Gazi University Journal of Science
GU J Sci
29(3):651-658 (2016)

New Results for α −Geraghty Type Contractıve Maps with Some Applıcatıons

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Received:01/05/2016 Accepted: 23/06/2016

ABSTRACT

In this paper, we have established some coupled fixed point theorems for α −Geraghty type contractive mappings in the context of partially ordered metric spaces. Applying these results, we have deduced fixed point results on metric spaces endowed with the graph. Also, the effectiveness of our work is validated with the help of a suitable example.

Keywords: *fixed point, partially ordered set, Geraghty-type contractive, connected graph*.

1. INTRODUCTION AND PRELIMINARIES

One of the most famous results in the fixed point theory is Banach contraction principle introduced by Banach [8]. Many authors have studied and proved the fixed point results in several directions; see, e.g., [9-14].

Let $\mathcal{L}$ be the class of functions $\gamma: [0,\infty) \to [0,1)$ with $\gamma(f_n) \to 1$ implies $f_n \to 0$. By using the function $\gamma \in \mathcal{L}$, Geraghty [4] proved the following remarkable theorem.

Theorem 1. ([4]) Let (X,d) be a complete metric space and $T: X \to X$ be an operatör. If T satisfies the following inequality

$$d(Tx,Ty) \leq \gamma\big(d(x,y)\big)d(x,y) \qquad (1.1)$$

for any $x, y \in X$, where $\gamma \in \mathcal{L}$, then T has a unique fixed point.

Amini-Harandi and Emami [2] presented the following results.

Theorem 2. ([8]) Let $(X,\preccurlyeq)$ be an ordered set endowed with a metric d and $T: X \to X$ be a given mapping. Suppose that the following conditions hold:

i. (X,d) is complete;

ii. (1) T is continuous or (2) if a nondecreasing sequence $\{x_n\}$ in X converges to some point $x \in X$, then $x_n \preccurlyeq x$ for all n;

iii. T is nondecreasing;

iv. there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;

v. there exists $\gamma \in \mathcal{L}$ such that for all $x, y \in X$ with $x \succcurlyeq y$,

$$d(Tx,Ty) \leq \gamma\big(d(x,y)\big)d(x,y).$$

Then T has a fixed point. Moreover, if for all $(x,y) \in X^2$ there exists a $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, we obtain uniqueness of the fixed point.

The concept of α −admissible and triangular α −admissible mappings were introduced by [6] and [7] respectively.

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Definition 1. ([6]) Let $T: X \to X$ and $\alpha: X^2 \to [0,\infty)$ be given mappings. We say that T is α −admissible if for all $x, y \in X$, we have

$\alpha(x,y) \geq 1 \Longrightarrow \alpha\big((Tx,Ty)\big) \geq 1.$

Definition 2. ([7]) An α −admissible map T is said to be triangular α −admissible if

$$\alpha(x,z) \geq 1 \text{ and } \alpha(z,y) \geq 1 \text{ imply } \alpha(x,y) \geq 1.$$

In 2014, Popescu [15] investigated the notion of triangular α −orbital admissible as follows.

Definition 3. ([15]) Let $T: X \to X$ be a mapping and $\alpha: X^2 \to [0,\infty)$ be a function. We say that T is α −orbital admissible if

$\alpha(x,Tx) \geq 1 \Longrightarrow \alpha\big((Tx,T^2x)\big) \geq 1.$

Moreover, T is called triangular α −orbital admissible if T is α −orbital admissible and

$\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ imply $\alpha(x,Ty) \geq 1.$

For more details on α −orbital admissible and triangular α −orbital admissible mappings, see e.g. [16-19].

Remark 1. As stated in [15] each α −admissible (respectively, triangular α −admissible) mapping is an α −orbital admissible (respectively, triangular α −orbital admissible) mapping. The converse is not true (for more details, see [15, Example 7]).

Mursaleen et al. [5] introduced a modified type admissibility that re-named (α) −admissible mapping.

Definition 4. ([5]) Let $F: X^2 \to X$ and $\alpha: X^2 \times X^2 \to [0,\infty)$ be two mappings. Then F is said to be (α) −admissible if

$$\alpha\big((x,y),(u,v)\big) \geq 1 \Rightarrow \alpha\Big(\big(F(x,y),F(y,x)\big),\big(F(u,v),F(v,u)\big)\Big) \geq 1$$

for all $x, y, u, v \in X$.

