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Common Fixed Point Theorem for Contractive Mappings in Complex Valued b-Metric Spaces

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ABSTRACT

In this paper, we have presented some common fixed point results for the contractive mappings on complex valued *b*-metric spaces and obtained a generalization of the theorem of Azam et al., [2], Klin-eam and Suanoom [7], Mukheimer [8], Rouzkard and Imdad [12], Sitthikul and Saejung [13] and Sintunavarat and Kumam[14]. Further, our result is supported by examples.

Keywords: *Common fixed point, complex valued b-metric space.*

1. INTRODUCTION

It is common that the contractive-type conditions are very important in the study a fixed point theory. The first important result of fixed points for contractive-type mapping was the well-known Banach-Caccioppoli theorem published for the first time in 1922 in [4].

The concept of a *b*-metric space was introduced by S. Czerwik [5, 6]. Recently, some authors have proved some fixed point theorems in these spaces; see for example [3, 9, 10]. In 2013, Rao et al., [11] introduce the concept of complex valued *b*-metric spaces which was more general than the well-known complex valued metric spaces that was introduced by Azam et al., [2] in 2011. Several authors studied many common fixed point results on complex valued metric spaces, see, for example, [1, 13, 14].

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In this work, we establish a common fixed point result for mappings having contraction property on complex valued *b*-metric spaces. Moreover, the obtained result is generalizations of recent results proved by Azam et al., [2], Klin-eam and Suanoom [7], Mukheimer [8] and Rouzkard and Imdad [12], Sitthikul and Saejung [13] and Sintunavarat and Kumam [14].

2. PRELIMINARIES

In this section, at first we recall some definitions and properties in complex valued b -metric spaces that used in this paper.

Let C be the set of complex numbers and $z_1; z_2 \in C$. Define a partial order $\preceq$ on C as follows:

$z_1 \preceq z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2), \mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$.

It follows that $z_1 \preceq z_2$ if and only if one of the following conditions is satisfied:

(i). $\mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(ii). $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(iii). $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(iv). $\mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we will write $z_1 \underset{\neq}{\prec} z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that it is obvious that the following statements hold:

(i). If $0 \preceq z_1 \precnsim z_2$ then $|z_1| < |z_2|$,

(ii). If $z_1 \preceq z_2$ and $z_2 \underset{\neq}{\prec} z_3$ then $z_1 \prec z_3$.

Definition 2.1. [2] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to C$ satisfies:

(a). $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(b). $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c). $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Definition 2.2. [11] Let X be a nonempty set and $s \ge 1$ a given real number. A function $d : X \times X \to C$ is called a b-metric provided that, for all $x, y, z \in X$,

(cvbm-1). $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(cvbm-2). $d(x, y) = d(y, x)$ for all $x, y \in X$;

(cvbm-3). $d(x, z) \preceq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a complex valued b -metric space.

Example 2.3. *[11] Let* $X = [0,1]$. *Define the mapping* $d : X \times X \to C$ *for all* $x, y \in X$ *, by*

$d(x, y) = |x - y|^2 + i\,|x - y|^2$.

The pair (X,d) *is called a complex valued b-metric space with* $s=2$.

Definition 2.4. [11] Let (X,d) be a complex valued *b*-metric space.

(i). A point $x\in X$ is called interior point of a set $A\subseteq X$ whenever there exists $0\prec r\in C$ such that $B(x,r)=\{y\in X: d(x,y)\prec r\}\subseteq A$.

(ii). A point $x\in X$ is called a limit point of *A* whenever for every $0\prec r\in C$, $B(x,r)\cap(A-X)\neq\varnothing$.

(iii). A subset $A\subseteq X$ is called open whenever each element of A is an interior point of A.

(iv). A subset $A\subseteq X$ is called close whenever each limit point of A is an interior of A.

(v). The family $F=\{B(x,r), x\in X, 0\prec r\}$; is a sub-basis for a Hausdorff topology τ on X We denote this complex topology by τ_c . Indeed, the topology τ_c is Hausdorff.

Definition 2.5. [11] Let (X,d) be a complex valued *b*-metric space. Let $\{x_n\}$ be a sequence in X and $x\in X$.

(i). If for every $c\in C$, with $0\prec c$ there is $n_0\in N$ such that $d(x_n,x)\prec c$, for all $n>n_0$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x, x is the limit point of $\{x_n\}$. We denote this by $Lim_n x_n = x$ or $x_n\to x$ as $n\to\infty$.

