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A New Kind of Legendre Matrix Polynomials

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ABSTRACT

The main aim of this paper is to introduce a new kind of Legendre matrix polynomials. Hypergeometric matrix representation of these matrix polynomials is given. The convergence properties and the integral form for the Legendre matrix polynomials are derived. The Legendre matrix differential equation of second order is established. Subsequently, Rodrigues formula, orthogonality property, matrix recurrence relation and types of generating matrix functions are then developed for the Legendre matrix polynomials. Furthermore, general families of bilinear and bilateral generating matrix functions for these matrix polynomials are obtained and their applications are presented. Finally, the composite Legendre matrix polynomials is introduced.

Keywords: *Hypergeometric matrix function; Legendre matrix polynomials; Legendre matrix differential equations; Rodrigues formula; Orthogonality; Matrix recurrence relation; Generating matrix functions.*

1. INTRODUCTION

Orthogonal matrix polynomials comprise an emerging field of study, with important results in both theory and applications being still is contained to appear in the literature. Theory of classical orthogonal polynomials are extended to orthogonal matrix polynomials, see for example, [2, 4, 5, 18, 19, 20, 24, 28, 29, 30, 34, 36]. In [1, 6, 9, 11, 14, 35, 37], the authors introduced and studied Jacobi matrix polynomials. In [3, 13], the authors introduced the Chebyshev matrix polynomials of the first and second kind and gave some results with Chebyshev matrix polynomials. Legendre matrix polynomials have been introduced and studied in [33]. In the scalar case, Jacobi polynomials $P^{(\alpha,\beta)}(x)$ are a class of classical orthogonal polynomials. They are orthogonal with respect to the weight $(1-x)^{\alpha}(1+x)^{\beta}$ on the interval $[-1,1]$. For $\alpha=\beta=0$, we have the classical Legendre polynomials. As in the corresponding problem for scalar polynomials, the problem of development of polynomials of Legendre matrix polynomials requires some new results about the matrix operational calculus not available in the literature. The Legendre ordinary differential equation is frequently encountered in physics and other technical fields.

The present investigation is motivated essentially by several recent works [13, 14]. Our main aim here at presenting a new class of Legendre matrix polynomials and their interesting properties. The structure of the paper is organized as follows: In Section 2 the Legendre

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matrix polynomials is defined and the hypergeometric matrix representation is given. The convergence, radius of convergence, an integral form are given and their connections with Legendre matrix differential equation of the second order are established. Rodrigues formula is given then developed for the Legendre matrix polynomials in Section 3. orthogonality properties of Legendre matrix polynomials are established in Section 4. Matrix recurrence relation for the Legendre matrix polynomials is obtained in Section 5. Formulas of generating matrix functions for the Legendre matrix polynomials are obtained, bilinear and bilateral generating matrix functions are derived for the Legendre matrix polynomials and some applications of the results obtained are presented in Section 6. Finally, the composite Legendre matrix polynomials and the convergence properties are investigated in Section 7.

1.1 Preliminaries

Throughout this study, we concerned with the matrix polynomials

$$P_n(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$$

where the coefficients A_0, $A_1, \ldots, A_n$ are members of $\mathbb{C}^{N\times N}$, the space of real or complex matrices of common order N, and x is a real number. The matrix polynomials $P_n(x)$ is of degree n if the matrix A_n is not zero matrix. For orthogonal matrix polynomials, the leading coefficient A_n being nonsingular matrix is important [15].Throughout this paper, if A is a matrix in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. The matrices I and $\mathbf{O}$ will denote the identity matrix and the null matrix (zero matrix) in $\mathbb{C}^{N\times N}$, respectively. In this expression, $\Re(z)$ is the real part of the complex number z. The two-norm of A is denoted by $\| A\|_2$, and is defined by

$$\| A\|_2 = \sup_{x\neq 0}\frac{\| Ax\|_2}{\| x\|_2}$$

where $\| x\|_2 = (x^T x)^{\frac{1}{2}}$ denotes the usual Euclidean norm of a vector x in $\mathbb{C}^N$.

Fact 1.1 *[16] If $f(z)$ and $g(z)$ are holomorphic functions of complex variable z, which are defined in an open set Ω of the complex plane, and A, B are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$ and $\sigma(B)\subset\Omega$, such that $AB = BA$, then from the properties of the matrix functional calculus, it follows that*

$$f(A)g(B) = g(B)f(A).$$

Definition 1.1 *[21] A matrix A in $\mathbb{C}^{N\times N}$ is called a positive stable if*

$$\Re(\mu) > 0, \quad \forall\ \mu\in\sigma(A). \tag{1.1}$$

Fact 1.2 *[22, 23] Let us denote the real numbers $M(A)$, $m(A)$ for a matrix A in $\mathbb{C}^{N\times N}$ such that*

$$\left\| e^{tA}\right\| \le e^{t\,M(A)}\sum_{i=0}^{N-1}\frac{\left(\|A\| N^{\frac{1}{2}} t\right)^i}{i!}; \quad t\ge 0 \tag{1.2}$$

and considering that $k^A = e^{A\ln k}$, one gets

$$\left\| n^{A}\right\| \le n^{M(A)}\sum_{i=0}^{N-1}\frac{\left(\|A\| N^{\frac{1}{2}}\ln n\right)^i}{i!}; \quad n\ge 1 \tag{1.3}$$

where

$$M(A) = \max\{\Re(z) : z\in\sigma(A)\}; \quad m(A) = \min\{\Re(z): z\in\sigma(A)\}. \tag{1.4}$$

Definition 1.2 *[21] If P is a positive stable matrix in $\mathbb{C}^{N\times N}$, then the Gamma matrix function $\Gamma(P)$ is defined by*

$$\Gamma(P) = \int_0^\infty e^{-t}t^{P-I}dt; \quad t^{P-I} = \exp((P-I)\ln t). \tag{1.5}$$

Notation 1.1 *[22] The reciprocal Gamma function $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of complex variable z. Then for any matrix A in $\mathbb{C}^{N\times N}$, the image of $\Gamma^{-1}(z)$ acting on A is denoted by the well-defined matrix $\Gamma^{-1}(A)$. Furthermore, if*

$$A + nI \quad \text{is an invertible matrix for every an non-negative integer } n \tag{1.6}$$

then $\Gamma(A)$ is an invertible matrix in $\mathbb{C}^{N\times N}$, its inverse matrix coincide with $\Gamma^{-1}(A)$. The Pochhammer symbol (shifted factorial) is defined as

$$\begin{aligned}(A)_n &= A(A+I)(A+2I)\ldots(A+(n-1)I)\\ &= \Gamma(A+nI)\Gamma^{-1}(A);\quad n\geq 1;\quad (A)_0 = I.\end{aligned} \tag{1.7}$$

Notation 1.2 *[13] Using (1.7), one gets*

$$(-nI)_k = \frac{(-1)^k\ n!}{(n-k)!} I, 0\leq k\leq n. \tag{1.8}$$

Theorem 1.1 *[21] Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying $\Re(z)>0$, for every eigenvalue $z\in\sigma(A)$ and $n\geq 1$ be an integer, then*

$$\Gamma(A) = \lim_{n\to\infty}(n-1)![(A)_n]^{-1} n^A \tag{1.9}$$

where $(A)_n$ is defined by (1.7).

Definition 1.3 *[21] If P and Q are positive stable matrices in $\mathbb{C}^{N\times N}$, then the Beta matrix function $B(P,Q)$ is defined as*

$$B(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I}\,dt. \tag{1.10}$$

Definition 1.4 *[22, 23] If A, B, and C are matrices in $\mathbb{C}^{N\times N}$ for which $C+nI$ is an invertible matrix for every integer $n\geq 0$, then the hypergeometric matrix function ${}_2F_1(A,B;C;z)$ is defined as follows*

$${}_2F_1(A,B;C;z) = \sum_{k=0}^{\infty}\frac{z^k}{k!}(A)_k(B)_k[(C)_k]^{-1}. \tag{1.11}$$

Also the hypergeometric matrix differential equation is defined as in the form (see Theorem 4 [22] and eq. (38) [23])

$$z(1-z)\frac{d^2W(z)}{dz^2} - zA\frac{dW(z)}{dz} + (C - z(B+I))\frac{dW(z)}{dz} - ABW(z) = \mathbf{O}; 0<|z|<1. \tag{1.12}$$

Fact 1.3 *For any matrix A in $\mathbb{C}^{N\times N}$, the hypergeometric matrix function ${}_0F_1(-;A;z)$ is given as*

$${}_0F_1(-;A;z) = \sum_{k=0}^{\infty}\frac{z^k}{k!}[(A)_k]^{-1} \tag{1.13}$$

where $A+kI$ is an invertible matrix for all integers $k\geq 0$ (see [22]).

