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An Hybrid Numerical Algorithm With Error Estimation For A Class Of Functional Integro-Differential Equations

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ABSTRACT

In this paper, a numerical algorithm based on Laguerre and Taylor polynomials is applied for solving a class of functional integro-differential equations. The considered problem is transfered to a matrix equation which corresponds to a system of linear algebraic equations by Hybrid collocation method under the mixed conditions. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments. Also, the approximate solutions are corrected by using the residual correction.

Key words: *Linear functional-differential equations, Taylor and Laguerre polynomials and series, Algorithms for functional approximation, Collocation methods.*

1. INTRODUCTION

The aim of this work is to present an efficient numerical scheme for solving a class of functional integro-differential equations with respect to the Laguerre and Taylor polynomials. They play an important role in the study of heat conduction, electromagnetic theory, electrical engineering, quantum mechanics, medicine, mechanical and mathematical statistics etc. [1-11]. They provide an inherent way to expand and interpret solutions to many types of important equations.

Recently, we have seen an increase in the application of these type of problems in biology, physics and engineering. In the field of a class of functional integro-differential equations, the computation of its solution has been a great challenge and importance due to the versatility of such equations in the mathematical modeling of processes in various application fields. Mainly, we deal with the following equation,

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(\lambda_k x+\mu_k) = g(x) + \int_a^b K_f(x,t)\, y(\alpha t+\beta)dt$$

$$+\int_a^{h(x)} K_v(x,t)y(t)dt,\ \ 0 \le a \le x, h(x), t \le b < \infty \qquad (1)$$

under the mixed conditions

$$\sum_{k=0}^{m-1} a_{jk}y^{(k)}(a) + b_{jk}y^{(k)}(b) = \lambda_j, \qquad j = 0,1,2,\dots,m-1 \qquad (2)$$

where $y(x)$ is an unknown function, the known functions $P_k(x)$, $g(x)$ g(x), $K_f(x,t)$ and $K_v(x,t)$ are defined on an

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interval $[a, b]$ and also, a_{jk} and b_{jk} are appropriate constants. Our aim is to find an approximate solution expressed in the form

$$y_N(x) = \sum_{n=0}^{N} a_n L_n(x),\ m \leq N. \tag{3}$$

where a_n are unknown coefficients, $L_n(x)$, $(n = 0,1, \dots)$ are Laguerre polynomials defined by

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \binom{n}{r} x^r \ . \tag{4}$$

This is the relation between the powers of x and the Laguerre polynomials [12-15].

To obtain a solution in the form (3) of the problem (1) under the conditions (2), we can use the collocation points defined by

$$x_i(x) = a + \frac{b-a}{N} i,\ i = 0,1,2, \dots, N \tag{5}$$

The remainder of the paper is organized as follows: In Section 2., some properties of Laguerre polynomials are given. A class of functional integro-differential equations and fundamental relations are presented in Section 3. The method of finding approximate solution and the algorithm for the calculation are described in Section 4. Residual error analysis is performed in Section 5. To support our finding, the result of numerical experiments using Maple 18 in Section 6. In Section 7, we have the conclusion of this article with a brief summary.

2. PROPERTIES OF LAGUERRE POLYNOMIALS

In this section, we consider some properties of Laguerre polynomials. Laguerre polynomials are one of the orthogonal polynomials. Orthogonal polynomials are very important in the area of mathematics. One of the orthogonal polynomials is Laguerre polynomials $L_n(x)$ which constitute complete orthogonal sets of functions on the semi-infinite interval $[0, \infty)$. Laguerre polynomials are defined by Rodriguez formula as

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x})$$

where e^{-x} is the weight function of orthogonal Laguerre polynomials. By using Leibniz formula and the formula

$$\frac{d^p x^q}{dx^p} = q(q-1) \dots (q-p+1) x^{q-p} = \frac{q!}{(q-p)!} x^{q-p}$$

we have the result

$$\frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) = \frac{e^x}{n!} \sum_{r=0}^{n} \frac{n!}{(n-r)!\, r!} \frac{n!}{r!} x^r (-1)^r e^{-x} = \sum_{r=0}^{n} (-1)^r \frac{n!\, x^r}{(r!)^2 (n-r)!} = L_n(x).$$

Then, we obtain Laguerre polynomials defined in (4) [16].

Laguerre polynomials $L_n(x)$ have analogous properties to those of the classical orthogonal polynomials. The generating function of Laguerre polynomials is defined as

$$T(x,t) = \frac{e^{\frac{-xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n, \quad |t| < 1.$$

We first differentiate the generating function with respect to; we obtain

$$(n+1) L_{n+1}(x) = (2n + 1 - x)\, L_n(x) - n\, L_{n-1}(x)$$

Differentiating the generating function with respect to x gives us the second recursion relation

$$x L'_n(x) = n\, L_n(x) - n\, L_{n-1}(x)$$

Then, we have the recursion relation is given as

$$L'_n(x) = -\sum_{r=0}^{n-1} L_r(x),$$

where we may show some Laguerre polynomials as

$$L_0(x) = 1,\ L_1(x) = 1 - x,\ L_2(x) = \frac{1}{2!}(x^2 - 4x + 2),\ L_3(x) = \frac{1}{3!}(-x^3 + 9x^2 - 18x + 6) \dots$$