Definition 5. Let $F: X^2 \to X$ and $\alpha: X^2 \times X^2 \to [0,\infty)$. We say that F is a (α) −orbital admissible if

$$\alpha\Big((x,y),\big(F(x,y),F(y,x)\big)\Big) \geq 1 \Longrightarrow$$

$$\alpha\Big(\big(F(x,y),F(y,x)\big),\big(F^2(x,y),F^2(y,x)\big)\Big) \geq 1.$$

Furthermore, F is called triangular (α) −orbital admissible if F is a (α) −orbital admissible and

$$\alpha\big((x,y),(u,v)\big) \geq 1 \text{ and } \alpha\Big((u,v),\big(F(u,v),F(v,u)\big)\Big) \geq 1 \Longrightarrow$$

$$\alpha\Big((x,y),\big(F(u,v),F(v,u)\big)\Big) \geq 1.$$

Remark 2. Notice that Definition 4 (resp. Definition 5) is the same with Definition 1 (resp. Definition 3) by selecting X^2.

Lemma 1. Let F be a triangular (α) −orbital admissible mapping. Assume that there exists x_1, $y_1 \in X$ such that

$$\alpha\Big((x_1,y_1),\big(F(x_1,y_1),F(y_1,x_1)\big)\Big) \geq 1 \text{ and } \alpha\Big((y_1,x_1),\big(F(y_1,x_1),F(x_1,y_1)\big)\Big) \geq 1.$$

Define sequences $\{x_n\}$ and $\{y_n\}$ by $x_n = F^n(x_1,y_1)$ and $y_n = F^n(y_1,x_1)$. Then

$$\alpha\big((x_n,y_n),(x_m,y_m)\big) \geq 1 \text{ and } \alpha\big((y_n,x_n),(y_m,x_m)\big) \geq 1$$

for all $m, n \in \mathbb{N}$ with $n < m$.

Proof. Following the lines of the proof of [15, Lemma 8] by selecting X^2, we conclude the results. We omit the details.

Definition 6. ([3]) Let (X,d) be a complete metric space, $\alpha: X^2 \times X^2 \to [0,\infty)$ be a function, and $F: X^2 \to X$ be a map. We say that the sequences $\{x_n\}$ and $\{y_n\}$ are (α) −regular if the following condition is satisfied:

If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha\big((x_n,y_n),(x_{n+1},y_{n+1})\big) \geq 1$ and $\alpha\big((y_n,x_n),(y_{n+1},x_{n+1})\big) \geq 1$ for all n, and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then there exists subsequences $\{x_{n(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that $\alpha\Big(\big(x_{n(k)},y_{n(k)}\big),(x,y)\Big) \geq 1$ and $\alpha\Big(\big(y_{n(k)},x_{n(k)}\big),(y,x)\Big) \geq 1$ for all k.

Definition 7. ([1]) An element $(x,y) \in X^2$ is said to be a coupled fixed point of the mapping $F: X^2 \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 8. ([1]) Let $(X,\preccurlyeq)$ be a partially ordered set and $F: X^2 \to X$ be a mapping. Then a map F is said to have the mixed monotone property if $F(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y; that is; for any $x, y \in X$,

$$x_1, x_2 \in X,\, x_1 \preccurlyeq x_2 \text{ implies } F(x_1,y) \preccurlyeq F(x_2,y)$$

and

$$y_1, y_2 \in X,\, y_1 \preccurlyeq y_2 \text{ implies } F(x,y_1) \succcurlyeq F(x,y_2).$$

2. MAIN RESULTS

Throughout this section, we said that the tripled $(X,\leq,d)$ is a partially ordered metric space, if the relation $\leq$ on X is a partial order and (X,d) is a metric space. Further, if (X,d) is complete in the sense of metric d, then the tripled $(X,\leq,d)$ is called partially ordered metric space. We start this section with the following definition.

