^2$, $(m = 0,1,2)$ *and*

$$|\lambda| < \frac{\alpha}{\max\limits_{0\le x\le l} \int\limits_0^l |M(x,\zeta)|\, d\zeta}.$$

Then the solution $u(x)$ *of the problem (1)-(2) satisfies the following estimates:*

$$\|v\|_\infty \le C, \tag{3}$$

$$\left|v^{(k)}(x)\right| \le C\left\{1 + \varepsilon^{-\frac{k}{2}}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}}\right)\right\}, \quad (k = 1,2), \quad 0 \le x \le 1. \tag{4}$$

Proof. The proof of Lemma 1 is by like approach as in [2, 10, 17]. □

Let ω_N be an equidistant mesh on $[0,l]$:

$$\omega_N = \{x_i = ih,\, i = 1,2,...,N-1,\, h = lN^{-1}\}, \quad \bar{\omega}_N = \omega_N \cup \{x_0 = 0,\, x_N = l\}.$$

We utilize the following difference approximations for any mesh function $q(x)$ defined on $\bar{\omega}_N$:

$$q_{x,i} = \frac{q_{i+1} - q_i}{h}, \qquad q_{\bar{x},i} = \frac{q_i - q_{i-1}}{h}, \qquad q_{\bar{x}x,i} = \frac{q_{x,i} - q_{\bar{x},i}}{h}.$$

For the equation (1), we begin with the following integral identity:

$$\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}} Lv(x)\psi_i(x)dx = \frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\psi_i(x)dx, \qquad 1 \le i \le N-1, \tag{5}$$

with the basis functions

$$\psi(x) = \begin{cases} \psi_i^{(1)}(x) \equiv \frac{sinh\gamma_i(x-x_i)}{sinh\gamma_i h}, & x \in (x_{i-1}, x_i), \\ \psi_i^{(2)}(x) \equiv \frac{sinh\gamma_i(x_{i+1}-x)}{sinh\gamma_i h}, & x \in (x_i, x_{i+1}), \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

where

$$\gamma_i = \sqrt{\frac{a(x_i)}{\varepsilon}}, \qquad \chi_i = \frac{1}{h}\int\limits_{x_{i-1}}^{x_{i+1}} \psi_i(x)dx = \frac{2tanh(\gamma_i h/2)}{\gamma_i h}.$$

We should remark that the functions $\psi_i^{(1)}$ and $\psi_i^{(2)}$ are the solutions to the following problems:

$$-\varepsilon\psi'' + a_i\psi = 0, \quad x_{i-1} < x < x_i, \quad \psi(x_{i-1}) = 0, \quad \psi(x_i) = 1,$$
$$-\varepsilon\psi'' + a_i\psi = 0, \quad x_i < x < x_{i+1} \quad \psi(x_i) = 1, \quad \psi(x_{i+1}) = 0.$$

By using the technique of the exact difference approximations [3, 4, 11, 24, 25] (see also [22], pp. 207-214), it follows that

$$
\begin{aligned}
&-\frac{\varepsilon}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}\psi_i(x)v''(x)dx+\frac{a_i}{\chi_i h}a_i\int\limits_{x_{i-1}}^{x_{i+1}}\psi_i(x)v(x)dx= \\
&-\frac{\varepsilon}{\chi_i}\left\{1+a_i\varepsilon^{-1}\int\limits_{x_{i-1}}^{x_i}\psi_i^{(1)}(x)(x-x_i)dx\right\}v_{\bar{x}x,i} \\
&+\frac{a_i}{\chi_i}\left\{h^{-1}\int\limits_{x_{i-1}}^{x_i}\psi_i^{(1)}dx+h^{-1}\int\limits_{x_i}^{x_{i+1}}\psi_i^{(2)}dx\right\}v_i=-\varepsilon\theta_i v_{\bar{x}x,i}+a_i v_i
\end{aligned}
$$

where

$$\theta_i=\frac{a_i\rho^2}{4sinh^2\left(\sqrt{a_i}\frac{\rho}{2}\right)},\quad \left(\rho=\frac{h}{\sqrt{\varepsilon}}\right). \tag{6}$$