3. FUNDAMENTAL MATRIX RELATIONS

Let us write Eq. (1), briefly, in the form

$$D(x) = g(x) + I_1(x) + I_2(x):$$

The functional differential part

$$D(x) = \sum_{k=0}^{m} P_k(x) y^{(k)}(\lambda_k x + \mu_k),$$

The functional Fredholm integral part

$$I_1(x) = \int_a^b K_f(x,t)\, y(\alpha t + \beta) dt,$$

The functional Volterra integral part

$$I_2(x) = \int_a^{h(x)} K_v(x,t)\, y(t) dt.$$

In this section, we convert these parts and the mixed conditions Eq. (2) into the matrix forms. Let us consider Eq. (1) and find the matrix forms of each term of the equation. We first consider the solution $y(x)$ and its derivative $y^{(k)}(x)$ defined by a truncated Laguerre series. Then, we can put the solution in the matrix form

$$y(x) = \mathbf{L}(x)\mathbf{A}\ , y^{(k)}(x) = \mathbf{L}^{(k)}(x)\mathbf{A} \tag{6}$$

where

$$\mathbf{L}(x) = [\, L_0(x)\ \ L_1(x) \quad \dots \quad L_N(x)]$$
$$\mathbf{L}^{(k)}(x) = \left[L_0^{(k)}(x)\ L_1^{(k)}(x) \quad \dots \quad L_N^{(k)}(x)\right]$$
$$\mathbf{L}(x) = [a_0\ a_1 \quad \dots \quad a_N]^T$$

By using (4) and taking $n = 0,1,\dots,N$ we obtain the matrix relation

$$\mathbf{L}^T(x) = \mathbf{H}\mathbf{X}^T(x) \text{ and } \mathbf{L}(x) = \mathbf{X}(x)\mathbf{H}^T \tag{7}$$

where

$$\mathbf{X}(x) = [1 \quad x^1 \quad \dots \quad x^N] \tag{8}$$

and

$$\mathbf{H} = \begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & 0 & \dots & 0 \\ \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^1}{1!}\binom{1}{1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^0}{0!}\binom{N}{0} & \frac{(-1)^1}{1!}\binom{N}{1} & \dots & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix}$$

Then, by taking into account (7), we have

$$\mathbf{L}^{(k)}(x) = \mathbf{X}^{(k)}(x)\mathbf{H}^T, k = 0,1,2,\dots \tag{9}$$

Moreover, it is clear that the relation between the matrix $\mathbf{X}(x)$ and its derivative $\mathbf{X}^{(k)}(x)$ is

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{B}^k \tag{10}$$

where

$$\mathbf{B}=\begin{bmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & N\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}$$

Hence, by means of Eq.(9) and Eq.(10) we have the matrix relation (6) in the form

$$\left[\,y^{(k)}(x)\right]=\mathbf{L}^{(k)}(x)\mathbf{A}=\mathbf{X}(x)\mathbf{B}^{k}\mathbf{H}^{T}\mathbf{A},\quad \mathrm{k}=0,1,2,\dots,\mathrm{m}. \tag{11}$$

3.1. Matrix representation of functional part $D(x)$

Firstly, by using the relations (9) and (10), by substituting $x=\lambda_k x+\mu_k$ into (11) [16-17], we obtain

$$[y^{(k)}(\lambda_k x+\mu_k)]=\mathbf{L}^{(k)}(\lambda_k x+\mu_k)\mathbf{A} \tag{12}$$

where

$$\begin{aligned}\mathbf{L}^{(k)}(\lambda_k x+\mu_k)\mathbf{A}=\mathbf{X}^{(k)}(\lambda_k x+\mu_k)\mathbf{H}^{T}\mathbf{A}\\ =\mathbf{X}(\lambda_k x+\mu_k)\mathbf{B}^{k}\mathbf{H}^{T}\mathbf{A}\\ =\mathbf{X}(x)\mathbf{B}_{(\lambda_k,\mu_k)}\mathbf{H}^{T}\mathbf{A}\end{aligned}$$

where

$$\mathbf{B}_{(\lambda_k,\mu_k)}=\begin{bmatrix}\binom{0}{0}(\lambda_k)^0(\mu_k)^0 & \binom{1}{0} & \dots & \binom{N}{0}(\lambda_k)^0(\mu_k)^N\\ 0 & \binom{1}{1} & \dots & \binom{N}{1}(\lambda_k)^1(\mu_k)^{N-1}\\ 0 & 0 & \dots & \binom{N}{2}(\lambda_k)^2(\mu_k)^{N-2}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & & \binom{N}{N}(\lambda_k)^N(\mu_k)^0\end{bmatrix}$$

Therefore, by using the relation (12), the matrix representation of the functional part $D(x)$ can be given by

$$D(x,t)=\sum_{k=0}^{m}\mathrm{P}_k(x)\,\mathbf{X}(x)\mathbf{B}_{(\lambda_k,\mu_k)}\mathbf{B}^{k}\mathbf{H}^{T}\mathbf{A} \tag{13}$$

3.2. Matrix representationof functional Fredholm integral part $I_1(x)$

In this section, we use Taylor series and the matrix representation of kernel function $K_f(x,t)$ becomes

$$K_f(x,t)=\mathbf{X}(x)\mathbf{K}_f\mathbf{X}^{T}(t) \tag{14}$$

so that

$$K_f(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}k_{ij}^{f}\,x^i t^j,$$

$$k_{ij}^{f}=\frac{1}{i!j!}\frac{\partial^{i+j}K_f(0,0)}{\partial x^i\partial t^j}\ .$$

Also, the expression $y(\alpha t+\beta)$ in the part $I_1(x)$ becomes

$$\begin{aligned}y(\alpha t+\beta)&=\mathbf{L}(\alpha t+\beta)\mathbf{A}\\ &=\mathbf{X}(\alpha t+\beta)\mathbf{H}^{T}\mathbf{A}\\ &=\mathbf{X}(t)\mathbf{B}_{(\alpha,\beta)}\mathbf{H}^{T}\mathbf{A}\end{aligned}$$