Definition 9. Let $(X,\leq,d)$ be a partially ordered metric space and $F: X^2 \to X$ be a mapping. Then a map F is said to α −Geraghty type contractive map if there exists two functions $\alpha: X^2 \times X^2 \to [0,\infty)$ and $\gamma \in \mathcal{L}$ such that

$$\alpha\big((x,y),(u,v)\big)\frac{d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)}{2} \leq \gamma\left(\frac{d(x,u)+d(y,v)}{2}\right)\left(\frac{d(x,u)+d(y,v)}{2}\right) \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.

Theorem 3. Let $(X,\preccurlyeq,d)$ be a partially ordered complete metric space and $F: X^2 \to X$ be a mapping having the mixed monotone property on X. Suppose that the following conditions are satisfied:

i. F is an α −Geraghty type contractive map;

ii. F is a triangular (α) −orbital admissible mapping;

iii. there exists $x_1, y_1 \in X$ such that

$$\alpha\Big((x_1,y_1),\big(F(x_1,y_1),F(y_1,x_1)\big)\Big) \geq 1 \text{ and } \alpha\Big((y_1,x_1),\big(F(y_1,x_1),F(x_1,y_1)\big)\Big) \geq 1,$$

iv. F is continuous.

If there exists $x_1, y_1 \in X$ such that $x_1 \leq F(x_1,y_1)$ and $y_1 \geq F(y_1,x_1)$, then F has a coupled fixed point, that is, there exists $x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$.

Proof. By assumption iii, there exists a point $x_1, y_1 \in X$ such that

$$\alpha\Big((x_1,y_1),\big(F(x_1,y_1),F(y_1,x_1)\big)\Big) \geq 1 \text{ and } \alpha\Big((y_1,x_1),\big(F(y_1,x_1),F(x_1,y_1)\big)\Big) \geq 1,$$

and $x_1 \leq F(x_1,y_1) = x_2$ and $y_1 \geq F(y_1,x_1) = y_2$. Moreover, we have a point $x_2, y_2 \in X$ such that $F(x_2,y_2) = x_3$ and $F(y_2,x_2) = y_3$. By induction, we have two sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$x_{n+1} = F(x_n,y_n) \text{ and } y_{n+1} = F(y_n,x_n) \text{ for all } n \geq 1. \tag{2.2}$$

Now, suppose that if $\big(x_{n_0+1}, y_{n_0+1}\big) = \big(x_{n_0}, y_{n_0}\big)$ for some n_0, then $F\big(x_{n_0}, y_{n_0}\big) = x_{n_0}$ and $F\big(y_{n_0}, x_{n_0}\big) = y_{n_0}$. Namely, F has a coupled fixed point. Now, we assume that $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$. From Lemma 1, we get

$$\alpha\big((x_n,y_n),(x_{n+1},y_{n+1})\big) \geq 1 \tag{2.3}$$

and similarly,

$$\alpha\big((y_n,x_n),(y_{n+1},x_{n+1})\big) \geq 1 \tag{2.4}$$

for all $n \in \mathbb{N}$. Now, we assume that $d(x_{n-1},x_n) + d(y_{n-1},y_n) \neq 0$ for all n. By (2.1) and using (2.3) and (2.4), we have

$$\big(d(x_n,x_{n+1}) + d(y_n,y_{n+1})\big) \times 2^{-1} \tag{2.5}$$

$$\leq \gamma\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right)\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right)$$

which implies

$$\left(d(x_n,x_{n+1})+d(y_n,y_{n+1})\right)\times 2^{-1}<\left(d(x_{n-1},x_n)+d(y_{n-1},y_n)\right)\times 2^{-1}.$$

Set $\omega_n \coloneqq d(x_n,x_{n+1})+d(y_n,y_{n+1})$, then sequence $\{\omega_n\}$ is monotone decreasing. Hence, there is some $\omega_n \geq 0$ such that $\lim_{n\to\infty}\omega_n=\omega$.