Thus

$$\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}\varepsilon v''(x)\,\psi_i(x)\,dx+\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}a(x)\,v(x)\,\psi_i(x)\,dx=-\varepsilon\theta_i v_{\bar{x}x,i}+a_i v_i+R_i^{(1)}, \tag{7}$$

with remainder term

$$R_i^{(1)}=\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}\left[a(x)-a(x_i)\right]v(x)\psi_i(x)dx. \tag{8}$$

Furthermore, for the right-side in (5) we get

$$\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}f(x)\psi_i(x)dx=f_i+R_i^{(2)}, \tag{9}$$

with remainder term

$$R_i^{(2)}=\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}\left[f(x)-f(x_i)\right]\psi_i(x)dx. \tag{10}$$

For integral term that include the kernel function, from (5), we have

$$\frac{\lambda}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}dx\psi_i(x)\int\limits_0^l M(x,\zeta)v(\zeta)d\zeta=\lambda\int\limits_0^l M(x_i,\zeta)v(\zeta)d\zeta$$

$$+\frac{\lambda}{\chi_i h}\int_{x_{i-1}}^{x_{i+1}} dx\psi_i(x)\int_{x_{i-1}}^{x_{i+1}}\left(\int_0^l \frac{\partial^2 M(\xi,\zeta)}{\partial\xi^2}v(\zeta)\,d\zeta\right)M_1(x,\xi)\,d\xi,$$

where

$$M_1(x,\xi)=T_1(x-\xi)-T_1(x_i-\xi)+(2h)^{-1}(x_{i+1}-\xi)(x_i-x),$$
$$T_1(\lambda)=\lambda,\quad \lambda\geq 0;\qquad T_1(\lambda)=0\quad \lambda<0.$$

We computed by using composite trapezoidal integration with the remainder term in integral form for the second integral term in the left side of the identity of (5):

$$\int_0^l M(x_i,\zeta)v(\zeta)d\zeta=\sum_{j=0}^N \hbar_j M_{ij}v_j+\frac{1}{2}\sum_{j=1}^N\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\big(M(x_i,\xi)v(\xi)\big)''d\xi,$$

where

$$\hbar_0=\hbar_N=\frac{h}{2},\qquad \hbar_i=h,\qquad 1\leq i\leq N-1.$$

Thus we get

$$\frac{\lambda}{\chi_i h}\int_{x_{i-1}}^{x_{i+1}} dx\psi_i(x)\int_0^l M(x,\zeta)v(\zeta)d\zeta=\lambda\sum_{j=0}^N \hbar_j M_{ij}v_j+R_i^{(3)}, \tag{11}$$

with remainder term

$$R_i^{(3)}=\frac{\lambda}{\chi_i h}\int_{x_{i-1}}^{x_{i+1}} dx\psi_i(x)\int_{x_{i-1}}^{x_{i+1}}\left(\int_0^l \frac{\partial^2 M(\xi,\zeta)}{\partial\xi^2}v(\zeta)\,d\zeta\right)M_1(x,\xi)\,d\xi$$
$$+\frac{1}{2}\lambda\sum_{j=1}^N\int_{x_{j-1}}^{x_j}(x_j-\xi)(x_{j-1}-\xi)\big(M(x_i,\xi)v(\xi)\big)''d\xi. \tag{12}$$

Combining (7), (9) and (11) in (5) we obtain the following difference scheme:

$$L_N v_i:=-\varepsilon\theta_i v_{\bar{x}x,i}+a_i v_i+\lambda\sum_{j=0}^N \hbar_j M_{ij}v_j+R_i=f_i,\qquad 1\leq i\leq N-1, \tag{13}$$

with remainder term

$$R_i=R_i^{(1)}+R_i^{(2)}+R_i^{(3)}, \tag{14}$$

where the remainder terms $R_i^{(1)}, R_i^{(2)}$ and $R_i^{(3)}$ are defined by (8), (10) and (12) respectively.