By substituting the relations (14) and the matrix form of $(\alpha t+\beta)$, we organize $I_1(x)$ as follows:

$$\begin{aligned}I_1(x)&=\int_a^b\mathbf{X}(x)\mathbf{K}_f\mathbf{X}^{T}(t)\mathbf{X}(t)\,\mathbf{B}_{(\alpha,\beta)}\mathbf{H}^{T}\mathbf{A}dt\\ &=\mathbf{X}(x)\mathbf{K}_f\int_a^b\mathbf{X}^{T}(t)\mathbf{X}(t)\,dt\mathbf{B}_{(\alpha,\beta)}\mathbf{H}^{T}\mathbf{A}\\ &=\mathbf{X}(x)\mathbf{K}_f\mathbf{Q}_f\mathbf{B}_{(\alpha,\beta)}\mathbf{H}^{T}\mathbf{A}\end{aligned}$$

where

$$\mathbf{Q}_f=\int_a^b \mathbf{X}^T(t)\mathbf{X}(t)\,dt=\int_a^b [t^{i+j}]\,dt=\left[q_{ij}^f\right],\quad i,j=0,1,\dots,N$$

$$q_{ij}^f=\frac{b^{i+j+1}-a^{i+j+1}}{i+j+1},i,j=0,1,\dots,N.$$

Hence, the matrix representation of functional Fredholm integral part can be given as

$I_1(x)=\mathbf{X}(x)\mathbf{K}_f\mathbf{Q}_f\mathbf{B}_{(\alpha,\beta)}\mathbf{H}^T\mathbf{A}.$ (15)

3.3. Matrix representation of functional Volterra integral part $I_2(x)$

Similarly, we use Taylor series and then the matrix representation of the kernel function $K_v(x,t)$ becomes

$K_v(x,t)=\mathbf{X}(x)\mathbf{K}_v\mathbf{X}^T(t)$ (16)

where

$$\mathbf{K}_v(x,t)=\left[k_{ij}^v(x)\right],\quad i,j=0,1,\dots,N.$$

Then, we consider Laguerre polynomials and using the relation (16), we obtain the Volterra part as

$$\begin{aligned}I_2(x)&=\int_a^{h(x)}\mathbf{X}(x)\mathbf{K}_v\mathbf{X}^T(t)\mathbf{L}(t)\,\mathbf{A}dt\\&=\mathbf{X}(x)\mathbf{K}_v\int_a^{h(x)}\mathbf{X}^T(t)\mathbf{X}(t)\,dt\mathbf{H}^T\mathbf{A}\\&=\mathbf{X}(x)\mathbf{K}_v\mathbf{Q}_v(x)\mathbf{H}^T\mathbf{A}\end{aligned}$$

where

$$\mathbf{Q}_v(x)=\int_a^{h(x)}\mathbf{X}^T(t)\mathbf{X}(t)\,dt=\int_a^{h(x)}[t^{i+j}]\,dt=\left[q_{ij}^v(x)\right],\quad i,j=0,1,\dots,N$$

$$q_{ij}^v(x)=\frac{h(x)^{i+j+1}-a^{i+j+1}}{i+j+1},\qquad i,j=0,1,\dots,N$$

Hence, the matrix representation of Volterra integral part can be given by

$I_2(x)=\mathbf{X}(x)\mathbf{K}_v\mathbf{Q}_v(x)\mathbf{H}^T\mathbf{A}.$ (17)

3.4. Matrix representation of the conditions

By using the relation (11) in (2), we can write

$\left[\,y^{(k)}(a)\right]=\mathbf{X}(a)(\mathbf{B})^k\mathbf{H}^T\mathbf{A}$ and $\left[\,y^{(k)}(b)\right]=\mathbf{X}(b)(\mathbf{B})^k\mathbf{H}^T\mathbf{A}$

and then, we obtain the matrix form for conditions (2)

$$\mathbf{U}_j\mathbf{A}=\sum_{k=0}^{m-1}\left(a_{jk}\mathbf{X}(a)+b_{jk}\mathbf{X}(b)\right)(\mathbf{B})^k\mathbf{H}^T\mathbf{A}=[\lambda_j]$$

where

$$\mathbf{U}_j=[u_{j0}\quad\cdots\quad u_{jN}],\qquad j,k=0,1,\dots,m-1.$$

4. METHOD OF SOLUTION

We are ready to construct the fundamental matrix equation corresponding to Eq.(1). For this purpose, by substituting the matrix relations (13), (15) and (17) into Eq.(1), we obtain the matrix equation

$$\sum_{k=0}^{m}\mathrm{P}_k(x)\,\mathbf{X}(x)\mathbf{B}_{(\lambda_k,\mu_k)}\mathbf{B}^k\mathbf{H}^T\mathbf{A}=g(x)+\mathbf{X}(x)\mathbf{K}_f\mathbf{Q}_f\mathbf{B}_{(\alpha,\beta)}\mathbf{H}^T\mathbf{A}$$

$$+\mathbf{X}(x)\mathbf{K}_v\mathbf{Q}_v(x)\mathbf{H}^T\mathbf{A}$$

Then, we substitute the collocation points defined by Eq.(5) into the previous equation and obtain the system

$$\sum_{k=0}^{m} \mathrm{P}_k(x_i)\,\mathbf{X}(x)\mathbf{B}_{(\lambda_k,\mu_k)}\mathbf{B}^k\mathbf{H}^T\mathbf{A} = g(x_i) + \mathbf{X}(x_i)\mathbf{K}_f\mathbf{Q}_f\mathbf{B}_{(\alpha,\beta)}\mathbf{H}^T\mathbf{A}$$

$$+\mathbf{X}(x_i)\mathbf{K}_v\mathbf{Q}_v(x_i)\mathbf{H}^T\mathbf{A}$$

Briefly, the fundamental matrix equation is gained as

$$\left(\sum_{k=0}^{m} \mathbf{P}_k\,\mathbf{X}(x)\mathbf{B}_{(\lambda_k,\mu_k)}\mathbf{B}^k\mathbf{H}^T - \mathbf{X}\mathbf{K}_f\mathbf{Q}_f\mathbf{B}_{(\alpha,\beta)}\mathbf{H}^T - \overline{\mathbf{X}}\overline{\mathbf{K}}_v\overline{\mathbf{Q}}_v\overline{\mathbf{H}}^T\right)\mathbf{A} = G \tag{18}$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & \dots & 0 \\ 0 & P_k(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_k(x_N) \end{bmatrix}, \overline{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(x_0) & 0 & \dots & 0 \\ 0 & \mathbf{X}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{X}(x_N) \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} 1 x_0 x_0^2 \dots x_0^N \\ 1 x_1 x_1^2 \cdots x_1^N \\ 1 x_2 x_2^2 \cdots x_2^N \\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots \\ 1 x_N x_N^2 \dots x_N^N \end{bmatrix}, \quad \overline{\mathbf{H}} = \begin{bmatrix} \mathbf{H}^T \\ \mathbf{H}^T \\ \mathbf{H}^T \\ \vdots \\ \mathbf{H}^T \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