We claim that $\omega=0$. Suppose, to the contrary, that $\omega>0$, then by (2.5), we obtain

$$\frac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{d(x_{n-1},x_n)+d(y_{n-1},y_n)}\leq \gamma\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right)<1,$$

which yields that $\lim_{n\to\infty}\gamma\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right)=1$. This implies

$$d(x_{n-1},x_n)\to 0 \text{ and } d(y_{n-1},y_n)\to 0 \text{ as } n\to\infty \ \ (\text{since } \gamma\in\mathcal{L}) \tag{2.6}$$

or

$$d(x_{n-1},x_n)+d(y_{n-1},y_n)\to 0 \text{ as } n\to\infty \ \ (\text{since } \gamma\in\mathcal{L}) \tag{2.7}$$

which is a contraction. Therefore $\omega=0$, that is

$$\lim_{n\to\infty}\{d(x_n,x_{n+1})+\ d(y_n,y_{n+1})\}=0. \tag{2.8}$$

Now, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose that, to the contrary, that at least one of $\{x_n\}$ or $\{y_n\}$ are not Cauchy sequences. Then there exists an $\varepsilon>0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}$ with $n(k)>m(k)\geq k$ such that

$$d\left(x_{n(k)},x_{m(k)}\right)+d\left(y_{n(k)},y_{m(k)}\right)\geq\varepsilon. \tag{2.9}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k)>m(k)\geq k$ and satisfying (2.9). Then

$$d\left(x_{n(k)-1},x_{m(k)}\right)+d\left(y_{n(k)-1},y_{m(k)}\right)<\varepsilon. \tag{2.10}$$

Let $\xi_k \coloneqq d\left(x_{n(k)},x_{m(k)}\right)+d\left(y_{n(k)},y_{m(k)}\right)$. From (2.8), (2.9), (2.10) and using triangle inequality, we have

$$\lim_{k\to\infty}\xi_k=\varepsilon. \tag{2.11}$$

By the triangle inequality, we obtain

$$\xi_k\leq\omega_{n(k)}+\omega_{m(k)}+d\left(x_{n(k)+1},x_{m(k)+1}\right)+d\left(y_{n(k)+1},y_{m(k)+1}\right).$$

By Lemma 1, $\alpha\left(\left(x_{n(k)},y_{n(k)}\right),\left(x_{m(k)},y_{m(k)}\right)\right)\geq 1$. Since $n(k)>m(k)$, $x_{n(k)}\geq x_{m(k)}$ and $y_{n(k)}\leq y_{m(k)}$, from (2.1) and (2.2)

$$\begin{aligned}
&\left(d\left(x_{n(k)+1},x_{m(k)+1}\right)+d\left(y_{n(k)+1},y_{m(k)+1}\right)\right)\times 2^{-1}\\
&=\left(d\left(F\left(x_{n(k)},y_{n(k)}\right),F\left(x_{m(k)},y_{m(k)}\right)\right)+d\left(F\left(y_{n(k)},x_{n(k)}\right),F\left(y_{m(k)},x_{m(k)}\right)\right)\right)\times 2^{-1}\\
&\leq\alpha\left(\left(x_{n(k)},y_{n(k)}\right),\left(x_{m(k)},y_{m(k)}\right)\right)\left(\frac{d\left(F\left(x_{n(k)},y_{n(k)}\right),F\left(x_{m(k)},y_{m(k)}\right)\right)+d\left(F\left(y_{n(k)},x_{n(k)}\right),F\left(y_{m(k)},x_{m(k)}\right)\right)}{2}\right)\\
&\leq\gamma\left(\frac{d\left(x_{n(k)},x_{m(k)}\right)+d\left(y_{n(k)},y_{m(k)}\right)}{2}\right)\left(\frac{d\left(x_{n(k)},x_{m(k)}\right)+d\left(y_{n(k)},y_{m(k)}\right)}{2}\right)\\
&=\gamma\left(\frac{d\left(x_{n(k)},x_{m(k)}\right)+d\left(y_{n(k)},y_{m(k)}\right)}{2}\right)\left(\frac{\xi_k}{2}\right)
\end{aligned}$$

which implies

$$\left(d\left(x_{n(k)+1},x_{m(k)+1}\right)+d\left(y_{n(k)+1},y_{m(k)+1}\right)\right)\leq\gamma\left(\frac{d\left(x_{n(k)},x_{m(k)}\right)+d\left(y_{n(k)},y_{m(k)}\right)}{2}\right)\xi_k\,.$$

Thus,

$$\xi_k \leq \omega_{n(k)} + \omega_{m(k)} + \gamma\left(\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}\right)\xi_k .$$