(ii). If for every $c\in C$, with $0\prec c$ there is $n_0\in N$ such that for all $n>n_0$, $d(x_n,x_{n+m})\prec c$, where $m\in N$, then $\{x_n\}$ is said to be Cauchy sequence.

(iii). If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete complex valued *b*-metric space.

Lemma 2.6. *[11] Let* (X,d) *be a complex valued b-metric space and let* $\{x_n\}$ *be a sequence in* X *. Then* $\{x_n\}$ *converges to* x *if and only if* $|d(x_n,x)|\to 0$ *as* $n\to\infty$.

Lemma 2.7. *[11] Let* (X,d) *be a complex valued b-metric space and let* $\{x_n\}$ *be a sequence in* X *. Then* $\{x_n\}$ *converges to* x *if and only if* $|d(x_n,x_{n+m})|\to 0$ *as* $n\to\infty$,. $m\in N$

Recently, many authors established some common fixed point results in complex valued metric and *b*-metric spaces.

Theorem 2.8. *[2] Let* (X,d) *be a complete complex valued metric space and let* λ_1,λ_2 *be nonnegative real numbers such that* $\lambda_1+\lambda_2<1$. *Suppose* $S,T:X\to X$ *are mappings satisfying*

$$\mathrm{d}(\mathrm{Sx},\mathrm{Ty})\preceq\lambda_1\mathrm{d}(\mathrm{x},\mathrm{y})+\frac{\lambda_2\mathrm{d}(\mathrm{x},\mathrm{Sx})\mathrm{d}(\mathrm{y},\mathrm{Ty})}{1+\mathrm{d}(\mathrm{x},\mathrm{y})},$$

for all $x,y\in X$. *Then* S *and* T *have a unique common fixed point in* X .

Theorem 2.9. *[8] Let* (X,d) *be a complete complex valued* b *-metric space with the coefficient* $s\geq 1$ *and* $S,T:X\to X$ *are mappings satisfying*

$$d(Sx, Ty) \preceq \lambda_1 d(x, y) + \frac{\lambda_2 d(x, Sx) d(y, Ty)}{1 + d(x, y)},$$

for all $x, y \in X$ *, where* λ_1, λ_2 *are nonnegative real with* $s\lambda_1 + \lambda_2 < 1$*. Then* S *and* T *have a unique common fixed point in* X *.*

Theorem 2.10. *[12] If* S *and* T *are self-mappings defined on a complete complex valued metric space* (X, d) *satisfying the condition*

$$d(Sx, Ty) \preceq \lambda_1 d(x, y) + \frac{\lambda_2 d(x, Sx) d(y, Ty) + \lambda_3 d(y, Sx) d(x, Ty)}{1 + d(x, y)},$$

for all $x, y \in X$ *, where* $\lambda_1, \lambda_2, \lambda_3$ *are nonnegative real with* $\lambda_1 + \lambda_2 + \lambda_3 < 1$*. Then* S *and* T *have a unique common fixed point in* X *.*

Theorem 2.11. *[7] If* S *and* T *are self-mappings defined on a complete complex valued metric space* (X, d) *satisfying the condition*

$$\begin{aligned} d(Sx, Ty) \preceq \lambda_1 d(x, y) &+ \frac{\lambda_2 d(x, Sx) d(y, Ty)}{1 + d(x, y)} \\ &+ \frac{\lambda_3 d(y, Sx) d(x, Ty)}{1 + d(x, y)} + \frac{\lambda_4 d(x, Sx) d(x, Ty)}{1 + d(x, y)} \\ &+ \frac{\lambda_5 d(y, Sx) d(y, Ty)}{1 + d(x, y)}, \end{aligned}$$

for all $x, y \in X$ *, where* λ_i *for* $i = 1,2,3,4,5$ *are nonnegative real with* $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 < 1$*. Then* S *and* T *have a unique common fixed point in* X *.*

3. MAIN RESULTS

We start this section by a proposition.

Proposition 3.1. *Let* (X, d) *be a complete complex valued b-metric space with the coefficient* $s \geq 1$*. If* $\{x_n\}$ *and* $\{y_n\}$ *are convergent to* x *and* y *respectively, then*

$$\frac{1}{s^2} |d(x, y)| \leq \liminf_{n \to \infty} |d(x_n, y_n)| \leq \limsup_{n \to \infty} |d(x_n, y_n)| \leq s^2 |d(x, y)|.$$

So if $\lim_{n \to \infty} |d(x, y)|$ *exists and equals to* $|d(x, y)|$.