The fundamental matrix equation (18) for Eq.(1) corresponds to a system of $(N+1)$ algebraic equation for the $(N+1)$ unknown coefficients $a_0, a_1, \dots, a_N$[18]. Briefly, we can write Eq.(18) as

$$\mathbf{WA} = \mathbf{G} \quad \text{or} \quad [\mathbf{W}; \boldsymbol{G}] \tag{19}$$

so that, for $i, j = 0,1,2, \dots, N$

$$\mathbf{W} = [\omega_{ij}] = \sum_{k=0}^{m} \mathbf{P}_k\,\mathbf{X}(x)\mathbf{B}_{(\lambda_k,\mu_k)}\mathbf{B}^k\mathbf{H}^T - \mathbf{X}\mathbf{K}_f\mathbf{Q}_f\mathbf{B}_{(\alpha,\beta)}\mathbf{H}^T - \overline{\mathbf{X}}\overline{\mathbf{K}}_v\overline{\mathbf{Q}}_v\overline{\mathbf{H}}^T$$

Then, it is seen from Section 3. 4 that the matrix form of the conditions (2) can be written as

$$\mathbf{U}_j\mathbf{A} = [\lambda_j] \text{ or } [\mathbf{U}_j; \lambda_j],\ j = 0,1,2, \dots, m-1 \tag{20}$$

To obtain the solution of Eq.(1) under the conditions (2), by replacing the rows in the matrix (20) by the last m rows of the matrix (19), we have the required augmented matrix

$$\widetilde{\mathbf{W}}\mathbf{A} = \widetilde{\mathbf{G}} \tag{21}$$

If $rank(\widetilde{\mathbf{W}}) = rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = N+1$, then we can write

$$\mathbf{A} = (\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}.$$

Thus, the coefficients a_n, $i = 0,1,2, \dots, N$ are uniquely determined by Eq.(21).

5. RESIDUAL CORRECTION AND ERROR ESTIMATION

In this section, we introduce an error estimation for the Laguerre polynomial solution (3) and it supports the idea of the corrected Laguerre polynomial solution with respect to the residual error function. Let us define Eq.(1)-(2) with L operator

$$L[y(x)] = \sum_{k=0}^{m} P_k(x) y^{(k)}(\lambda_k x + \mu_k) - \int_a^b K_f(x,t)\, y(\alpha t + \beta)dt$$

$$-\int_a^{h(x)} K_v(x,t) y(t)dt = g(x),\ 0 \le a \le x, h(x), t \le b < \infty$$

Then, we get the residual function of the Laguerre collocation method as

$$R_N(x) = L[y_N(x)] - g(x) \tag{22}$$

where $y_N(x)$ is the approximate solution (3) of Eq. (1) with conditions (2). Thus, $y_N(x)$ satisfies the problem (1)-(2). Also, the error function $E_N(x)$ can be defined as

$$E_N(x) = y(x) - y_N(x)$$

where $y(x)$ is the exact solution of Eq.(1) with conditions (2). Then, by using Eq. (1), (2) and (22), we obtain the error differential equation

$$L[E_N(x)] = L[y(x)] - L[y_N(x)] = -R_N(x)$$

with the homogeneous conditions

$$\sum_{k=0}^{m-1} a_{jk} E_N^{(k)}(a) + b_{jk} E_N^{(k)}(b) = 0, j = 0,1, \dots, m-1.$$

Therefore, by solving this error problem with the technique introduced in Section 4, we get the approximation

$$E_{N,M}(x) = \sum_{n=0}^{M} a_n^* L_n(x)$$

to $E_N(x)$. Consequently, by means of the polynomials $y_N(x)$ and $E_{N,M}(x)$, $M > N$, we obtain the corrected Laguerre polynomial solution $y_{N,M}(x) = y_N(x) + E_{N,M}(x)$.

Also, we construct the corrected error function

$$E_{N,M}^*(x) = E_N(x) - E_{N,M}(x) = y(x) - y_{N,M}(x)$$

and the estimated error function $E_{N,M}(x)$.[20 − 23].

5.1. Algorithm

In this section, we consider the method in two steps which is solved by Maple 18. Algorithm 1, shows the calculation of approximate solution. Algorithm 2, shows the each step of residual error function and it supports to the corrected error and the corrected Laguerre polynomial solution.

Algorithm 1

Step 1

1. Given coefficients are P_k, kernel functions are $K_f(x,t), K_v(x,t)$ and the coefficients of the conditions a_{jk}, b_{jk}.
2. The set on $[0,1]$.
3. Develop the algorithm for matrix system which is constructed by collocation points in Eq.(5).
4. Compute $[\mathbf{W}, \mathbf{G}]$ then stop.

Step 2

1. Construct the augmented matrix by given conditions and the system is done.
2. If $rank(\widetilde{\mathbf{W}}) = rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = N+1$ then run.
3. Output: **A** unknown matrix.
4. Compute approximate solution with truncated Laguerre series.
5. End of the Algorithm 1.