Based on (13) we achieve the following difference approximate for approximating (1)-(2):

$$L_N y_i:=-\varepsilon\theta_i y_{\bar{x}x,i}+a_i y_i+\lambda\sum_{j=0}^N \hbar_j M_{ij}y_j=f_i,\qquad 1\leq i\leq N-1, \tag{15}$$

$$y_0 = A, \quad y_N = B, \tag{16}$$

where θ_i is defined by (6).

3. Error Analysis

For the error function $z_i = y_i - v_i$ $(i = 0, 1, ..., N)$ considering (13) and (15), we get

$$L_N z_i := R_i, \qquad 1 \leq i \leq N-1, \tag{17}$$

$$z_0 = 0, \quad z_N = 0, \tag{18}$$

where the remainder term R_i is defined by (14).

Theorem 1. *Let* $\frac{\partial^m M}{\partial x^m} \in C^2\left[0, l\right]^2$, $(m = 0, 1, 2)$, $M(x, 0) = M(x, l) = 0$*;* $a, f \in C^2\left[0, l\right]$, $a'(0) = a'(l) = 0$, *and*

$$|\lambda| < \frac{\alpha}{\max\limits_{1 \leq i \leq N} \sum\limits_{j=0}^{N} \hbar_j |M_{ij}|}.$$

Then for the error of the scheme (15)-(16), we have

$$||y - v||_{\infty, \bar{\omega}_N} \leq Ch^2.$$

Proof. Applying the discrete maximum principle to discrete problem (17) and (18), we get

$$\begin{aligned}\|z\|_{\infty,\bar{\omega}_N} &\leq \alpha^{-1} \left\| R - \lambda \sum_{j=0}^{N} \hbar_j M_{ij} z_j \right\|_{\infty, \omega_N} \\ &\leq \alpha^{-1} \|R\|_{\infty,\omega_N} + \alpha^{-1} |\lambda| \max_{1 \leq i \leq N} \sum_{j=0}^{N} \hbar_j |M_{ij}| \, \|z\|_{\infty,\bar{\omega}_N}.\end{aligned}$$

Hence

$$\|z\|_{\infty,\bar{\omega}_N} \leq \frac{\alpha^{-1} \|R\|_{\infty,\omega_N}}{1 - \alpha^{-1} |\lambda| \max\limits_{1 \leq i \leq N} \sum\limits_{j=0}^{N} \hbar_j |M_{ij}|},$$

which leads to

$$\|z\|_{\infty,\overline{\omega}_N} \leq C \|R\|_{\infty,\omega_N}. \tag{19}$$

Now we estimate the remainder terms $R_i^{(1)}$, $R_i^{(2)}$ and $R_i^{(3)}$ separately.
First we will show that, for $R_i^{(1)}$ the estimate

$$\left| R_i^{(1)} \right| \leq Ch^2, \tag{20}$$

holds. Using relations

$$v(x) = v(x_i) + (x - x_i)\, v'(\eta_i), \quad \eta_i \in (x_i, x),$$

$$a(x) = a(x_i) + (x - x_i)\, a'(x_i) + \frac{(x - x_i)^2}{2} a''(\xi_i), \quad \xi_i \in (x_i, x)$$

and

$$\int_{x_{i-1}}^{x_{i+1}} (x - x_i)\, \psi_i(x)\, dx = 0,$$

we take

$$\begin{aligned} R_i^{(1)} &= \frac{1}{\chi_i h} \int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)]\, v(x) \psi_i(x) dx = \frac{a'(x_i)\, v(x_i)}{\chi_i h} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)\, \psi_i(x)\, dx \\ &+ \frac{a'(x_i)}{\chi_i h} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, v'(\eta_i(x))\, \psi_i(x)\, dx \\ &+ \frac{1}{2\chi_i h} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, a''(\xi_i(x))\, v(x)\, \psi_i(x)\, dx \\ &\equiv \frac{a'(x_i)}{\chi_i h} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, v'(\eta_i(x))\, \psi_i(x)\, dx \\ &+ \frac{1}{2\chi_i h} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, a''(\xi_i(x))\, v(x)\, \psi_i(x)\, dx. \end{aligned} \tag{21}$$