Fact 1.4 *[22] The relation*

$$(1-z)^{-A} = {}_1F_0(A;-;z) = \sum_{n=0}^{\infty}\frac{1}{n!}(A)_n z^n;\quad |z|<1 \tag{1.14}$$

is valid for $A\in\mathbb{C}^{N\times N}$.

Fact 1.5 *[22] If n is large enough so that $n>\|C\|$, then by the perturbation lemma [16], for C in $\mathbb{C}^{N\times N}$ we have the following inequality*

$$\left\|(C+nI)^{-1}\right\| \leq \frac{1}{n-\|C\|}. \tag{1.15}$$

Fact 1.6 *[17] Let A and B be matrices in $\mathbb{C}^{N\times N}$ and from (1.7), one can easily obtain the following inequalities*

$$\|AB\| \leq \|A\|\|B\|,\ \left\|A^n\right\| \leq \|A\|^n,\ \|(A)_n\| \leq (\|A\|)_n. \tag{1.16}$$

Theorem 1.2 *[13] Let B and C are matrices in $\mathbb{C}^{N\times N}$ such that $C+nI$ is an invertible matrix for all integers $n\geq 0$. Suppose that C and $C-B$ are positive stable matrices with $BC=CB$, the relation*

$$_2F_1\left(-nI, B; C; z\right) = (1-z)^n\ {_2F_1}\left(-nI, C-B; C; -\frac{z}{1-z}\right) \tag{1.17}$$

is valid for $|z|<1$ and $\left|\frac{z}{1-z}\right|<1.$

Lemma 1.1 *[21] Let P and Q be positive stable matrices in $\mathbb{C}^{N\times N}$ such that*

$$\Re(z) > -1 \quad \text{and} \quad \Re(w) > -1, \quad \text{forall} \quad z\in\sigma(P), \quad w\in\sigma(Q) \text{ and } PQ=QP. \tag{1.18}$$

By virtue of (1.10), one can obtain

$$\int_{-1}^{1}(1+x)^{P-I}(1-x)^{Q-I}\,dx = 2^{P+Q-I}B(P,Q). \tag{1.19}$$

Lemma 1.2 *[22] Let P, Q and $Q+P$ be positive stable matrices in $\mathbb{C}^{N\times N}$ satisfying $PQ=QP$ and $P+nI$, $Q+nI$ and $P+Q+nI$ are invertible matrices for all nonnegative integers n. Then we have*

$$B(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q). \tag{1.20}$$

At the end we recall that [12], if $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n\geq 0$, $k\geq 0$, then the following relations are satisfied

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n-k),\\
\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}A(k,n-k),\\
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}A(k,n-mk); m\in N.
\end{aligned} \tag{1.21}$$

Similarly, we can write

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k),\\
\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+2k),\\
\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+mk); m\in N.
\end{aligned} \tag{1.22}$$

For the purpose of this work, we recall the following definitions of the Jacobi and Legendre matrix polynomials.

Definition 1.5 *[14] Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition*

$$\Re(z) > -1, \quad \forall\, z \in \sigma(A) \quad \text{and} \quad \Re(w) > -1, \quad \forall\, w \in \sigma(B). \tag{1.23}$$

For $n \geq 0$, the Jacobi matrix polynomials $P_n^{A,B}(x)$ is defined by the hypergeometric matrix function

$$P_n^{A,B}(x) = \frac{(B+I)_n}{n!}\, {}_2F_1\left(A+B+(n+1)I, -nI; B+I; \frac{1-x}{2}\right). \tag{1.24}$$

Definition 1.6 *[31, 33] The Legendre matrix polynomials is defined by*

$$P_n(x,A) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2n-2k)!\left(x\sqrt{2A}\right)^{n-2k}}{2^{2n-2k}\, k!(n-k)!(n-2k)!}, \tag{1.25}$$

where A be a positive stable matrix in $\mathbb{C}^{N\times N}$.

2. LEGENDRE MATRIX POLYNOMIALS

Definition 2.1 *Let C be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition*

$$0 < \Re(\lambda) < 1, \quad \text{for all } \lambda \in \sigma(C). \tag{2.1}$$

The Legendre matrix polynomials $P_n(x,C)$ is defined in the form

$$P_n(x,C) = \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!(n-k)!}\left(\frac{1-x}{2}\right)^k \Gamma^{-1}(C+kI)\Gamma(C), n \geq 0 \tag{2.2}$$

such that $C+kI$ is an invertible matrix for all integers $k \geq 0$ and $\left|\frac{1-x}{2}\right| < 1$.

Theorem 2.1 *Let C be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (2.1), the hypergeometric matrix representations for the Legendre matrix polynomials is given as:*

$$\begin{aligned} P_n(x,C) &= W\left(\frac{1-x}{2}\right) = {}_2F_1\left(-nI, (n+1)I; C; \frac{1-x}{2}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-nI)_k ((n+1)I)_k [(C)_k]^{-1}}{k!}\left(\frac{1-x}{2}\right)^k \end{aligned} \tag{2.3}$$

where $(-nI)_k$ is given by (1.8).

Proof. This can be proved using (1.7), (1.8) and (2.2) and the above expressions.

We are going to study the convergence properties of Legendre matrix polynomials of degree n. For this aim, we denote

$$\alpha(k) = \left\|(C)^{-1}\right\|\left\|(C+I)^{-1}\right\|\ldots\left\|(C+(k-1)I)^{-1}\right\|; \quad k > 1. \tag{2.4}$$

From (1.15), (1.16) and (2.4) for $k > \|C\|$, we have

$$\left\|\frac{1}{k!}\left(\frac{1-x}{2}\right)^k (-nI)_k\,((1+n)I)_k\,[(C)_k]^{-1}\right\|$$
$$\leq \frac{1}{k!}\left|\frac{1-x}{2}\right|^k (\|-nI\|)_k\,(\|(1+n)I\|)_k\,\alpha(k).$$

Using the ratio test and the relation (1.9) for the series $\sum_{k=0}^{\infty}\frac{1}{k!}\left|\frac{1-x}{2}\right|^k (\|-nI\|)_k\,(\|(1+n)I\|)_k\,\alpha(k)$ as the following

$$\lim_{k\to\infty}\left|\frac{\|U_{k+1}\|}{\|U_k\|}\frac{\left(\frac{1-x}{2}\right)^{k+1}}{\left(\frac{1-x}{2}\right)^k}\right| = \lim_{k\to\infty}\left|\frac{(\|-nI\|)_{k+1}(\|(1+n)I\|)_{k+1}\alpha(k+1)k!}{(\|-nI\|)_k\,(\|(1+n)I\|)_k\,\alpha(k)(k+1)!}\frac{\left(\frac{1-x}{2}\right)^{k+1}}{\left(\frac{1-x}{2}\right)^k}\right|$$
$$\leq \lim_{k\to\infty}\frac{(\|-nI\|+k)(\|(1+n)I\|+k)}{(k-\|C\|)(k+1)}\left|\frac{1-x}{2}\right| = \left|\frac{1-x}{2}\right|$$

where

$$\|U_k\| \leq \frac{(\|-nI\|)_k\,(\|(1+n)I\|)_k\,\alpha(k)}{k!}.$$

It is easy to see that the power series (2.2) is absolutely convergent for $\left|\frac{1-x}{2}\right| < 1$, divergent for $\left|\frac{1-x}{2}\right| > 1$ and absolutely convergent for $\left|\frac{1-x}{2}\right| = 1$ under the condition $m(C) > M(-nI) + M((1+n)I)$, where $M(-nI)$, $M((1+n)I)$ and $m(C)$ are defined in (1.4).

The radius of convergence R is given from [26]

$$\frac{1}{R} = \limsup_{k\to\infty}\left(\|U_k\|\right)^{\frac{1}{k}} = \lim_{k\to\infty}\sup\left(\left\|\frac{(-nI)_k\,((1+n)I)_k\,[(C)_k]^{-1}}{k!}\right\|\right)^{\frac{1}{k}}$$
$$= \limsup_{k\to\infty}\left[\left\|\frac{k^{nI}(-nI)_k}{(k-1)!}(k-1)!k^{-nI}\,\frac{k^{-(1+n)I}((1+n)I)_n}{(k-1)!}(k-1)!k^{(1+n)I}\right.\right.$$
$$\left.\left.\frac{k^{-C}}{(k-1)!}(k-1)![(C)_k]^{-1}k^{C}\,\frac{1}{k!}\right\|\right]^{\frac{1}{k}}$$
$$= \limsup_{k\to\infty}\left[\left\|(\Gamma^{-1}(-nI)\Gamma^{-1}((1+n)I)\Gamma(C))k^{I-C}\,\frac{(k-1)!}{k!}\right\|\right]^{\frac{1}{k}} \leq \limsup_{k\to\infty}\left[\left\|k^{I-C}\,\frac{1}{k}\right\|\right]^{\frac{1}{k}} = 1$$

i.e. the radius of convergence of the Legendre series is one and it is regular in a circle of radius one.