Algorithm 2

1. Given $y_N(x), E_N(x)$ and truncated Laguerre series (4).
2. If $L(y(x)) = y_N(x)$ then run.
3. Compute $L(E_N(x))$ with respect to $E_N(0)$ conditions.
4. Obtain $E_{N,M}(x)$.
5. Output: $E_{N,M}^*(x)$.
6. End of the Algorithm 2. [24 − 28].

6. ILLUSTRATIVE EXAMPLES

In this section, several numerical examples are given to illustrate the accuracy and the effectiveness of the method. All of them are performed on the computer by using Maple 18.

Example 6.1 [29]

Let us first consider the functional differential equation with variable coefficients

$$y'(x)+y(x)+e^{(x-1)}y(x-1)=1, \qquad 0\le x\le 1$$

under the condition

$$y(0)=1$$

and we seek the approximate solution $y_N(x)$ as a truncated Laguerre series

$$y_N(x)=\sum_{r=0}^{N} a_r L_r(x),\ 0\le x<\infty.$$

Here,

$$P_0(x)=1, P_1(x)=1, P_2(x)=e^{(x-1)}, g(x)=1$$

Then, for $N=3$, the collocation points are $x_0=0, x_1=\frac{1}{3}, x_2=\frac{2}{3}, x_3=1$. The fundamental matrix $\mathbf{W}$ and the condition matrix $[\mathbf{U}_0;\lambda_0]$ are computed as

$$\mathbf{W}=\mathbf{P}_0\mathbf{X}\mathbf{H}^T+\mathbf{P}_1\mathbf{X}\mathbf{B}+\mathbf{P}_2\mathbf{X}\mathbf{B}_{(\lambda_1,\mu_0)}\mathbf{H}^T$$

and

$$[\mathbf{U}_0;\lambda_0]=[1\quad 1\quad 1\quad 1\quad ;\quad 1]$$

Then, we have the augmented matrix

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\begin{bmatrix} \frac{31769}{23225} & \frac{26633}{36198} & \frac{3086}{10731} & \frac{5298}{62587} & ; & 1 \\ \frac{8911}{5888} & \frac{25894}{49571} & \frac{10276}{299695} & \frac{3795}{295984} & ; & 1 \\ \frac{121321}{70678} & \frac{16957}{58734} & -\frac{5639}{26799} & -\frac{5989}{124691} & ; & 1 \\ 1 & 1 & 1 & 1 & ; & 1 \end{bmatrix}$$

Subsequently, we have

$$\mathbf{A}=\begin{bmatrix} \frac{11964}{33989} & \frac{37143}{39928} & -\frac{20361}{29090} & \frac{32351}{77453} \end{bmatrix}$$

and we obtain the approximate solution of the problem for $N=3$ as

$$y_3(x)=1-0.783443646x+0.2765627224x^2-0.06961426068x^3$$

We follow the same steps for $N=6$ and $N=10$. The exact solution of this problem is $y(x)=e^{-x}$.

Table 1. Numerical solution of Example 6.1 for different N.

		Present Method and Corrected Laguerre polynomial solution					
x	Exact Solution	y_3	$y_{3,7}$	y_6	$y_{6,7}$	y_{10}	$y_{10,12}$
0.0	1.00000	1.00000	1.00000	0.99999	1.00000	1.00000	1.00000
0.1	0.90484	0.92435	0.88576	0.90634	0.90375	0.90502	0.90465
0.2	0.81873	0.85382	0.78444	0.82142	0.81677	0.81907	0.81840
0.3	0.74082	0.78798	0.69472	0.74443	0.73819	0.74127	0.74037
0.4	0.67032	0.72642	0.61548	0.67461	0.66720	0.67085	0.66979
0.5	0.60653	0.66872	0.54574	0.61128	0.60307	0.60712	0.60594
0.6	0.54881	0.61446	0.48463	0.55382	0.54517	0.54943	0.54819
0.7	0.49659	0.56323	0.43144	0.50167	0.49289	0.49722	0.49596
0.8	0.44933	0.51460	0.38552	0.45432	0.44570	0.44995	0.44871
0.9	0.40657	0.46817	0.34637	0.41133	0.40311	0.40716	0.40598
1.0	0.36788	0.42350	0.31354	0.37226	0.36469	0.36842	0.36734

Table 1 shows the comparison between exact and approximate solutions for $N=3,6$ and 10 values. The results of residual error function support to evaluate the corrected Laguerre polynomial solutions for different N and M values. Figure 1 shows the comparison between the exact and approximate solutions for different N values. Figure 2 shows the absolute error and absolute residual error functions. Mainly, this comparison shows the relation between corrected Laguerre polynomial solution for different N and M values.

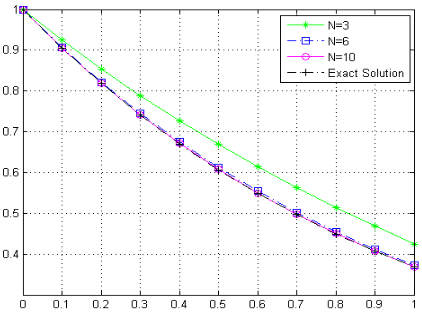

Fig.1. Comparison between approximate solutions for $N = 3, 6, 10$ values and exact solution for Example 6.1.

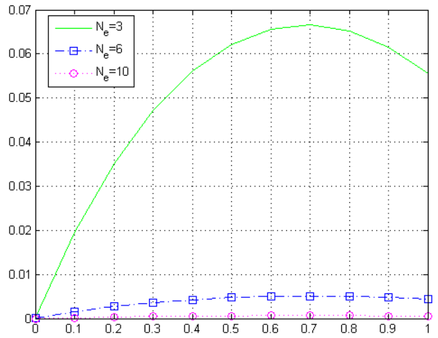

Fig.2. Comparison between absolute error functions $N_e = E_N$ for the N values of Example 6.1.

Table 2. Comparison of the actual and estimated absolute errors for Example 6.1.