Since $a \in C^2[0, l]$, $|v(x)| \leq C$ and $|x - x_i| \leq h$ for the second term in the right side of (21), we have

$$\begin{aligned} \frac{1}{2\chi_i h} \left| \int_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, a''(\xi_i(x))\, v(x)\, \psi_i(x)\, dx \right| &\leq \frac{Ch^2}{\chi_i h} \int_{x_{i-1}}^{x_{i+1}} \psi_i(x)\, dx \\ &= O(h^2). \end{aligned} \tag{22}$$

Next, according to Lemma 1, we take the following inequality

$$\begin{aligned} |v'(\eta_i)| &\leq C \left\{ 1 + \frac{1}{\sqrt{\varepsilon}} \left(e^{-\frac{\sqrt{\alpha}\eta_i}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l - \eta_i)}{\sqrt{\varepsilon}}} \right) \right\} \\ &\leq C \left\{ 1 + \frac{1}{\sqrt{\varepsilon}} \left(e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l - x_{i+1})}{\sqrt{\varepsilon}}} \right) \right\}, \qquad 1 < i < N - 1. \end{aligned}$$

Hence, for the first term in the right side of (21), we have

$$\frac{1}{\chi_i h}\left|a'(x_i)\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2 v'(\eta_i(x))\,\psi_i(x)\,dx\right| \le \frac{C}{\chi_i h}\left|a'(x_i)\right|\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,dx$$

$$+\frac{C}{\sqrt{\varepsilon}\chi_i h}\left|a'(x_i)\right|\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}dx$$

$$+\frac{C}{\sqrt{\varepsilon}\chi_i h}\left|a'(x_i)\right|\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,e^{-\frac{\sqrt{\alpha}x_{i+1}}{\sqrt{\varepsilon}}}dx. \tag{23}$$

We can easily view that the first term in the right side of (23) is that $O(h^2)$. From $a'(0)=0$ and $xe^{-x}\le e^{-\frac{x}{2}}$, $(x\ge 0)$ for the second term of (21), we have

$$\left|\frac{C}{\sqrt{\varepsilon}\chi_i h}a'(x_i)\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}dx\right|$$

$$\le \frac{C}{\sqrt{\varepsilon}\chi_i h}\left|a''(\bar{\xi}_i)\right|e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,dx$$

$$\le Ch^2\frac{x_i}{\sqrt{\varepsilon}}e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}$$

$$\le Ch^2\frac{x_i}{x_{i-1}}\frac{x_{i-1}}{\sqrt{\varepsilon}}e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}$$

$$\le Ch^2 i\,(i-1)^{-1}\,e^{-\frac{\sqrt{\alpha}x_{i-1}}{2\sqrt{\varepsilon}}}$$

$$\le Ch^2,\qquad i>1.$$

The same evaluation is achieved for the third term in the right side of (23) from $a'(l)=0$, for $i<N-1$. Thus, identity (21) is proved for $i=2,3,...,N-2$.

Also for $i=1$, using relations

$$a(x)=a(x_1)+(x-x_1)\,a'(x_1)+\frac{(x-x_1)^2}{2}a''(\xi_1),\quad \xi_1\in(x_1,x)$$

and

$$v(x)=v(x_0)+\int\limits_{x_0}^{x}v'(\xi)\,d\xi,$$

we get

$$R_1^{(1)}=\frac{1}{\chi_1 h}a'(x_1)\int\limits_{x_0}^{x_2}(x-x_1)\left[\int\limits_{x_0}^{x}v'(\xi)\,d\xi\right]\psi_1(x)\,dx$$

$$+\frac{1}{2\chi_1 h}\int_{x_0}^{x_2}(x-x_1)^2\,a''(\xi_1(x))\,v(x)\,\psi_1(x)\,dx. \tag{24}$$