Then we have

$$\frac{\xi_k - \omega_{n(k)} - \omega_{m(k)}}{\xi_k} \leq \gamma\left(\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}\right) < 1.$$

From (2.8) and (2.11), we have

$$\gamma\left(\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}\right) \to 1 \text{ (as } k \to \infty).$$

We have

$$\lim_{n\to\infty} d(x_{n(k)}, x_{m(k)}) = 0 = \lim_{n\to\infty} d(y_{n(k)}, y_{m(k)}),$$

that means

$$\lim_{n\to\infty}\{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})\} = 0$$

which is a contradiction. This show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is a complete metric space, there exists $x, y \in X$ such that

$$x_n \to x \text{ and } y_n \to y \text{ (as } n \to \infty). \tag{2.12}$$

Note that in the hypothesis of the theorem F is given continuous. Taking the limit as $n \to \infty$ in (2.2), by (2.12), we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = F(x, y).$$

Similarly, we show that

$$y = F(y, x).$$

Therefore $x = F(x, y)$ and $y = F(y, x)$.

Theorem 4. Let $(X, \preccurlyeq, d)$ be a partially ordered complete metric space and $F: X^2 \to X$ be a mapping having the mixed monotone property on X. Suppose that the following conditions are satisfied:

i. F is an α −Geraghty type contractive map;

ii. F is a triangular (α) −orbital admissible mapping;

iii. there exists $x_1, y_1 \in X$ such that

$$\alpha\left((x_1, y_1), (F(x_1, y_1), F(y_1, x_1))\right) \geq 1 \text{ and } \alpha\left((y_1, x_1), (F(y_1, x_1), F(x_1, y_1))\right) \geq 1,$$

iv. $\{x_n\}$ and $\{y_n\}$ are (α) −regular.

If there exists $x_1, y_1 \in X$ such that $x_1 \leq F(x_1, y_1)$ and $y_1 \geq F(y_1, x_1)$, then F has a coupled fixed point, that is, there exists $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Proof. Let $x_1, y_1 \in X$ be such that $\alpha\left((x_1, y_1), (F(x_1, y_1), F(y_1, x_1))\right) \geq 1$ and $\alpha\left((y_1, x_1), (F(y_1, x_1), F(x_1, y_1))\right) \geq 1$.

Following the lines in the proof of Theorem 3, we know that $x_n \to x \in X$ and $y_n \to y \in X$ (as $n \to \infty$). From (2.3), (2.4) and assumption iv, then there exists subsequences $\{x_{n(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that

$$\alpha\left((x_{n(k)}, y_{n(k)}), (x, y)\right) \geq 1 \text{ and } \alpha\left((y_{n(k)}, x_{n(k)}), (y, x)\right) \geq 1 \tag{2.13}$$

for all k. By (2.13), we have

$$\begin{aligned}
&\left(d(F(x,y), x) + d(F(y,x), y)\right) \times 2^{-1} \\
&\leq \left(d(F(x,y), x_{n(k)+1}) + d(x_{n(k)+1}, x) + d(F(y,x), y_{n(k)+1}) + d(y_{n(k)+1}, y)\right) \times 2^{-1} \\
&= \frac{d(x_{n(k)+1}, x) + d(y_{n(k)+1}, y)}{2} + \frac{d\left(F(x,y), F(x_{n(k)}, y_{n(k)})\right) + d\left(F(y,x), F(y_{n(k)}, x_{n(k)})\right)}{2} \\
&\leq \frac{d(x_{n(k)+1}, x) + d(y_{n(k)+1}, y)}{2} + \alpha\left((x,y), (x_{n(k)}, y_{n(k)})\right)\frac{d\left(F(x,y), F(x_{n(k)}, y_{n(k)})\right) + d\left(F(y,x), F(y_{n(k)}, x_{n(k)})\right)}{2}
\end{aligned}$$

$$\leq \frac{d(x_{n(k)+1},x)+d(y_{n(k)+1},y)}{2}+\gamma\left(\frac{d(x,x_{n(k)})+d(y,y_{n(k)})}{2}\right)\left(\frac{d(x,x_{n(k)})+d(y,y_{n(k)})}{2}\right)$$
$$\leq \frac{d(x_{n(k)+1},x)+d(y_{n(k)+1},y)}{2}+\frac{d(x,x_{n(k)})+d(y,y_{n(k)})}{2}$$

which implies

$$d(F(x,y),x)+d(F(y,x),y)\leq d(x_{n(k)+1},x)+d(y_{n(k)+1},y)+d(x,x_{n(k)})+d(y,y_{n(k)})$$
$$\to 0 \text{ as } k\to\infty.$$

Therefore $x=F(x,y)$ and $y=F(y,x)$. Thus we proved that F has a coupled fixed point.