The actual absolute and estimated absolute errors						
x	E_3	$E_{3,7}$	E_6	$E_{6,7}$	E_{10}	$E_{10,12}$
0.0	0.0000000	4.800E-09	0.500E-09	2.400E-09	0.0000000	0.347E-13
0.1	1.951E-02	1.907E-02	1.499E-03	1.091E-03	1.186E-04	0.105E-09
0.2	3.509E-02	3.930E-02	2.694E-03	1.960E-03	3.351E-04	3.346E-04
0.3	4.716E-02	4.610E-02	3.616E-03	2.631E-03	4.498E-04	4.492E-04
0.4	5.610E-02	5.484E-02	4.293E-03	3.125E-03	5.341E-04	5.333E-04
0.5	6.219E-02	6.079E-02	4.749E-03	3.456E-03	5.910E-04	5.900E-04
0.6	6.565E-02	6.418E-02	5.007E-03	3.643E-03	6.230E-04	6.220E-04
0.7	6.664E-02	6.515E-02	5.083E-03	3.699E-03	6.325E-04	6.314E-04
0.8	6.527E-02	6.381E-02	4.994E-03	3.634E-03	6.214E-04	6.204E-04
0.9	6.160E-02	6.020E-02	4.755E-03	3.451E-03	5.917E-04	5.908E-04
1.0	5.563E-02	5.434E-02	4.381E-03	3.188E-03	5.451E-04	5.442E-04

Table 2 shows the corrected absolute errors for $N = 3, 6\ 10$ and $M = 7,\ 12$. The results support the idea when N and M values are chosen large enough, the absolute error and residual error decrease.

Example 6.2 [30]

Let us find the Laguerre series solution of the following second order linear functional Fredholm-integro differential equation with variable coefficients

$$y''(x) - e^{(x+2)}y'(x+1) - e^{-x} = \int_0^1 y(t-1)dt\,, \qquad 0 \le x \le 1$$

with initial conditions $y(0) = 2$, $y'(0) = -1$ .The exact solution of this problem is $y(x) = 1 + e^{-x}$. By means of our method and Maple 18; the approximate solutions of the problem for $N = 2, 8\ 10$ are computed and the obtained results are given in Table 3.

Table 3. Numerical solutions of Example 6.2 for different N.

		Present Method and Corrected Laguerre polynomial solution					
x	Exact Solution	y_2	$y_{2,7}$	y_8	$y_{8,16}$	y_{10}	$y_{10,16}$
0.0	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000
0.1	1.90484	1.90393	1.90102	1.90490	1.90393	1.90393	1.90393
0.2	1.81873	1.81574	1.80501	1.81988	1.81574	1.81574	1.81574
0.3	1.74082	1.73541	1.71328	1.74513	1.73541	1.73541	1.73541
0.4	1.67032	1.66295	1.62689	1.68054	1.66295	1.66295	1.66295
0.5	1.60653	1.59836	1.54656	1.62571	1.59836	1.59836	1.59836
0.6	1.54881	1.54164	1.47263	1.57998	1.54164	1.54164	1.54164
0.7	1.49659	1.49279	1.40500	1.54245	1.49279	1.49279	1.49279
0.8	1.44933	1.45181	1.34318	1.51209	1.45181	1.45181	1.45181
0.9	1.40657	1.41870	1.28635	1.48779	1.41870	1.41870	1.41870
1.0	1.36788	1.39345	1.23339	1.46842	1.39345	1.39345	1.39345

Table 3 shows us comparison between exact solution, approximate solutions for different N values and corrected Laguerre polynomial solutions with respect to the residual error analysis. Figure 3 shows us comparison between exact and approximate solutions for different N values. Figure 4 shows the comparison of absolute error function of Example 6.2. for the same N values.

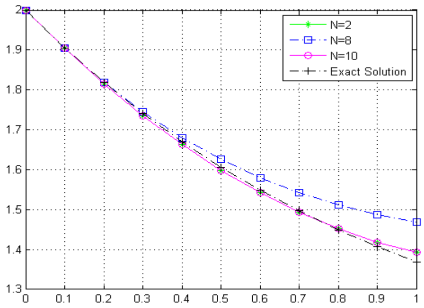

Fig.3. Numerical and exact solutions of Example 6.2 for $N = 2, 8\ 10$

Example 6.3 [31]

We consider the functional Volterra-integro differential equation

$$y''(x) - 3y'(x) + xy(x) + y(x-1) = -x^2 - 2x + 9 + \int_0^x y(t)dt,$$

$$0 \leq x \leq 1$$

under the mixed conditions $y(0) = 1$, $y'(0) = -2$ and we obtain the exact solution $y(x) = -2x$ of the problem for $N = 4$.

Example 6.4 [29]

Let us send the Laguerre series solution of the following fourth order linear functional Volterra-integro differential equation with variable coefficients

$y^{(4)}(x) - y(x) = x(1 + e^x) + 3e^x - \int_0^x y(t)dt\, , 0 \leq x \leq 1$

under the mixed conditions $y(0) = 1$, $y'(1) = 1 + e$, $y''(0) = 2$, $y'(1) = 3e$ with the exact solution $y(x) = 1 + xe^x$. We obtain the approximate solutions of the problem for $N = 4, 8, 14$.