From (22), the second term in the right side of (24) will be $O(h^2)$. From $a'(0)$ and Lemma 1, we can evaluate the first as following

$$\left|\frac{a'(x_1)}{\chi_1 h}\int_{x_0}^{x_2}(x-x_1)\left[\int_{x_0}^{x}v'(\xi)\,d\xi\right]\psi_1(x)\,dx\right| \le |a'(x_1)|\,h\int_{x_0}^{x_2}|v'(x)|\,dx$$

$$\le Cx_1 h\,|a''(\bar{\eta}_1)|\int_{x_0}^{x_2}\left\{1+\frac{1}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}}\right)\right\}dx$$

$$\le Ch^2\left\{h+\frac{1}{\sqrt{\varepsilon}}\int_{x_0}^{x_2}e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}}\,dx\right\}$$

$$\le Ch^2\left\{h+\sqrt{\alpha}^{-1}\left(1-e^{\frac{2\sqrt{\alpha}h}{\sqrt{\varepsilon}}}\right)\right\}=O\left(h^2\right).$$

Thus,

$$\left|R_1^{(1)}\right|=O\left(h^2\right)$$

are proved. The proof of $\left|R_{N-1}^{(1)}\right| = O\left(h^2\right)$ is similar. So, the inequality (20) is proved.

Next, it is not difficult to see that, for $f\in C^2[0,l]$

$$\left|R_i^{(2)}\right|=O\left(h^2\right),\qquad 1\le i\le N-1. \tag{25}$$

Finally, for $R_i^{(3)}$ we have

$$\left|R_i^{(3)}\right| \le \left|\frac{\lambda}{\chi_i h}\int_{x_{i-1}}^{x_{i+1}}dx\psi_i(x)\int_{x_{i-1}}^{x_{i+1}}\left(\int_0^l\frac{\partial^2 M(\xi,\zeta)}{\partial\xi^2}v(\zeta)\,d\zeta\right)M_1(x,\xi)\,d\xi\right|$$

$$+\left|\frac{1}{2}\lambda\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\left(M(x_i,\xi)\,v(\xi)\right)''d\xi\right|. \tag{26}$$

By virtue of boundedness of $\frac{\partial^2 M}{\partial x^2}$, $v(x)$ and $|M_1(x,\zeta)|\le Ch$ the first term in the right side of (26) will be $O(h^2)$.

Rearranging the second term in the right side of (26) gives

$$\frac{1}{2}|\lambda|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\left|\left(M(x_i,\xi)\,v(\xi)\right)\right|''d\xi$$

$$\leq \frac{1}{2}\left|\lambda\right|\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left|M''\left(x_i,\xi\right)\right|\left|v\left(\xi\right)\right|d\xi$$
$$+\left|\lambda\right|\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left|M'\left(x_i,\xi\right)\right|\left|v'\left(\xi\right)\right|d\xi$$
$$+\frac{1}{2}\left|\lambda\right|\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left|M\left(x_i,\xi\right)\right|\left|v''\left(\xi\right)\right|d\xi. \qquad (27)$$

Hence, from $|v(x)| \leq C$ and $\frac{\partial^2 M}{\partial x^2} \in C^2[0,l]$ for the first term on the right side (27) will be $O(h^2)$.
For the second term in the right side (27), we have the estimate

$$\left|\lambda\right|\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left|M'\left(x_i,\xi\right)\right|\left|v'\left(\xi\right)\right|d\xi \leq \left|\lambda\right|h^2\int\limits_0^l\left|M'\left(x_i,\xi\right)\right|\left|v'\left(\xi\right)\right|d\xi$$
$$\leq \left|\lambda\right|h^2\int\limits_0^l\left\{\left|M'\left(x_i,\xi\right)\right|\left|v\left(\xi\right)\right|+\left|M\left(x_i,\xi\right)\right|\left|v'\left(\xi\right)\right|\right\}d\xi.$$

From here using Lemma 1 it is obtained the estimate

$$\left|\lambda\right|\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left|M'\left(x_i,\xi\right)\right|\left|v'\left(\xi\right)\right|d\xi$$
$$\leq C\left|\lambda\right|h^2\int\limits_0^l\left(1+1/\sqrt{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)\right)d\xi$$
$$\leq Ch^2. \qquad (28)$$