To get an integral form for the Legendre matrix polynomials of degree n. Suppose that C is a matrix in $C^{N\times N}$, such that

$$((n+1)I)C = C((n+1)I) \tag{2.5}$$

and

$$(n+1)I, C \text{ and } C-(n+1)I \text{ are positive stable matrices.} \tag{2.6}$$

From (1.7) and (2.6) one gets the following

$$\begin{aligned}
&((n+1)I)_k[(C)_k]^{-1} \\
&= \Gamma((n+1)I+kI)\Gamma^{-1}((n+1)I)[\Gamma(C+kI)\Gamma^{-1}(C)]^{-1} \\
&= \Gamma^{-1}((n+1)I)\Gamma^{-1}(C-(n+1)I)\Gamma(C-(n+1)I)\Gamma((n+1+k)I)\Gamma^{-1}(C+kI)\Gamma(C).
\end{aligned} \tag{2.7}$$

Using Lemma 2 and the relation (2.13) of [21], we can get

$$\begin{aligned}
\int_0^1 t^{(n+1)I+(k-1)I}(1-t)^{C-(n+1)I-I}\,dt = \quad & B((n+1)I+kI, C-(n+1)I) \\
= \quad & \Gamma(C-(n+1)I)\Gamma((n+1+k)I)\Gamma^{-1}(C+kI).
\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we have

$$((n+1)I)_k[(C)_k]^{-1} = \Gamma^{-1}((n+1)I)\Gamma^{-1}(C-(n+1)I)\left(\int_0^1 t^{(n+k)I}(1-t)^{C-(n+1)I-I}\,dt\right)\Gamma(C)$$

where $\Re(c-n-1)>0$ for all $c-n-1\in\sigma(C-(n+1)I)$ and $\Re(n+1+k)>0$ for all $n+1+k\in\sigma((n+1+k)I)$.

Using the above expressions, we have the proof of the following theorem.

Theorem 2.2 *Legendre matrix polynomials has the following integral form*

$$\begin{aligned}
P_n(x,C) = \quad & \left(\int_0^1 {}_1F_0\left(-nI;-;\frac{(1-x)t}{2}\right)t^{(n+1)I-I}(1-t)^{C-(n+1)I-I}\,dt\right) \\
& \Gamma^{-1}((n+1)I)\Gamma^{-1}(C-(n+1)I)\Gamma(C).
\end{aligned} \tag{2.9}$$

Next, we get the matrix differential equation in a different way and prove the following theorem.

Theorem 2.3 *Let* C *be a matrix in* $\mathbb{C}^{N\times N}$ *satisfying the condition (2.1). For* $n\geq 0$*, the Legendre matrix polynomials* $Y(x)=P_n(x,C)$ *satisfy the matrix differential equation of second order as the following*

$$(1-x^2)Y''(x)+2\left[(1-x)I-C\right]Y'(x)+n(n+1)Y(x)=\mathbf{O};\quad 0<\left|\frac{x-1}{2}\right|<1, '=\frac{d}{dx}. \tag{2.10}$$

Proof. From (2.3), it follows

$$\begin{aligned}
W'\left(\frac{1-x}{2}\right) &= \frac{d}{dx}\left[{}_2F_1\left(-nI,(n+1)I;C;\frac{1-x}{2}\right)\right] = -2\frac{d}{dx}P_n(x,C), \\
W''\left(\frac{1-x}{2}\right) &= \frac{d^2}{dx^2}\left[{}_2F_1\left(-nI,(n+1)I;C;\frac{1-x}{2}\right)\right] = 4\frac{d^2}{dx^2}P_n(x,C).
\end{aligned} \tag{2.11}$$

The hypergeometric matrix function ${}_2F_1\left(-nI,(n+1)I;C;\frac{1-x}{2}\right)$ is a solution of (1.12) with $A=-nI$, $B=(n+1)I$, in $0<|z|<1$. For $z=\frac{x-1}{2}$, one can see that $P_n(x,C)$ satisfies (2.10) in $0<\left|\frac{1-x}{2}\right|<1$. Thus the proof is completed.

Postmultiplying (2.10) by $\left(\frac{1+x}{1-x}\right)^{I-C}$ and making up the relation, we have the following corollary.

Corollary 2.1 *For* $n \geq 0$ *, the Legendre matrix polynomials* $Y(x) = P_n(x,C)$ *is a solution of the matrix differential equation*

$$\frac{d}{dx}\left[Y'(x)(1-x)^2\left(\frac{1+x}{1-x}\right)^{I-C}\right] + n(n+1)Y(x)\left(\frac{1+x}{1-x}\right)^{I-C} = \mathbf{O};\ 0 < \left|\frac{1-x}{2}\right| < 1 \tag{2.12}$$

Remark 2.1 *From (2.10) it follows that* $P_n(x,C)$ *is a matrix polynomial in* x *of degree* n *precisely and* $P_n(1,C) = I$.

3. RODRIGUES FORMULA FOR THE LEGENDRE MATRIX POLYNOMIALS

Theorem 3.1 *Let* C *be a matrix in* $\mathbb{C}^{N\times N}$ *satisfying the condition (2.1) and let* $P_n(x,C)$ *be the Legendre matrix polynomials. Then the following formula holds for* $n \geq 0$ *and* $\left|\frac{1-x}{2}\right| < 1$

$$P_n(x,C) = \frac{(-1)^n[(C)_n]^{-1}}{2^n}\left(\frac{1+x}{1-x}\right)^{C-I} D^n\left[(1-x^2)^n\left(\frac{1-x}{1+x}\right)^{C-I}\right]. \tag{3.1}$$

Proof. Since for $\left|\frac{1-x}{2}\right| < 1$, $P_n(x,C) = {}_2F_1\left(-nI, (n+1)I; C; \frac{1-x}{2}\right)$ by theorem 2.1 one gets

$$P_n(x,C) = \left(\frac{x+1}{2}\right)^n {}_2F_1\left(-nI, C-(n+1)I; C; \frac{x-1}{x+1}\right). \tag{3.2}$$

From (1.7), we have

$$(C-(n+1)I)_k = (-1)^k (I-C)_{n+1}[(I-C)_{n+1-k}]^{-1}. \tag{3.3}$$

Equation (3.2) can be written as in the form

$$\begin{aligned} P_n(x,C) &= \sum_{k=0}^{n} \frac{(-nI)_k (C-(n+1)I)_k [(C)_k]^{-1}}{k!}\left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k} \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{2^n}\binom{n}{k}(C-(n+1)I)_k[(C)_k]^{-1}(x-1)^k(x+1)^{n-k} \\ &= \sum_{k=0}^{n} \frac{1}{2^n}\binom{n}{k}(C-(n+1)I)_k[(C)_k]^{-1}(1-x)^k(1+x)^{n-k} \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{2^n}\binom{n}{k}(I-C)_{n+1}[(I-C)_{n+1-k}]^{-1}[(C)_k]^{-1}(1-x)^k(1+x)^{n-k}. \end{aligned} \tag{3.4}$$

Consider the differential operator $D^k(f(x)) = \frac{d^k f(x)}{dx^k}$ for an arbitrary matrix C in $\mathbb{C}^{N\times N}$. From (1.7) and routine calculation [13], we have

$$D^s x^{C+mI} = (C+mI)(C+(m-1)I)\ldots(C+(m-s+1)I)x^{C+(m-s)I}, s = 0,1,2,\ldots.$$

or

$$D^s x^{C+mI} = (C+I)_m[(C+I)_{m-s}]^{-1} x^{C+(m-s)I}, \tag{3.5}$$

for non-negative integers s and m

From (3.5), we obtain

$$D^k (1+x)^{-C+(n+1)I} = (I-C)_{n+1}[(I-C)_{n+1-k}]^{-1}(1+x)^{-C+(n+1-k)I} \tag{3.6}$$

and

$$D^{n-k}(1-x)^{C+(n-1)I} = (-1)^{n-k}(C)_n[(C)_k]^{-1}(1-x)^{C+(k-1)I}. \tag{3.7}$$