Table 4. Numerical solution of Example 6.4 for different N

Present Method and Corrected Laguerre polynomial solution							
x	Exact Solution	y_4	$y_{4,7}$	y_8	$y_{8,10}$	y_{14}	$y_{14,20}$
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	1.110517	1.096623	1.124362	1.110517	1.110517	1.110517	1.110517
0.2	1.244280	1.217634	1.270833	1.244280	1.244280	1.244280	1.244280
0.3	1.404957	1.367789	1.441993	1.404957	1.404957	1.404957	1.404957
0.4	1.596729	1.552241	1.641056	1.596729	1.596729	1.596729	1.596729
0.5	1.824360	1.776546	1.871995	1.824360	1.824360	1.824360	1.824360
0.6	2.093271	2.046658	2.139702	2.093271	2.093271	2.093271	2.093271
0.7	2.409626	2.368933	2.450155	2.409626	2.409626	2.409626	2.409626
0.8	2.780432	2.750126	2.810611	2.780432	2.780432	2.780432	2.780432
0.9	3.213642	3.197390	3.229823	3.213642	3.213642	3.213642	3.213642
1.0	3.718281	3.718281	3.718282	3.718281	3.718281	3.718281	3.718281

Table 4 shows the comparison of the exact solution with approximate solutions for $N = 4, 8$ and 14 in Example 6.4. Absolute error comparisons are also given in Table 5. Figure 4 shows the comparison between approximate solutions for different N values.

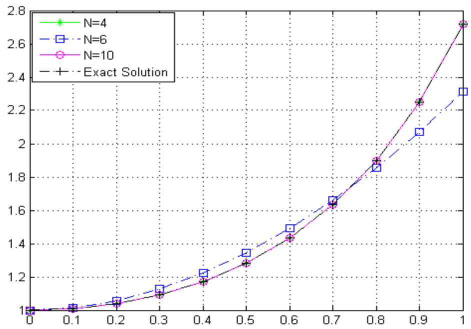

Fig.4. Numerical and exact solutions of Example 6.4 for $N = 4, 6, 10$.

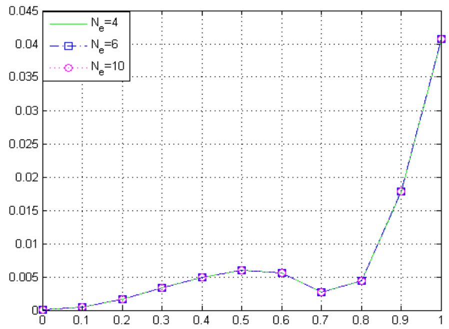

Fig.5. Error function $N_e = E_N$ of Example 6.4 for various N.

Table 5. Comparison of the actual and estimated absolute errors for Example 6.4.

	The actual absolute and estimated absolute errors					
x	E_4	$E_{4,7}$	E_8	$E_{8,10}$	E_{14}	$E_{14,20}$
0.0	0.00000000	0.13800E-06	0.10000E-07	0.40000E-10	0.20000E-08	0.20000E-08
0.1	0.13893E-02	0.13845E-02	0.42520E-05	0.39154E-06	0.59000E-07	0.59097E-07
0.2	0.26645E-02	0.26552E-02	0.82590E-05	0.75750E-06	0.10200E-06	0.10256E-06
0.3	0.37168E-02	0.37036E-02	0.11741E-04	0.10723E-05	0.12900E-06	0.12821E-06
0.4	0.44488E-02	0.44326E-02	0.14431E-04	0.13110E-05	0.13600E-06	0.13593E-06
0.5	0.47814E-02	0.47634E-02	0.16063E-04	0.14462E-05	0.12400E-06	0.12482E-06
0.6	0.46612E-02	0.46431E-02	0.16371E-04	0.14539E-05	0.95000E-07	0.94250E-07
0.7	0.40692E-02	0.40528E-02	0.15094E-04	0.13118E-05	0.42000E-07	0.42334E-07
0.8	0.30306E-02	0.30178E-02	0.12014E-04	0.10091E-05	0.33000E-07	0.31061E-07
0.9	0.16252E-02	0.16180E-02	0.70790E-05	0.55554E-06	0.12700E-06	0.12868E-06
1.0	0.50000E-08	0.24748E-06	0.72800E-06	0.56500E-09	0.24800E-06	0.24815E-06

Table 5 shows the corrected absolute errors by our method for $N = 4, 8, 14$ and $M = 7, 10, 20$. The results support the idea when N and M values are chosen large enough, the absolute error and residual error decrease.

Example 6.5 [30]

Let us find the Laguerre series solution of the following first order linear Volterra-integro differential equation with variable coefficients

$$y'(x) + y(x) = 1 + 2x + \int_0^x x(1+2x)e^{t(x-t)}y(t)dt,$$

$$0 \le x \le 1$$

with condition $y(0) = 1$ and exact solution $y(x) = e^{x^2}$. We obtain the approximate solution of Example 6.5 for $N = 4, 6, 10$ in Table 6.

Table 6. Numerical solution of Example 6.4 for different N

	Present Method and Corrected Laguerre polynomial solution						
x	Exact Solution	y_4	E_4	y_6	E_6	y_{10}	E_{10}
0.0	1.000000	0.999999	0.20000E-8	1.000000	0.90000E-8	1.000000	0.60000E-8
0.1	1.010050	1.014624	0.45745E-3	1.014686	0.46358E-3	1.010050	0.46365E-3
0.2	1.040810	1.057518	0.16707E-2	1.057640	0.16830E-2	1.040810	0.16830E-2
0.3	1.094174	1.127388	0.33214E-2	1.127501	0.33327E-2	1.094174	0.33328E-2
0.4	1.173510	1.223178	0.49667E-2	1.223241	0.49730E-2	1.173510	0.49731E-2
0.5	1.284025	1.344067	0.60042E-2	1.344102	0.60076E-2	1.284025	0.60077E-2
0.6	1.433329	1.489472	0.56143E-2	1.489543	0.56214E-2	1.433329	0.56214E-2
0.7	1.632316	1.659045	0.26728E-2	1.659191	0.26875E-2	1.632316	0.26875E-2
0.8	1.896480	1.852674	0.43806E-2	1.852800	0.43680E-2	1.896480	0.43679E-2
0.9	2.247907	2.070485	0.17742E-1	2.070218	0.17768E-1	2.247907	0.17768E-1
1.0	2.718281	2.312839	0.40544E-1	2.311364	0.40691E-1	2.718281	0.40693E-1

Table 6 shows the comparison between exact, approximate Laguerre polynomial solutions and error functions for the different N and M values. Figure 8 shows the comparison of absolute error functions for the same N values for Example 6.5.