For the third term in the right side (27), by virtue of (4) for $k=2$, we have

$$\frac{1}{2}\left|\lambda\right|\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left|M\left(x_i,\xi\right)\right|\left|v''\left(\xi\right)\right|d\xi$$
$$\leq C\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left|M\left(x_i,\xi\right)\right|\left\{1+\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right\}d\xi$$
$$\leq Ch^2\left\{1+\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left|M(x_i,\xi)\right|\left(\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)\right\}.$$

Taking into account the relations (the partial derivatives are estimated at intermediate points, as required by the mean value theorem, as indicated by the bar.)

$$M(x_i,\xi)=M(x,0)+\frac{\partial\bar{M}}{\partial\xi}\xi,\qquad M(x,0)=0,$$

we get

$$Ch^2\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}|M(x_i,\xi)|\,\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}\,d\xi=Ch^2\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left|M(x_i,0)+\frac{\partial\bar{M}}{\partial\xi}\xi\right|\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}$$

$$\leq Ch^2\int\limits_0^l\frac{\xi}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}\,d\xi,$$

from which after taking into consideration $xe^{-x}\leq e^{-\frac{x}{2}}$, we obtain

$$Ch^2\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}|M(x_i,\xi)|\,\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}\,d\xi\leq Ch^2\frac{1}{\sqrt{\alpha}}\int\limits_0^l\frac{1}{\sqrt{\varepsilon}}e^{-\frac{\sqrt{\alpha}\xi}{2\sqrt{\varepsilon}}}\,d\xi$$

$$=Ch^2\frac{2}{\alpha}\left(1-e^{-\frac{\sqrt{\alpha}l}{\sqrt{\varepsilon}}}\right)\leq Ch^2.$$

Analogously, after using the relation

$$M(x_i,\xi)=M(x_i,l)+\frac{\partial\bar{M}}{\partial\xi}(\xi-l),\qquad M(x,l)=0,$$

it is not difficult to confirm that

$$Ch^2\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}|M(x_i,\xi)|\,\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\,d\xi\leq Ch^2.$$

Therefore, we obtain

$$\frac{1}{2}\left|\lambda\right|\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left|M\left(x_i,\xi\right)\right|\left|v''\left(\xi\right)\right|d\xi\leq Ch^2. \tag{29}$$

Thus, it can be easily seen that the first term in the right side of (26) is that $O(h^2)$. In addition, after taking into account (28) and (29) we obtain

$$\left|R_i^{(3)}\right|\leq Ch^2. \tag{30}$$

From (20), (25) and (30), we have

$$|R_i|\leq Ch^2. \tag{31}$$

The bound (19) together with (31) finish the proof. □

4. Numerical Calculates

In this section, theoretical calculates are tested on one sample.
Our particular example is

$$Lv := -\varepsilon v''(x) + \left(2 - cos^2(\pi x)\right) v(x) + \frac{1}{2}\int_0^1 \left(e^{xsin(\pi\zeta)} - 1\right) v(\zeta)d\zeta = (1+x)^{-1}, \qquad (0 < x < 1),$$

$$v(0) = 1, \qquad v(1) = 0.$$

The exact solution to this problem is unknown. For this reason, we estimate errors and calculate solutions using the double-mesh method, which compares the obtained solution to a solution computed on a mesh that is twice as fine. We introduce the maximum point-wise errors and the computed ε-uniform maximum point-wise errors as

$$e_\varepsilon^N = \max_i |y_i^{\varepsilon,N} - \tilde{y}_{2i}^{\varepsilon,2N}|_{\infty,\overline{\omega}_N}, \quad e^N = \max_\varepsilon e_\varepsilon^N,$$

where $\tilde{y}_{2i}^{\varepsilon,2N}$ is the approximate solution of the related method on the mesh

$$\tilde{\omega}_{2N} = \{x_{\frac{i}{2}} : i = 0, 1, ..., 2N\}, \quad x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2} \quad \text{for} \quad 0 \le i \le N-1.$$

We also describe the computed ε-uniform the rates of convergence and the rates of convergence as follows

$$p_\varepsilon^N = \frac{\ln\left(\frac{e_\varepsilon^N}{e_\varepsilon^{2N}}\right)}{\ln 2}, \quad p^N = \frac{\ln\left(\frac{e^N}{e^{2N}}\right)}{\ln 2}.$$

The rate of convergence of the difference approximation is significantly in agreement with the theoretical analysis, as shown in the Table 1.