To ensure the uniqueness of the coupled fixed point of an α −Geraghty type contractive map, we shall consider the following hypothesis.

(A) (x,y), $(s,t)\in\mathcal{F}$, either $\alpha((x,y),(s,t))\geq 1$ and $\alpha((y,x),(t,s))\geq 1$ or $\alpha((s,t),(x,y))\geq 1$ and $\alpha((t,s),(y,x))\geq 1$.

Here, $\mathcal{F}$ denotes the set of coupled fixed points of F.

Theorem 5. In Theorem 3 (resp. in Theorem 4), additionally assume that the condition (A) is hold. Then the coupled fixed point is unique.

Proof. Suppose that (x^*,y^*) and (u^*,v^*) are two coupled fixed point of F. Then

$$\begin{aligned}
&\left(d(x^*,u^*)+d(y^*,v^*)\right)\times 2^{-1}\\
&=\left(d(F(x^*,y^*),F(u^*,v^*))+d(F(y^*,x^*),F(v^*,u^*))\right)\times 2^{-1}\\
&\leq \alpha((x^*,y^*),(u^*,v^*))\left(\frac{d(F(x^*,y^*),F(u^*,v^*))+d(F(y^*,x^*),F(v^*,u^*))}{2}\right)\\
&\leq \gamma\left(\frac{d(x^*,u^*)+d(y^*,v^*)}{2}\right)\left(\frac{d(x^*,u^*)+d(y^*,v^*)}{2}\right)\\
&\leq \left(d(x^*,u^*)+d(y^*,v^*)\right)\times 2^{-1},
\end{aligned}$$

which is a contradiction.

Example 1. Let $X=[0,\infty)$, and let $d(x,y)=|x-y|$ for all $x,y\in X$. Consider the mapping $\alpha: X^2\times X^2\to[0,\infty)$ defined as

$$\alpha((x,y),(u,v))=\begin{cases}1, & if\ x\geq u, y\leq v\ or\ x\leq u, y\geq v,\\ 0, & otherwise.\end{cases}$$

Let $\gamma(t)=\frac{8}{9}\in\mathcal{L}$. Define $F: X^2\to X$ as follows

$$F(x,y)=\frac{7x-y}{9}$$

for all $x,y\in X$. Assume $\alpha((x,y),(u,v))=1$. Without loss of generality, suppose that $x\geq u$, $y\leq v$. Then

$$\begin{aligned}
&\alpha((x,y),(u,v))\left(\left(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))\right)\times 2^{-1}\right)\\
&=\left(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))\right)\times 2^{-1}\\
&=\frac{1}{18}|7(x-u)-(y-v)|+\frac{1}{18}|3(y-v)-(x-u)|\\
&\leq\frac{4}{9}[|x-u|+|y-v|]\\
&=\frac{8}{9}\times\frac{[|x-u|+|y-v|]}{2}\\
&=\gamma\left(\left(d(x,u)+d(y,v)\right)\times 2^{-1}\right)\left(\left(d(x,u)+d(y,v)\right)\times 2^{-1}\right)
\end{aligned}$$

holds for all $x\geq u$ and $y\leq v$. Otherwise,

$$\alpha((x,y),(u,v))\left(\left(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))\right)\times 2^{-1}\right)$$
$$=0$$

$$\leq \gamma\left(\left(d(x,u)+d(y,v)\right)\times 2^{-1}\right)\left(\left(d(x,u)+d(y,v)\right)\times 2^{-1}\right).$$

Moreover, F is a triangular (α) −orbital admissible, $\alpha\left((0,0),\left(F(0,0),F(0,0)\right)\right)\geq 1$ and $\alpha\left((0,0),\left(F(0,0),F(0,0)\right)\right)\geq 1$. Therefore, by Theorem 3, the point $(0,0)$ is the coupled fixed point of F.