Proof. Suppose $\{x_n\}$ and $\{y_n\}$ are convergent to x *and* y respectively. Then, by (*cvbm* -3) we have,

$$\begin{aligned} d(x_n, y_n) &\preceq s\left(d(x_n, x) + d(x, y_n)\right) \\ &\preceq s\left[d(x_n, x) + s\left(d(x, y) + d(y, y_n)\right)\right] \\ &= s d(x_n, x) + s^2 d(x, y) + s^2 d(y, y_n), \end{aligned}$$

consequently,

$$\operatorname*{Lim\,sup}_{n\to\infty} | d(x_n, y_n) | \le s^2 | d(x, y) | . \tag{3.1}$$

Also

$$\begin{aligned} d(x, y) &\preceq s\left(d(x, x_n) + d(x_n, y)\right) \\ &\preceq s\left[d(x, x_n) + s\left(d(x_n, y_n) + d(y_n, y)\right)\right] \\ &= sd(x_n, x) + s^2 d(x_n, y_n) + s^2 d(y_n, y), \end{aligned}$$

consequently,

$$\frac{1}{s^2} | d(x, y) | \le \operatorname*{Lim\,inf}_{n\to\infty} | d(x_n, y_n) | . \tag{3.2}$$

Consequently, from (3.1) and (3.2) we have

$$\frac{1}{s^2} | d(x, y) | \le \operatorname*{Lim\,inf}_{n\to\infty} | d(x_n, y_n) | \le \operatorname*{Lim\,sup}_{n\to\infty} | d(x_n, y_n) | \le s^2 | d(x, y) | .$$ □

The following theorem is the first main result of this paper.

Theorem 3.2. *Let* (X, d) *be a complete complex valued b-metric space with the coefficient* $s \ge 1$ *and* $S, T : X \to X$. *Suppose there exist mappings* $\lambda_i : X \times X \to [0,1)$ *where* $i = 1,2,3,4,5$ *such that*

$$\begin{aligned} &(a)\ \lambda_i(TSx, y) \le \lambda_i(x, y) \text{ and } \lambda_i(x, STy) \le \lambda_i(x, y), \\ &(b)\ s\lambda_1(x, y) + \lambda_2(x, y) + \lambda_3(x, y) + (s^2 + s)(\lambda_4(x, y) + \lambda_5(x, y)) < 1, \\ &(c)\ d(Sx, Ty) \preceq \lambda_1(x, y) d(x, y) + \lambda_2(x, y)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} \\ &\qquad + \lambda_3(x, y)\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)} + \lambda_4(x, y)\frac{d(x, Sx)d(x, Ty)}{1 + d(x, y)} \\ &\qquad + \lambda_5(x, y)\frac{d(y, Sx)d(y, Ty)}{1 + d(x, y)}, \end{aligned} \tag{3.3}$$

for all $x, y \in X$. *Then there exists a unique common fixed point* $u \in X$ *such that* $u = Su = Tu$.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n};\quad x_{2n+2} = Tx_{2n+1}\quad \text{for all}\quad n = 0,1,2,\ldots. \tag{3.4}$$

From condition (*a*), for $x, y \in X$, $n = 0,1,2,\ldots$ and $i = 1,2,3,4,5,$ we have

$$\lambda_i(x_{2n}, y) = \lambda_i(TSx_{2n-2}, y) \le \lambda_i(x_{2n-2}, y) \le \lambda_i(TSx_{2n-4}, y) \le \cdots \le \lambda_i(x_0, y). \tag{3.5}$$

Similarly, we obtain

$$\lambda_i(x, x_{2n+1}) = \lambda_i(x, STx_{2n-1}) \le \lambda_i(x, x_{2n-1}) \le \lambda_i(x, STx_{2n-3}) \le \cdots \le \lambda_i(x, x_1). \tag{3.6}$$