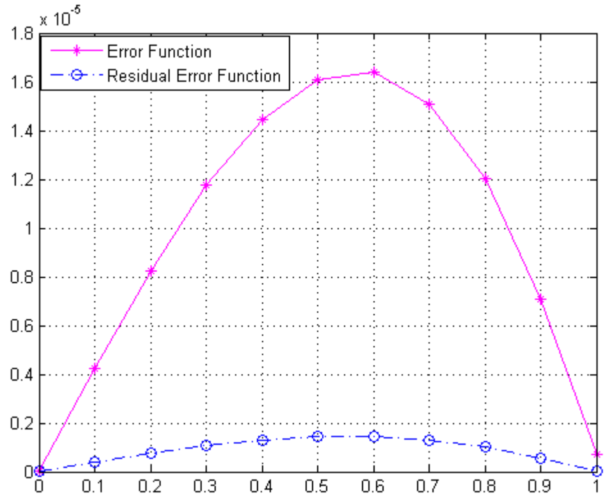

Fig.8. Comparison of absolute error and residual error functions for $y_{8,10}$ for Example 6.5.

Example 6.6 [30]

Consider the linear functional Volterra-integro differential equation

$$y^{(4)}(x)-y(x)=-\frac{2}{3}+\frac{x}{3}+\frac{11}{3}x^2-\frac{9}{3}x^3e^x+\frac{1}{3}\int_0^x te^{(x-t)}y(t)dt,$$

$$0\le x\le 1$$

with conditions $y(0)=1,\ \ y'(0)=1\ ,y''(0)=2\ ,y'''(0)=3$ with the exact solution $y(x)=1+xe^x$. We obtain the approximate solution of the problem for $N=5,8,9$ in Table 7.

Table 7. Numerical solution of Example 6.6 for different N.

		Present Method and Corrected Laguerre polynomial solution					
x	Exact Solution	y_5	$y_{5,7}$	y_8	$y_{8,10}$	y_9	$y_{9,10}$
0.0	1.000000	0.999999	1.000000	0.999999	1.000000	1.000000	1.000000
0.1	1.110517	1.110517	1.110517	1.110517	1.110517	1.110517	1.110517
0.2	1.244280	1.244282	1.244286	1.244281	1.244282	1.244281	1.244282
0.3	1.404957	1.404969	1.404993	1.404963	1.404975	1.404963	1.404974
0.4	1.596729	1.596771	1.596853	1.596753	1.596801	1.596754	1.596801
0.5	1.824360	1.824455	1.824646	1.824430	1.824570	1.824430	1.824570
0.6	2.093271	2.093430	2.093747	2.093435	2.093765	2.093436	2.093765
0.7	2.409626	2.409795	2.410132	2.409960	2.410629	2.409961	2.410629
0.8	2.780432	2.780407	2.780457	2.781038	2.782249	2.781039	2.782249
0.9	3.213642	3.212935	3.214342	3.214644	3.216647	3.214645	3.216647
1.0	3.718281	3.715924	3.720613	3.719813	3.722877	3.719814	3.722877

Table 7 shows the comparison between exact, approximate solutions and corrected Laguerre polynomial solutions for the different N and M values. Figure 8 shows the comparison of absolute error functions for the same N values.

Example 6.7 [29]

Consider the Volterra-Fredholm integral equation with functional arguments

$$x^2y(x)+e^xy\big(h(x)\big)=f(x)+\lambda_1\int_0^{h(x)} K_1(x,t)y(t)dt$$

$$+\lambda_2 \int_0^{h(x)} K_2(x,h(t))y(h(t))dt$$

where $K_1(x,t) = e^{(x+t)}, K_2(x,t) = e^{x-h(t)}$,

$h(x) = 2x,\ a = 0, b = 1$,

$f(x) = x^2\sin(x) + e^x\sin(2x) - \frac{1}{2}\lambda_1[e^{3x}(\sin(2x) + \cos(2x)) - e^x] - \quad -\frac{1}{4}\lambda_2[e^x - e^{x-2}(\sin(2) + \cos(2))]$.

We consider $(\lambda_1,\lambda_2) = (1,0)$ and exact solution $y(x) = \sin(x)$. We obtain the approximate solution of the problem for $N = 2,5,8,9$.

Table 8. Comparison of the errors for Example 6.7

N	Present Method	Taylor Collocation Method	Taylor Polynomial Method	Lagrange Collocation Method
2	1.86 E-02	3.77E-002	6.05E-002	3.77E-002
5	8.83E-05	2.18E-005	5.06E-005	2.18E-005
8	6.54E-09	1.89E-008	6.27E-007	1.61E-007
9	6.39E-08	3.16E-008	2.98E-008	1.04E-005

Table 8 shows the comparisons of the errors for present method, Taylor collocation method, Taylor polynomial method and Lagrange collocation method.

7. CONCLUSION

In recent years, the studies of functional integro-differential equations have attracted the attention of many mathematicians and physicists. [30-33].Laguerre collocation methods are used to solve the mentioned class of functional integro-differential equations numerically. It is observed that the method has the best advantage when the known functions in equation can be expanded to Laguerre series. Another advantage of the method is that the Laguerre polynomial coefficients of the solution are found very easily by using computer programs. Shorter computation time and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy. To get the best approximating solution of the equation, we have been improved the Laguerre collocation method with the aid of the residual error function for solving a class of functional integro-differential equations. As a result, the power of the employed method is confirmed. We assured the correctness of the obtained solutions by putting them back into the original equation with the aid of Maple 18. As a future work, the method can also be extended to the system of functional integro-differential equations with variable coefficients and their residual error analysis, but some modifications are required.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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