From (3.4), (3.6) and (3.7) for $\left|\frac{1-x}{2}\right| < 1$ one gets

$$\begin{aligned} P_n(x,C) &= \sum_{k=0}^{n} \frac{(-1)^k}{2^n} \binom{n}{k} (I-C)_{n+1}[(I-C)_{n+1-k}]^{-1}[(C)_k]^{-1}(-1)^{n-k}(1+x)^{C-I}(1-x)^{-C+I} \\ & (I-C)_{n+1-k}(C)_k[(C)_n]^{-1}[(I-C)_{n+1}]^{-1} D^k(1+x)^{-C+(n+1)I} D^{n-k}(1-x)^{C+(n-1)I} \\ &= \sum_{k=0}^{n} \frac{(-1)^n}{2^n}\binom{n}{k}(1+x)^{C-I}(1-x)^{-C+I}[(C)_n]^{-1} D^k(1+x)^{-C+(n+1)I} D^{n-k}(1-x)^{C+(n-1)I} \\ &= \frac{(-1)^n[(C)_n]^{-1}}{2^n}(1+x)^{C-I}(1-x)^{-C+I}\sum_{k=0}^{n}\binom{n}{k} D^k(1+x)^{-C+(n+1)I} D^{n-k}(1-x)^{C+(n-1)I}. \end{aligned} \tag{3.8}$$

Using Leibnitz rule for the n-th derivative of a product, equation (3.8) yields the Rodrigues formula

$$P_n(x,C) = \frac{(-1)^n[(C)_n]^{-1}}{2^n}(1+x)^{C-I}(1-x)^{I-C} D^n\left((1-x^2)^n(1+x)^{I-C}(1-x)^{C-I}\right) \tag{3.9}$$

or equivalently

$$P_n(x,C) = \frac{[(C)_n]^{-1}}{2^n}(1+x)^{C-I}(1-x)^{I-C} D^n\left((x^2-1)^n(1+x)^{I-C}(1-x)^{C-I}\right). \tag{3.10}$$

Thus the formula (3.1) is established.

Remark 3.1 *The formula 3.1 is called Rodrigues formula for the Legendre matrix polynomials.*

4. ORTHOGONALITY FOR LEGENDRE MATRIX POLYNOMIALS

From Corollary 3.1, the Legendre matrix polynomials satisfied

$$\frac{d}{dx}\left[P_n'(x,C)(1-x)^2(1+x)^{I-C}(1-x)^{C-I}\right] + n(n+1)P_n(x,C)(1+x)^{I-C}(1-x)^{C-I} = \mathbf{O}. \tag{4.1}$$

Replaced n by m, we

$$\frac{d}{dx}\left[P_m'(x,C)(1-x)^2(1+x)^{I-C}(1-x)^{C-I}\right] + m(m+1)P_m(x,C)(1+x)^{I-C}(1-x)^{C-I} = \mathbf{O}. \tag{4.2}$$

Premultiplying (4.1) by $P_m(x,C)$ and (4.2) by $P_n(x,C)$ and subtracting we have

$$\begin{aligned} &\frac{d}{dx}\left[\left(P_m(x,C)P_n'(x,C) - P_n(x,C)P_m'(x,C)\right)(1-x)^2(1+x)^{I-C}(1-x)^{C-I}\right] \\ &+(n(n+1)-m(m+1))P_m(x,C)P_n(x,C)(1+x)^{I-C}(1-x)^{C-I} = \mathbf{O}. \end{aligned} \tag{4.3}$$

Integrating (4.3) between $x=-1$ and $x=1$, we obtain

$$(m(m+1)-n(n+1))\int_{-1}^{1}P_m(x,C)P_n(x,C)(1+x)^{I-C}(1-x)^{C-I}dx$$
$$=\lim_{x\to 1^-}\left[\left(P_m(x,C)P_n'(x,C)-P_n(x,C)P_m'(x,C)\right)(1-x)^2(1+x)^{I-C}(1-x)^{C-I}\right] \tag{4.4}$$
$$-\lim_{x\to -1^+}\left[\left(P_m(x,C)P_n'(x,C)-P_n(x,C)P_m'(x,C)\right)(1-x)^2(1+x)^{I-C}(1-x)^{C-I}\right].$$

Lemma 4.1 *For the matrix* C *in* $\mathbb{C}^{N\times N}$ *satisfies the condition (2.1), we have*

$$\lim_{x\to 1^-}(1+x)^{2I-C}=2^{2I-C},$$
$$\lim_{x\to -1^+}(1-x)^{C}=2^{C}. \tag{4.5}$$

Using (4.5), we have

$$\lim_{x\to 1^-}(1+x)^{2I-C}(1-x)^{C}(P_m(x,C)P_n'(x,C)-P_n(x,C)P_m'(x,C))=\mathbf{O},$$
$$\lim_{x\to -1^+}(1-x)^{C}(1+x)^{2I-C}(P_m(x,C)P_n'(x,C)-P_n(x,C)P_m'(x,C))=\mathbf{O}. \tag{4.6}$$

From (4.4) and (4.6) one gets the property of orthogonality

$$\int_{-1}^{1}P_m(x,C)P_n(x,C)(1+x)^{I-C}(1-x)^{C-I}dx=0, n\neq m. \tag{4.7}$$

That is, the Legendre matrix polynomials form an orthogonal set over $(-1,1)$ with respect to the weight matrix function $(1+x)^{I-C}(1-x)^{C-I}$.

In case of $m=n$, we put

$$g_n(C)=\int_{-1}^{1}(1+x)^{I-C}(1-x)^{C-I}[P_n(x,C)]^2dx \tag{4.8}$$

Using Rodrigues formula and integration by parts, we have a second derivation of the property of orthogonality (4.7).

From (3.1), we obtain

$$(1+x)^{I-C}(1-x)^{C-I}P_n(x,C) \quad =\frac{(-1)^n[(C)_n]^{-1}}{2^n}D^n((1-x^2)^n(1+x)^{I-C}(1-x)^{C-I}) \tag{4.9}$$

Therefore, we get

$$\int_{-1}^{1}(1+x)^{I-C}(1-x)^{C-I}P_n(x,C)P_m(x,C)dx=\frac{(-1)^n[(C)_n]^{-1}}{2^n}$$
$$\int_{-1}^{1}D^n((1+x)^{(n+1)I-C}(1-x)^{C+(n-1)I})P_m(x,C)dx. \tag{4.10}$$

Integrate n times by parts on the right-hand side by (4.10), we have

$$\int_{-1}^{1}(1+x)^{I-C}(1-x)^{C-I}P_n(x,C)P_m(x,C)dx$$
$$=\frac{(-1)^{2n}[(C)_n]^{-1}}{2^n}\int_{-1}^{1}(1+x)^{(n+1)I-C}(1-x)^{C+(n-1)I}D^nP_m(x,C)dx. \tag{4.11}$$

If $n\neq m$ without loads of generality, we obtain

$$\int_{-1}^{1}(1+x)^{I-C}(1-x)^{C-I}P_n(x,C)P_m(x,C)dx=\mathbf{O}, \quad n>m. \tag{4.12}$$

From (4.11), (4.8) and (2.10), we have

$$P_m(x,C) = {}_2F_1\left(-mI, (m+1)I; C; \frac{1-x}{2}\right)$$

The formula for the derivative of hypergeometric matrix functions gives

$$D^n P_n(x,C) = \frac{(-1)^n(-nI)_n((n+1)I)_n[(C)_n]^{-1}}{2^n}\ {}_2F_1\left(\mathbf{O}, (2n+1)I; C+nI; \frac{1-x}{2}\right)$$

from which we have

$$D^n P_n(x,C) = \frac{(2n)![(C)_n]^{-1}}{2^n} \tag{4.13}$$

or in the equivalent form

$$D^n P_n(x,C) = 2^n n!\left(\frac{1}{2}\right)_n [(C)_n]^{-1}. \tag{4.14}$$

Now (4.11) with $n = m$ yields

$$g_n(C) = \int_{-1}^{1}(1+x)^{I-C}(1-x)^{C-I}[P_n(x,C)]^2\,dz$$
$$= \frac{[(C)_n]^{-1}}{2^n}\frac{(2n)![(C)_n]^{-1}}{2^n}\int_{-1}^{1}(1-x^2)^n(1+x)^{(n+1)I-C}(1-x)^{C+(n-1)I}\,dx.$$

Using Lemma **1.1**, we get

$$\int_{-1}^{1}(1+x)^{(n+1)I-C}(1-x)^{C+(n-1)I}\,dx = 2^{2n+1}B(C+nI,(n+2)I-C)$$
$$= \frac{2^{2n+1}}{(2n+1)!}\Gamma(C+nI)\Gamma((n+2)I-C).$$

Hence, we have

$$g_n(C) = \frac{[(C)_n]^{-1}}{2^n}\frac{(2n)![(C)_n]^{-1}}{2^n}\frac{2^{2n+1}}{(2n+1)!}\Gamma(C+nI)\Gamma((n+2)I-C)$$

or

$$g_n(C) = \frac{2}{(2n+1)}[(C)_n]^{-1}\Gamma(C)\Gamma((n+2)I-C). \tag{4.15}$$

From (4.13) and (4.14), we conclude that

$$\begin{aligned} P_n(x,C) &= \frac{(2n)![(C)_n]^{-1}}{2^n n!}x^n + \Pi_{n-2}(x) \\ &= 2^n n!(\frac{1}{2})_n[(C)_n]^{-1}x^n + \Pi_{n-2}(x) \end{aligned} \tag{4.16}$$

in which $\Pi_{n-2}(z)$ is a matrix polynomial in x of degree $n-2$. Thus we have the proof of the following theorem.