5. Conclusion

In this paper, we described a new second-order difference scheme, which was constructed on the uniform mesh by using composite trapezoidal rule for integral term involving kernel function to solve linear FIDEs with singular perturbation. We tested the technique on one example with various values of ε and N to demonstrate the appropriateness of the method. Numerical investigations can be sustained for more sophisticated types such as partial integro-differential equations, nonlinear, delay form, higher dimensional, etc.

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

ε	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
1	0,00873286	0,00220145	0,00055304	0,00013874	0,00003471
	1,988	1,993	1,995	1,999	
10^{-2}	0,01038533	0,00262165	0,00065997	0,00016591	0,00004165
	1,986	1,99	1,992	1,994	
10^{-4}	0,01110798	0,00281381	0,00071081	0,00017931	0,00004514
	1,981	1,985	1,987	1,99	
10^{-6}	0,01173901	0,00297572	0,00075327	0,00019055	0,00004817
	1,98	1,982	1,983	1,984	
10^{-8}	0,01168824	0,00296902	0,00075366	0,00019131	0,00004853
	1,977	1,978	1,978	1,979	
e^N	0,01173901	0,00297572	0,00075366	0,00019131	0,00004853
p^N	1,98	1,982	1,978	1,979	

Table 1. Maximum point-wise errors and convergence rates for various ε and N values.

Declaration of Competing Interests The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Amiraliyev, G. M., Durmaz, M. E., Kudu, M., Uniform convergence results for singularly perturbed Fredholm integro-differential equation, *J. Math. Anal.*, 9(6) (2018), 55–64.

[2] Amiraliyev, G. M., Durmaz, M. E., Kudu, M., Fitted second order numerical method for a singularly perturbed Fredholm integro-differential equation, *Bull. Belg. Math. Soc. Simon Steven.*, 27(1) (2020), 71–88. https://doi.org/10.36045/bbms/1590199305

[3] Amiraliyev, G. M., Durmaz, M. E., Kudu, M., A numerical method for a second order singularly perturbed Fredholm integro-differential equation, *Miskolc Math. Notes.*, 22(1) (2021), 37–48. https://doi.org/10.18514/MMN.2021.2930

[4] Amiraliyev, G. M., Mamedov, Y. D., Difference schemes on the uniform mesh for singularly perturbed pseudo-parabolic equations, *Turk. J. Math.*, 19 (1995), 207–222

[5] Brunner, H., Numerical Analysis and Computational Solution of Integro-Differential Equations, Contemporary Computational Mathematics-A Celebration of the 80th Birthday of Ian Sloan (J. Dick et al., eds.), Springer, Cham, 2018, 205–231. https://doi.org 10.1007/978-3-319-72456-0_11

[6] Chen, J., He, M., Zeng, T., A multiscale Galerkin method for second-order boundary value problems of Fredholm integro differential equation II: Efficient algorithm for the discrete linear system, *J. Vis. Commun. Image R.*, 58 (2019), 112–118. https://doi.org/10.1016/j.jvcir.2018.11.027

[7] Chen, J., He, M., Huang, Y., A fast multiscale Galerkin method for solving second order linear Fredholm integro-differential equation with Dirichlet boundary conditions, *J. Comput. Appl. Math.*, 364 (2020), 112352. https://doi.org/10.1016/j.cam.2019.112352

[8] Dehghan, M., Chebyshev finite difference for Fredholm integro-differential equation, *Int. J. Comput. Math.*, 85 (1) (2008), 123–130. https://doi.org/10.1080/00207160701405436