3. SOME APPLICATIONS

In this section, we give the existence results for coupled fixed points of α −Geraghty contraction type mapping endowed with a directed graph.

Let (X,d) be a metric space, Δ be a diagonal of X^2, and G be a directed graph with no parallel edges such that the set $V(G)$ of its vertices coincides with X and $\Delta\subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. That is, G is determined by $\left(V(G),E(G)\right)$. Furthermore, denote by G^{-1} the graph obtained from G by reversing the direction of the edges in G. Hence, $E(G^{-1})=\{(x,y)\in X^2:(y,x)\in E(G)\}$.

Definition 10. ([20]) $F:X^2\to X$ is edge preserving if $(x,a)\in E(G)$, $(y,b)\in E(G^{-1})\Longrightarrow\left(F(x,y),F(a,b)\right)\in E(G)$ and $\left(F(y,x),F(b,a)\right)\in E(G^{-1})$.

Definition 11. ([20]) Let (X,d) be a complete metric space, G be a directed graph, and $F:X^2\to X$ be a mapping. Then

i. F is called G −continuous if for all $(a_*,b_*)\in X^2$ and for any sequence $\{n_i\}_i\in\mathbb{N}$ of positive integers such that $F\left(x_{n_i},y_{n_i}\right)\to a_*$, $F\left(y_{n_i},x_{n_i}\right)\to b_*$ as $i\to\infty$ and $\left(F\left(x_{n_i},y_{n_i}\right),F\left(x_{n_i+1},y_{n_i+1}\right)\right),\left(F\left(y_{n_i},x_{n_i}\right),F\left(y_{n_i+1},x_{n_i+1}\right)\right)\in E(G)$, we have that

$$\lim_{i\to\infty}F\left(F\left(x_{n_i},y_{n_i}\right),F\left(y_{n_i},x_{n_i}\right)\right)=F(a_*,b_*)$$

and

$$\lim_{i\to\infty}F\left(F\left(y_{n_i},x_{n_i}\right),F\left(x_{n_i},y_{n_i}\right)\right)=F(b_*,a_*);$$

ii. (X,d,G) has property A_1 if for any sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq X$ with $x_n\to x$ as $n\to\infty$ and $(x_n,x_{n+1})\in E(G)$ for $n\in\mathbb{N}$, then $(x_n,x)\in E(G)$;

iii. (X,d,G) has property A_2 if for any sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq X$ with $x_n\to x$ as $n\to\infty$ and $(x_n,x_{n+1})\in E(G^{-1})$ for $n\in\mathbb{N}$, then $(x_n,x)\in E(G^{-1})$.

Consider the set $\mathcal{F}$ of all coupled fixed points of mappings $F:X^2\to X$ and the set $(X^2)_F$ as follows:

$$\mathcal{F}=\{(x,y)\in X^2:x=F(x,y)\ and\ y=F(y,x)\}$$

and

$$(X^2)_F=\left\{(x,y)\in X^2:\left(x,F(x,y)\right)\in E(G),\left(y,F(y,x)\right)\in E(G^{-1})\right\}.$$

Theorem 6. Let (X,d) be complete metric space endowed with a directed graph G, and let $F:X^2\to X$ be a α −Geraghty type contractive mapping. Suppose that:

i. F is G −continuous;

or

ii. The tripled (X,d,G) has a properties A_1 and A_2.

Under these conditions, $\mathcal{F}\neq\emptyset\Leftrightarrow(X^2)_F\neq\emptyset$.

Proof. Along the lines of the proof of Theorem 3 and Theorem 4, we get the desired results. In view of the analogy, we skip the details of the proof.

Remark 3. In this case where $(X,d,\preccurlyeq)$ is partially ordered complete metric space, letting $E(G)=\{(x,y)\in X\times X:x\preccurlyeq y\}$, we obtain Theorem 3 and Theorem 4.

Acknowledgement 1.

The authors would like to express their sincere appreciation to the referees for their very helpful suggestions and many kind comments.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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