Theorem 4.1 *Let C be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (2.1). Then for any nonnegative integers n and m, we have*

$$\int_{-1}^{1}(1+x)^{I-C}(1-x)^{C-I}P_n(x,C)P_m(x,C)dx=\begin{cases}\mathbf{O}, & n\neq m;\\ \dfrac{2}{(2n+1)}[(C)_n]^{-1}\Gamma(C)\Gamma((n+2)I-C), & n=m.\end{cases} \tag{4.17}$$

Remark 4.1 *The family* $\{P_n(x,C)\}_{n\geq 0}$ *is orthogonal in* $(-1,1)$ *with respect to the weight matrix function* $(1+x)^{I-C}(1-x)^{C-I}$.

Remark 4.2 *The leading matrix coefficient of* $P_n(x,C)$ *for* $n=0$ *is* I *and an invertible matrix* $\dfrac{(-1)^n(2n)!}{2^n n!}\Gamma^{-1}(C+nI)\Gamma(C)$ *if* $n\geq 1$.

Making use of the hypergeometric matrix representation (2.3) in the familiar property of orthogonality (4.17), and setting $x=1-2t,\quad 0<t<1$, we obtain

$$\int_0^1 t^{C-I}(1-t)^{I-C}\ {}_2F_1\left(-mI,(m+1)I;C;t\right)\ {}_2F_1\left(-nI,(n+1)I;C;t\right)dt$$
$$=\begin{cases}\mathbf{O}, & n\neq m;\\ \dfrac{1}{(2n+1)}[(C)_n]^{-1}\Gamma(C)\Gamma((n+2)I-C), & n=m.\end{cases} \tag{4.18}$$

In view of the hypergeometric matrix function representation (2.3) and the property of orthogonality (4.17) with $x=1-\dfrac{2}{t},\quad 1<t<\infty$, we have

$$\int_1^\infty t^{-2}(t-1)^{I-C}\ {}_2F_1\left(-mI,(m+1)I;C;\frac{1}{t}\right)\ {}_2F_1\left(-nI,(n+1)I;C;\frac{1}{t}\right)dt$$
$$=\begin{cases}\mathbf{O}, & n\neq m;\\ \dfrac{1}{(2n+1)}[(C)_n]^{-1}\Gamma(C)\Gamma((n+2)I-C), & n=m.\end{cases} \tag{4.19}$$

Thus, if we employ the hypergeometric representation (2.3) on the left-hand side of the property of orthogonality (4.17) and set $x=\dfrac{1-t}{1+t},\quad 0<t<\infty$, we get

$$\int_0^\infty (1+t)^{-2}t^{C-I}\ {}_2F_1\left(-mI,(m+1)I;C;\frac{t}{t+1}\right)\ {}_2F_1\left(-nI,(n+1)I;C;\frac{t}{t+1}\right)dt$$
$$=\begin{cases}\mathbf{O}, & n\neq m;\\ \dfrac{1}{(2n+1)}[(C)_n]^{-1}\Gamma(C)\Gamma((n+2)I-C), & n=m.\end{cases} \tag{4.20}$$

5. MATRIX RECURRENCE RELATION

For this purpose, let $F(x)$ be a matrix polynomial of degree n as [15]

$$F(x)=\sum_{k=0}^{n}A_kP_k(x,C);A_k\in\mathbb{C}^{N\times N}. \tag{5.1}$$

Using Theorem 4.1 and (5.1), then

$$\int_{-1}^{1} F(x)(1+x)^{I-C}(1-x)^{C-I} P_n(x,C)dx = \mathbf{O}. \tag{5.2}$$

From (5.1), we get

$$(1-x)P_n(x,C) = \sum_{k=0}^{n+1} A_k P_k(x,C) \tag{5.3}$$

Using Theorem 4.1, we have

$$\int_{-1}^{1}(1-x)P_n(x,C)(1+x)^{I-C}(1-x)^{C-I} P_k(x,C)dx = A_k H_k; k+1<n \tag{5.4}$$

where

$$H_k = \begin{cases} 2\Gamma(C)\Gamma(2I-C), & k=0; \\ \frac{2}{2k+1}[(C)_k]^{-1}\Gamma(C)\Gamma((k+2)I-C), & k>0. \end{cases}$$

Since H_k is an invertible matrix for $k \geq 0$, by (5.4) one gets $A_k = \mathbf{O}$ for $k < n-1$ and by (5.3), we have

$$(1-x)P_n(x,C) = A_{n-1}P_{n-1}(x,C) + A_n P_n(x,C) + A_{n+1}P_{n+1}(x,C). \tag{5.5}$$

From (5.3), one gets

$$A_k = \left[\int_{-1}^{1}(1-x)P_k(x,C)(1+x)^{I-C}(1-x)^{C-I} P_n(x,C)dx\right] H_k^{-1}.$$

Identifying the matrix coefficients of each power of $(1+x)$ on both sides of (5.5) one gets a more explicit three-terms matrix recurrence relation for the Legendre matrix polynomials as the following

$$(1-x)P_n(x,C) = A_{n-1}P_{n-1}(x,C) + A_n P_n(x,C) + A_{n+1}P_{n+1}(x,C). \tag{5.6}$$

Identifying equal powers of $(1-x)$ in (5.6), leads to explicit expressions for the recurrence coefficient matrices A_{n-l}, A_n, and A_{n+l} as in the form

$$A_{n+1} = \frac{n+1}{2n+1}I, \quad A_n = I, \quad A_{n-1} = \frac{1}{2n+1}((n+1)I - C). \tag{5.7}$$

Thus we have the following theorem.

Theorem 5.1 *Let C be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (2.1). Then the Legendre matrix polynomials satisfies the three-term matrix recurrence relation*

$$A_{n+1}P_{n+1}(x,C) = \left[(1+x)I - A_n\right]P_n(x,C) - A_{n-1}P_{n-1}(x,C); n \geq 1, P_{-1}(x,C) = 0, P_0(x,C) = I. \tag{5.8}$$

6.GENERATING MATRIX FUNCTIONS FOR THE LEGENDRE MATRIX POLYNOMIALS

In this section, we present some types of generating matrix functions, new matrix polynomials, matrix differential equation, bilinear and bilateral generating matrix functions for the Legendre matrix polynomials. We first state our results as the following.

Theorem 6.1 *Let C be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $0 < \Re(\lambda) < 1$ for all $\lambda \in \sigma(C)$. Then*

$$\sum_{n=0}^{\infty} P_n(x,C)t^n = (1-t)^{-1} {}_1F_1\left(\frac{1}{2}I; C; \frac{2t(x-1)}{(1-t)^2}\right); |t|<1, \left|\frac{2t(x-1)}{(1-t)^2}\right| < 1. \tag{6.1}$$

Proof. From (2.2) and (1.22), we have

$$\sum_{n=0}^{\infty} P_n(x,C)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)!}{k!n!}\left(\frac{1-x}{2}\right)^k t^{n+k}[(C)_k]^{-1}$$
$$== \sum_{k=0}^{\infty}\left(\frac{1}{2}I\right)_k 2^k (x-1)^k (1-t)^{-(2k+1)} t^k [(C)_k]^{-1} = (1-t)^{-1}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)_k 2^k \left(\frac{x-1}{(1-t)^2}\right)^k t^k [(C)_k]^{-1}.$$

Thus the proof is completed.