[9] Doolan, E. R., Miller, J. J. H., Schilders, W. H. A., Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.
[10] Durmaz, M. E., Amiraliyev, G. M., A robust numerical method for a singularly perturbed Fredholm integro-differential equation, *Mediterr. J. Math.*, 18(24) (2021), 1–17. https://doi.org/10.1007/s00009-020-01693-2
[11] Durmaz, M. E., Amiraliyev, G. M., Kudu, M., Numerical solution of a singularly perturbed Fredholm integro differential equation with Robin boundary condition, *Turk. J. Math.*, 46(1) (2022), 207–224. https://doi.org/10.3906/mat-2109-11
[12] Farrell, P. A., Hegarty, A. F., Miller, J. J. H., O'Riordan, E., Shishkin, G. I., Robust Computational Techniques for Boundary Layers, Chapman Hall/CRC, New York, 2000. https://doi.org/10.1201/9781482285727
[13] Jalilian, R., Tahernezhad, T., Exponential spline method for approximation solution of Fredholm integro-differential equation, *Int. J. Comput. Math.*, 97(4) (2020), 791–801. https://doi.org/10.1080/00207160.2019.1586891
[14] Jalius, C., Majid, Z. A., Numerical solution of second-order Fredholm integrodifferential equations with boundary conditions by quadrature-difference method, *J. Appl. Math.*, (2017). https://doi.org/10.1155/2017/2645097
[15] Kadalbajoo, M. K., Gupta, V., A brief survey on numerical methods for solving singularly perturbed problems, *Appl. Math. Comput.*, 217 (2010), 3641–3716. https://doi.org/10.1016/j.amc.2010.09.059
[16] Karim, M. F., Mohamad, M., Rusiman, M. S., Che-him, N., Roslan, R., Khalid, K., ADM for solving linear second-order Fredholm integro-differential equations, *Journal of Physics*, (2018), 995. https://doi.org/10.1088/1742-6596/995/1/012009
[17] Kudu, M., Amirali, I., Amiraliyev, G. M., A finite-difference method for a singularly perturbed delay integro-differential equation, *J. Comput. Appl. Math.*, 308 (2016), 379–390. https://doi.org/10.1016/j.cam.2016.06.018
[18] Miller, J. J. H., O'Riordan, E., Shishkin, G. I., Fitted Numerical Methods for Singular Perturbation Problems, World Scientific, Singapore, 1996.
[19] Nayfeh, A. H., Introduction to Perturbation Techniques, Wiley, New York, 1993.
[20] O'Malley, R. E., Singular Perturbations Methods for Ordinary Differential Equations, Springer, New York, 1991. https://doi.org/10.1007/978-1-4612-0977-5
[21] Roos, H. G., Stynes, M., Tobiska, L., Numerical Methods for Singularly Perturbed Differential Equations, Springer-Verlaq, Berlin, 1996. https://doi.org/10.1007/978-3-662-03206-0
[22] Samarskii, A. A., The Theory of Difference Schemes(1st ed.), CRC Press, 2001. https://doi.org/10.1201/9780203908518
[23] Shahsavaran, A., On the convergence of Lagrange interpolation to solve special type of second kind Fredholm integro differential equations, *Appl. Math. Sci.*, 6(7) (2012), 343–348.
[24] Yapman, Ö., Amiraliyev, G. M., Amirali, I., Convergence analysis of fitted numerical method for a singularly perturbed nonlinear Volterra integro-differential equation with delay, *J. Comput. Appl. Math.*, 355(2019), 301309. https://doi.org/10.1016/j.cam.2019.01.026
[25] Yapman, Ö., Amiraliyev, G. M., A novel second–order fitted computational method for a singularly perturbed Volterra integro-differential equation, *Int. J. Comput. Math.*, 97(6) (2020), 1293–1302. https://doi.org/10.1080/00207160.2019.1614565
[26] Xue, Q., Niu, J., Yu, D., Ran, C., An improved reproducing kernel method for Fredholm integro-differential type two-point boundary value problems, *Int. J. Comput. Math.*, 95(5) (2018), 1015–1023. https://doi.org/10.1080/00207160.2017.1322201