Theorem 6.2 *Let* C *be a matrix in* $\mathbb{C}^{N\times N}$ *satisfying the condition* $0 < \Re(\lambda) < 1$ *for all* $\lambda \in \sigma(C)$. *Then the generating matrix function for the Legendre matrix polynomials is given as*

$$\sum_{n=0}^{\infty} \frac{1}{n!}[(2I-C)_n]^{-1} P_n(x,C)t^n = {}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right);$$
$$\left|\frac{1}{2}t(x-1)\right| < 1, \left|\frac{1}{2}t(x+1)\right| < 1. \tag{6.2}$$

Proof. From (3.2), (3.3) and (1.22), we get

$$\sum_{n=0}^{\infty} \frac{1}{n!}[(2I-C)_n]^{-1} P_n(x,C)t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{2^n n!}[(2I-C)_n]^{-1} \frac{(-nI)_k (I-C)_{n+1}[(I-C)_{n+1-k}]^{-1}[(C)_k]^{-1}}{k!}(1-x)^k (1+x)^{n-k}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k}{2^n k!(n-k)!}[(2I-C)_n]^{-1}(I-C)_{n+1}[(I-C)]^{-1}[(2I-C)_{n-k}]^{-1}[(C)_k]^{-1}(1-x)^k (1+x)^{n-k}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k}{2^n k!(n-k)!}[(2I-C)_{n-k}]^{-1}[(C)_k]^{-1}(1-x)^k (1+x)^{n-k} t^n$$
$$= \left[\sum_{n=0}^{\infty} \frac{1}{2^n n!}[(2I-C)_n]^{-1}(1+x)^n t^n\right]\left[\sum_{k=0}^{\infty} \frac{1}{2^k k!}[(C)_k]^{-1}(x-1)^k t^k\right].$$

Thus, the proof is completed.

Theorem 6.3 *Let* C *be a matrix in* $\mathbb{C}^{N\times N}$ *satisfying the condition* $0 < \Re(\lambda) < 1$ *for all* $\lambda \in \sigma(C)$. *Then the derived generating matrix function for new matrix polynomials* $\Phi_n(x,C)$ *is given as*

$${}_0F_1(-;2I-C;xt)\ {}_0F_1(-;C;-t) = \sum_{n=0}^{\infty} \Phi_n(x,C)t^n. \tag{6.3}$$

Proof. Let $\Phi_n(x,C) = \frac{1}{n!}[(2I-C)_n]^{-1}(x+1)^n P_n\left(\frac{x-1}{x+1},C\right)$ and from (3.2), we obtain (6.3). Thus, we have the proof of the theorem.

Consider the differential operator $\theta = z\frac{d}{dz}$ to the matrix function $Y = {}_0F_1(-;2I-C;z)$ as the following

$$\theta(\theta I+I-C)\ {}_0F_1(-;2I-C;z)=\sum_{k=1}^{\infty}\frac{(kI+I-C)}{(k-1)!}z^k[(2I-C)_k]^{-1}$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}z^{k+1}[(2I-C)_k]^{-1}=z\ {}_0F_1(-;2I-C;z).$$

The matrix function ${}_0F_1$ is a solution of the matrix differential equation

$$[\theta(\theta I+I-C)-z\,I]\ {}_0F_1(-;2I-C;z)=0.$$

Put $z=xt$ and $\theta=x\frac{d}{dx}$ we have

$$[\theta(\theta I+I-C)-xt\,I]\ {}_0F_1(-;2I-C;-xt)=0$$

or in the equivalent form

$$\theta(\theta I+I-C)\ {}_0F_1(-;2I-C;-xt)=xt\ {}_0F_1(-;2I-C;xt).$$

Next we operate with $\theta(\theta I+I-C)\ {}_0F_1(-;2I-C;-xt)$ on both members of (6.3):

$$\theta(\theta I+I-C)\ {}_0F_1(-;2I-C;xt)\ {}_0F_1(-;C;-t)=xt\ {}_0F_1(-;2I-C;xt)\ {}_0F_1(-;C;-t)$$

and

$$\theta(\theta I+I-C)\sum_{n=0}^{\infty}\Phi_n(x,C)t^n=x\sum_{n=0}^{\infty}\Phi_n(x,C)t^{n+1}=x\sum_{n=1}^{\infty}\Phi_{n-1}(x,C)t^n.$$

Therefore $\theta(\theta I+I-C)\Phi_0(x,C)=0$ and

$$\theta(\theta I+I-C)\Phi_n(x,C)-x\Phi_{n-1}(x,C)=0;n\geq 1.$$

These results are summarized in the following theorem.

Theorem 6.4 *The new matrix polynomials $\Phi_n(x,C)$ satisfy the following matrix differential equation*

$$\theta(\theta I+I-C)\Phi_n(x,C)-x\Phi_{n-1}(x,C)=0;n\geq 1. \tag{6.4}$$

This last major properties completes developed here are several families of bilinear and bilateral generating matrix functions for the Legendre matrix polynomials derived from generating matrix functions (6.2), then using Theorem 6.2 and given explicitly by (2.2) without using Lie algebraic techniques but, with the help of the similar method as considered in [1]. We state our results as the following.

Theorem 6.5 *Corresponding to a non-vanishing matrix function $\Omega_\mu(y_1,y_2,\ldots,y_s)$ consisting of s complex variables $y_1,y_2,\ldots,y_s$, $s\in N$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1,y_2,\ldots,y_s;z)=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,y_2,\ldots,y_s)z^k;a_k\neq 0,\mu,\nu\in\mathbb{C} \quad (6.5)$$

and

$$\Psi_{n,m,\mu,\nu}(x;y_1,y_2,\ldots,y_s;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k\frac{1}{(n-mk)!}[(2I-C)_{n-mk}]^{-1}P_{n-mk}(x,C)\Omega_{\mu+\nu k}(y_1,y_2,\ldots,y_s)\eta^k;n,m\in N$$

(6.6)

where C is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $0<\Re(\lambda)<1$ for all $\lambda\in\sigma(C)$ and (as usual) $[\alpha]$ represents the greatest integer in $\alpha\in R$. Then we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y_1,y_2,\ldots,y_s;\frac{\eta}{t^m}\right)t^n = {}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y_1,y_2,\ldots,y_s;\eta). \tag{6.7}$$

Proof. For convenience, let S denote the first member of the assertion (6.7). Then, plugging the matrix polynomials $\Psi_{n,m,\mu,\nu}\left(x;y_1,y_2,\ldots,y_s;\frac{\eta}{t^m}\right)$.

From (6.5) and (6.7), we obtain

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y_1,y_2,\ldots,y_s;\frac{\eta}{t^m}\right)t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}a_k\frac{1}{(n-mk)!}[(2I-C)_{n-mk}]^{-1}P_{n-mk}(x,C)\Omega_{\mu+\nu k}(y_1,y_2,\ldots,y_s)\eta^k t^{n-mk}. \end{aligned} \tag{6.8}$$

Using (6.8) and (1.22), we can write

$$\begin{aligned} &\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y_1,y_2,\ldots,y_s;\frac{\eta}{t^m}\right)t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\frac{1}{n!}[(2I-C)_n]^{-1}P_n(x,C)\Omega_{\mu+\nu k}(y_1,y_2,\ldots,y_s)\eta^k t^n \\ &= \left[\sum_{n=0}^{\infty}\frac{1}{n!}[(2I-C)_n]^{-1}P_n(x,C)t^n\right]\left[\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,y_2,\ldots,y_s)\eta^k\right] \\ &= {}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y_1,y_2,\ldots,y_s;\eta) \end{aligned}$$

which completes the proof of the theorem.

Expressing the multivariable matrix function $\Omega_{\mu+\nu k}(y_1,y_2,\ldots,y_s)$, $k\in N_0$ and $s\in N$ in terms of simpler matrix function of one and more variables, we can give further applications to Theorem 6.5. For example, if we set $s=1$ and $\Omega_{\mu+\nu k}(y)=L_{\mu+\nu k}^{(A,\lambda)}(y)$ in Theorem 6.5, where the Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ defined as [8, 10]

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n}\frac{(-1)^k(A+I)_n[(A+I)_k]^{-1}\lambda^k x^k}{k!(n-k)!}, n\geq 0 \tag{6.9}$$

and

$$(1-t)^{-(A+I)}\exp\left(\frac{-\lambda xt}{1-t}\right)=\sum_{n=0}^{\infty}L_n^{(A,\lambda)}(x)t^n,(x,t)\in R,|t|<1 \tag{6.10}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $-k\notin\sigma(A)$ for every integer $k>0$ and λ is a complex parameter with $\Re(\lambda)>0$.

In the following, we obtain the result which provides a class of bilateral generating matrix functions for the Laguerre matrix polynomials and modified Laguerre matrix polynomials.

Corollary 6.1 *Let* $\Lambda_{\mu,\nu}(y;z)=\sum_{k=0}^{\infty}a_k L_{\mu+\nu k}^{(A,\lambda)}(y)z^k; a_k\neq 0, \mu,\nu\in N_0$ *and*

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k\frac{1}{(n-mk)!}[(2I-C)_{n-mk}]^{-1}P_{n-mk}(x,C)L_{\mu+\nu k}^{(A,\lambda)}(y)\eta^k; n,m\in N$$

where C is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $0<\Re(\lambda)<1$ for all $\lambda\in\sigma(C)$. Then we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n={}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y;\eta) \quad (6.11)$$

provided that each member of (6.11) exists.

Remark 6.1 *Using the generating matrix function (6.10) for the Laguerre matrix polynomials and taking* $a_k=1$, $\mu=0$ *and* $\nu=1$, *we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}\frac{1}{(n-mk)!}[(2I-C)_{n-mk}]^{-1}P_{n-mk}(x,C)L_k^{(A,\lambda)}(y)\eta^k t^{n-mk}$$
$$={}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right)(1-\eta)^{-(A+I)}\exp\left(\frac{-\lambda y\eta}{1-\eta}\right)$$

Let us consider the modified Laguerre matrix polynomials

$$f_n^{(A,\lambda)}(x)=\sum_{k=0}^{n}\frac{(A)_k(\lambda x)^{n-k}}{k!(n-k)!}=\sum_{k=0}^{n}\frac{(A)_{n-k}(\lambda x)^k}{k!(n-k)!} \quad (6.12)$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $-k\notin\sigma(A)$ for every integer $k>0$ and λ is a complex parameter with $\Re(\lambda)>0$. Here the modified Laguerre matrix polynomials $f_n^{(A,\lambda)}(x)$ are generated as [10, 32]

$$(1-t)^{-A}e^{\lambda xt}=\sum_{n=0}^{\infty}f_n^{(A,\lambda)}(x)t^n;(x,t)\in R,|t|<1. \quad (6.13)$$

Corollary 6.2 *Let* $\Lambda_{\mu,\nu}(y;z)=\sum_{k=0}^{\infty}a_k f_{\mu+\nu k}^{(A,\lambda)}(y)z^k; a_k\neq 0, \mu,\nu\in N_0$ *and*

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k\frac{1}{(n-mk)!}[(2I-C)_{n-mk}]^{-1}P_{n-mk}(x,C)f_{\mu+\nu k}^{(A,\lambda)}(y)\eta^k; n,m\in N.$$

Then

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n={}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y;\eta) \quad (6.14)$$

if each member of (6.14) exists.

Remark 6.2 *Using the generating matrix function (6.13) for the modified Laguerre matrix polynomials and taking* $a_k=1$, $\mu=0$ *and* $\nu=1$, *we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}\frac{1}{(n-mk)!}[(2I-C)_{n-mk}]^{-1}P_{n-mk}(x,C)f_k^{(A,\lambda)}(y)\eta^k t^{n-mk}$$
$$={}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right)(1-\eta)^{-A}e^{\lambda y\eta}.$$

Corollary 6.3 *Let* $\Lambda_{\mu,\nu}(y;z)=\sum_{k=0}^{\infty}a_k\frac{1}{k!}[(2I-B)_k]^{-1}P_{\mu+\nu k}(y,B)z^k;a_k\neq 0,\mu,\nu\in N_0$ *and*

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k\frac{1}{k!(n-mk)!}[(2I-C)_{n-mk}]^{-1}[(2I-B)_k]^{-1}P_{n-mk}(x,C)P_{\mu+\nu k}(y,B)\eta^k;n,m\in N$$

where B is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $0<\Re(\lambda)<1$ for all $\lambda\in\sigma(B)$. Then

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n={}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y;\eta). \quad (6.15)$$

if each member of (6.15) exists.

Remark 6.3 *Using the generating matrix function (6.2) for the Legendre matrix polynomials and taking* $a_k=1$, $\mu=0$ *and* $\nu=1$*, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}\frac{1}{k!(n-mk)!}[(2I-C)_{n-mk}]^{-1}[(2I-B)_k]^{-1}P_{n-mk}(x,C)P_k(y,B)\eta^k t^{n-mk}$$
$$={}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right)$$
$${}_0F_1\left(-;2I-B;\frac{1}{2}\eta(y-1)\right){}_0F_1\left(-;B;\frac{1}{2}\eta(y+1)\right).$$

Corollary 6.4 *Let* $\Lambda_{\mu,\nu}(y;z)=\sum_{k=0}^{\infty}a_k\Phi_{\mu+\nu k}(y,B)z^k;a_k\neq 0,\mu,\nu\in N_0$ *and*

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k\frac{1}{(n-mk)!}[(2I-C)_{n-mk}]^{-1}P_{n-mk}(x,C)\Phi_{\mu+\nu k}(y,B)\eta^k;n,m\in N$$

where B is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $0<\Re(\lambda)<1$ for all $\lambda\in\sigma(B)$. Then

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n={}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y;\eta) \quad (6.16)$$

if each member of (6.16) exists.

Remark 6.4 *Using the generating matrix function (6.3) for* $\Phi_k(y,B)$ *and taking* $a_k=1$, $\mu=0$ *and* $\nu=1$*, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}\frac{1}{(n-mk)!}[(2I-C)_{n-mk}]^{-1}P_{n-mk}(x,C)\Phi_k(y,B)\eta^k t^{n-mk}$$
$$=\left[\sum_{n=0}^{\infty}\frac{1}{n!}[(2I-C)_n]^{-1}P_n(x,C)t^n\right]\left[\sum_{k=0}^{\infty}\Phi_k(y,B)\eta^k\right]$$
$$={}_0F_1\left(-;2I-C;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;C;\frac{1}{2}t(x+1)\right){}_0F_1(-;2I-B;y\eta)\,{}_0F_1(-;B;-\eta).$$

7. COMPOSITE LEGENDRE MATRIX POLYNOMIALS

Let us introduce in the following notations [27]

$$\underline{k}=(k_1,k_2,\ldots,k_i),\qquad (\underline{k})=k_1+k_2+\ldots+k_i,$$
$$(\underline{k})!=k_1!k_2!\ldots k_i!,\qquad \underline{C}=(C_1,C_2,\ldots,C_i),$$
$$(\underline{C})_{\underline{k}}=(C_1)_{k_1}(C_2)_{k_2}\ldots(C_i)_{k_i}$$
$$\left(\frac{1-x}{\underline{2}}\right)^{\underline{k}}=\left(\frac{1-x_1}{2}\right)^{k_1}\left(\frac{1-x_2}{2}\right)^{k_2}\ldots\left(\frac{1-x_i}{2}\right)^{k_i}$$

and $\underline{P}=(P_1,P_2,\ldots,P_i)$.

Suppose that

$$P_l(x_l,C_l)=\sum_{k_l\geq 0}\frac{1}{(k_l)!}(-n_lI)_{k_l}((n_l+1)I)_{k_l}[(C_l)_{k_l}]^{-1}\left(\frac{1-x_l}{2}\right)^{k_l},\ l=1,2,\ldots,i, \tag{7.1}$$

are i Legendre matrix polynomials with square complex matrices $C_1,\ C_2\ ,\ldots,C_i$ in $\mathbb{C}^{N\times N}$ such that the condition (2.1) is valid.

Consider the Legendre matrix polynomials $\underline{P}_{\underline{n}}(\underline{x},\underline{C})$, which is defined as

$$\underline{P}_{\underline{n}}(\underline{x},\underline{C})={}_2F_1\left(-\underline{n}I,\underline{(n+1)}I;\underline{C};\frac{1-x}{\underline{2}}\right)=\sum_{\underline{k}\geq 0}\frac{1}{(\underline{k})!}(-\underline{n}I)_{\underline{k}}(\underline{(1+n)}I)_{\underline{k}}[(\underline{C})_{\underline{k}}]^{-1}\underline{\left(\frac{1-x}{2}\right)}^{\underline{k}}. \tag{7.2}$$

We call this function the composite Legendre matrix polynomials of several complex variables $z_1,z_2,\ldots,z_i$.

For calculating the radius of convergence R of the series (7.2). We recall the relation (1.3.10) of [26] and keeping in mind that $\sigma_{\underline{k}}\geq 1$. Hence

$$
\begin{aligned}
\frac{1}{R} &= \limsup_{(\underline{k})\to\infty}\left(\frac{\left\|U_{\underline{k}}\right\|}{\sigma_{\underline{k}}}\right)^{\frac{1}{(\underline{k})}} = \limsup_{(\underline{n})\to\infty}\left(\frac{\left\|(-\underline{n}I)_{\underline{k}}((\underline{1+n})I)_{\underline{k}}[(\underline{C})_{\underline{k}}]^{-1}\right\|}{(\underline{k})!}\right)^{\frac{1}{(\underline{k})}}\left(\frac{1}{\sigma_{\underline{k}}}\right)^{\frac{1}{(\underline{k})}}\\
&\le \limsup_{(\underline{k})\to\infty}\left(\frac{\left\|(-n_1I)_{k_1}\dots(-n_iI)_{k_i}((n_1+1)I)_{k_1}\dots((n_i+1)I)_{k_i}\right\|}{k_1!k_2!\dots k_i!}\right)^{\frac{1}{(\underline{k})}}\\
&\quad \left(\left\|[(C_1)_{k_1}]^{-1}\dots[(C_i)_{k_i}]^{-1}\right\|\right)^{\frac{1}{(\underline{k})}}\\
&\le \limsup_{(\underline{k})\to\infty}\left(\left\|\left(\frac{k_1^{n_1I}(-n_1I)_{k_1}}{(k_1-1)!}\right)(k_1-1)!k_1^{-n_1I}\dots\left(\frac{k_i^{-n_iI}(-n_iI)_{k_i}}{(k_i-1)!}\right)(k_i-1)!k_i^{-n_iI}\right.\right.\\
&\quad \left(\frac{k_1^{-(n_1+1)I}((n_1+1)I)_{k_1}}{(k_1-1)!}\right)(k_1-1)!k_1^{(n_1+1)I}\dots\left(\frac{k_i^{-(n_i+1)I}((n_i+1)I)_{k_i}}{(k_i-1)!}\right)(k_i-1)!k_i^{(n_i+1)I}\\
&\quad \left.\left.\frac{k_1^{C_1}}{(k_1-1)!}(k_1-1)![(C_1)_{k_1}]^{-1}k_1^{-C_1}\dots\frac{k_i^{C_i}}{(k_i-1)!}(k_i-1)![(C_i)_{k_i}]^{-1}k_i^{-C_i}\right\|\frac{1}{k_1!k_2!\dots k_i!}\right)^{\frac{1}{(\underline{k})}}\\
&\le \limsup_{(\underline{k})\to\infty}\left(\left\|\Gamma^{-1}(-n_1I)\right\|\dots\left\|\Gamma^{-1}(-n_iI)\right\|\left\|\Gamma^{-1}((n_1+1)I)\right\|\dots\left\|\Gamma^{-1}((n_i+1)I)\right\|\right)\\
&\quad \left(\left\|\Gamma(C_1)\right\|\dots\left\|\Gamma(C_i)\right\|\right)^{\frac{1}{(\underline{k})}}\\
&\quad \limsup_{(\underline{k})\to\infty}\left(\frac{\left\|k_1^{-n_1I}\right\|\dots\left\|k_i^{-n_iI}\right\|\left\|k_1^{(n_1+1)I}\right\|\dots\left\|k_i^{(n_i+1)I}\right\|\left\|k_1^{-C_1}\right\|\dots\left\|k_i^{-C_i}\right\|}{k_1k_2\dots k_i}\right)^{\frac{1}{(\underline{k})}}
\end{aligned}
\tag{7.3}
$$

where

$$
\sigma_{\underline{k}} = \begin{cases}\left(\dfrac{k_1+\dots+k_i}{k_1}\right)^{\frac{k_1}{2}}\left(\dfrac{k_1+\dots+k_i}{k_2}\right)^{\frac{k_2}{2}}\dots\left(\dfrac{k_1+\dots+k_i}{k_i}\right)^{\frac{k_i}{2}} & ,\underline{k}\neq 0\\ 1 & ,\underline{k}=0.\end{cases}
$$

If k, k_l and $\mu_l\ k$ are positive numbers, we can write

$$k_l = \mu_l k,\ l=1,2,\dots,i.$$

Substitute from (1.3), (1.4) into (7.3) one gets

$$\frac{1}{R} \leq \limsup_{k(\mu_1+\ldots+\mu_i)\to\infty} \left\{ (\mu_1 k)^{M(-n_1 I)} \sum_{j=0}^{N-1} \frac{\left(\|-n_1 I\| N^{\frac{1}{2}} \ln \mu_1 k\right)^j}{j!} \ldots (\mu_i k)^{M(-n_i I)} \sum_{j=0}^{N-1} \frac{\left(\|-n_i I\| N^{\frac{1}{2}} \ln \mu_i k\right)^j}{j!} \right.$$

$$(\mu_1 k)^{M((n_1+1)I)} \sum_{j=0}^{N-1} \frac{\left(\|(n_1+1)I\| N^{\frac{1}{2}} \ln \mu_1 k\right)^j}{j!} \ldots (\mu_i k)^{M((n_i+1)I)} \sum_{j=0}^{N-1} \frac{\left(\|(n_i+1)I\| N^{\frac{1}{2}} \ln \mu_i k\right)^j}{j!}$$

$$\left. (\mu_1 k)^{-m(C_1)} \sum_{j=0}^{N-1} \frac{\left(\|C_1\| N^{\frac{1}{2}} \ln \mu_1 k\right)^j}{j!} \ldots (\mu_i k)^{-m(C_i)} \sum_{j=0}^{N-1} \frac{\left(\|C_i\| N^{\frac{1}{2}} \ln \mu_i k\right)^j}{j!} \frac{1}{(\mu_1 k)! \ldots (\mu_i k)!} \right\}^{\frac{1}{k(\mu_1+\ldots+\mu_i)}}. \tag{7.4}$$

Since

$$\sum_{j=0}^{N-1} \frac{\left(\|A\| N^{\frac{1}{2}} \ln \mu k\right)^j}{j!} \leq (N \ln \mu k)^{N-1} \sum_{j=0}^{N-1} \frac{\left(\|A\|\right)^j}{j!} = (N \ln \mu k)^{N-1} e^{\|A\|}.$$

Thus, we have

$$\frac{1}{R} \leq \limsup_{k(\mu_1+\ldots+\mu_i)\to\infty} \left\{ k^{M(-n_1 I)+\ldots+M(-n_i I)} k^{M((n_1+1)I)+\ldots+M((n_i+1)I)} k^{-m(B_1)-\ldots-m(B_i)} \right\}^{\frac{1}{k(\mu_1+\ldots+\mu_i)}}$$

$$\limsup_{k(\mu_1+\ldots+\mu_i)\to\infty} \left((N \ln \mu_1 k)^{N-1} e^{\|(-n_1 I)\|} \ldots (N \ln \mu_i k)^{N-1} e^{\|-n_i I\|} \right)^{\frac{1}{k(\mu_1+\ldots+\mu_i)}}$$

$$\limsup_{k(\mu_1+\ldots+\mu_i)\to\infty} \left((N \ln \mu_1 k)^{N-1} e^{\|(n_1+1)I\|} \ldots (N \ln \mu_i k)^{N-1} e^{\|(n_i+1)I\|} \right)^{\frac{1}{k(\mu_1+\ldots+\mu_i)}}$$

$$\limsup_{k(\mu_1+\ldots+\mu_i)\to\infty} \left((N \ln \mu_1 k)^{N-1} e^{\|C_1\|} \ldots (N \ln \mu_i k)^{N-1} e^{\|C_i\|} \right)^{\frac{1}{k(\mu_1+\ldots+\mu_i)}} = 1,$$

i.e. the radius of convergence of the composite Legendre matrix polynomials is one and it is regular in the hypersphere $\bar{S}_R$; $R = 1$ (c.f. [26]).

8. CONCLUDING COMMENTS

The material developed in Sections 2-7 provides several important properties of the Legendre matrix polynomials introduced in (2.2), under the condition $0 < \Re(\lambda) < 1$ for all $\lambda \in \sigma(C)$ on the matrix C in $\mathbb{C}^{N\times N}$. The Legendre matrix polynomials $P_n(x,C)$ are first shown to satisfy the second-order matrix differential equation (2.10). Rodrigues formula and the property of orthogonality of Legendre matrix polynomials are established (Theorem 3.1 and Theorem 4.1). The matrix recurrence relation for the $P_n(x,C)$ is derived (Theorem 5.1). New families of matrix generating functions, new matrix polynomials and identities concerning the Legendre matrix polynomials are derived (Theorem 6.1, Theorem 6.2 and Theorem 6.3). The families of bilinear and bilateral generating matrix functions in the sense provided by Theorem 6.5 are established.

Definition 8.1 *If C is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $0 < \Re(\lambda) < 1$ for all $\lambda \in \sigma(C)$. We will define* *the associated Legendre matrix polynomials in the form*

$$P_n^m(x,C)=(-1)^m(1-x)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x,C)$$

or, equivalently,

$$P_n^m(x,C)=\frac{1}{\Gamma(1-m)}\left(\frac{1+x}{1-x}\right)^{\frac{m}{2}}{}_2F_1\left(-nI,(n+1)I;C+mI;\frac{1-x}{2}\right).$$

In the forthcoming investigation, we will discuss further associated Legendre matrix polynomials and discuss future direction of the present line of work. We will more deeply analyze in the matrix theory of the so far introduced matrix polynomials and discuss in detail their properties.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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