From inequality (3.3), we have

$$\begin{aligned} d(x_{2k+1}, x_{2k}) = d(STx_{2k-1}, Tx_{2k-1}) &\preceq \lambda_1(Tx_{2k-1},x_{2k-1})d(Tx_{2k-1},x_{2k-1}) \\ &+ \lambda_2(Tx_{2k-1},x_{2k-1})\frac{d(Tx_{2k-1},STx_{2k-1})d(x_{2k-1},Tx_{2k-1})}{1+d(Tx_{2k-1},x_{2k-1})} \\ &+ \lambda_3(Tx_{2k-1},x_{2k-1})\frac{d(x_{2k-1},STx_{2k-1})d(Tx_{2k-1},Tx_{2k-1})}{1+d(Tx_{2k-1},x_{2k-1})} \\ &+ \lambda_4(Tx_{2k-1},x_{2k-1})\frac{d(Tx_{2k-1},STx_{2k-1})d(Tx_{2k-1},Tx_{2k-1})}{1+d(Tx_{2k-1},x_{2k-1})} \\ &+ \lambda_5(Tx_{2k-1},x_{2k-1})\frac{d(x_{2k-1},STx_{2k-1})d(x_{2k-1},Tx_{2k-1})}{1+d(Tx_{2k-1},x_{2k-1})}, \end{aligned} \tag{3.7}$$

and hence

$$\begin{aligned} d(x_{2k+1}, x_{2k}) = d(STx_{2k-1}, Tx_{2k-1}) &\preceq \lambda_1(x_{2k},x_{2k-1})d(x_{2k},x_{2k-1}) \\ &+ \lambda_2(x_{2k},x_{2k-1})\frac{d(x_{2k},x_{2k+1})d(x_{2k-1},x_{2k})}{1+d(x_{2k},x_{2k-1})} \\ &+ \lambda_3(x_{2k},x_{2k-1})\frac{d(x_{2k-1},x_{2k+1})d(x_{2k},x_{2k})}{1+d(Tx_{2k-1},x_{2k-1})} + \\ &+ \lambda_4(x_{2k},x_{2k-1})\frac{d(x_{2k},x_{2k+1})d(x_{2k},x_{2k})}{1+d(x_{2k},x_{2k-1})} + \\ &+ \lambda_5(x_{2k},x_{2k-1})\frac{d(x_{2k-1},x_{2k+1})d(x_{2k-1},x_{2k})}{1+d(x_{2k},x_{2k-1})}. \end{aligned} \tag{3.8}$$

Since $|d(x_{2k}, x_{2k-1})| \leq 1 + |d(x_{2k}, x_{2k-1})|$, from (3.8) with using (3.5) and (3.6) we get

$$\begin{aligned} |d(x_{2k+1}, x_{2k})| = |d(STx_{2k-1}, Tx_{2k-1})| &\leq \lambda_1(x_0, x_{2k-1})|d(x_{2k}, x_{2k-1})| \\ &+ \lambda_2(x_0, x_{2k-1})|d(x_{2k}, x_{2k+1})| + \lambda_5(x_0, x_{2k-1})|d(x_{2k-1}, x_{2k+1})| \\ &\leq \lambda_1(x_0, x_1)|d(x_{2k}, x_{2k-1})| + \lambda_2(x_0, x_1)|d(x_{2k}, x_{2k+1})| \\ &+ \lambda_5(x_0, x_1)\left[s|d(x_{2k}, x_{2k-1})| + s|d(x_{2k}, x_{2k+1})|\right], \end{aligned}$$

which implies that

$$|d(x_{2k+1}, x_{2k})| \leq \frac{\lambda_1(x_0, x_1) + s\lambda_5(x_0, x_1)}{1 - \lambda_2(x_0, x_1) - s\lambda_5(x_0, x_1)} |d(x_{2k-1}, x_{2k})|. \tag{3.9}$$

Since $s \geq 1$, , we have

$$\begin{aligned} &\lambda_1(x, y) + \lambda_2(x, y) + (s+s)\lambda_5(x, y) \leq \\ &s\lambda_1(x, y) + \lambda_2(x, y) + \lambda_3(x, y) + (s^2 + s)(\lambda_4(x, y) + \lambda_5(x, y)) < 1. \end{aligned}$$

Therefore, set $\mu_1 = \dfrac{\lambda_1(x_0, x_1) + s\lambda_5(x_0, x_1)}{1 - \lambda_2(x_0, x_1) - s\lambda_5(x_0, x_1)}$, where $\mu_1 < 1.$

Similarly, we get

$$d(x_{2k+2}, x_{2k+1}) = d(Sx_{2k}, TSx_{2k}) \preceq \lambda_1(x_0, x_1)d(x_{2k}, x_{2k+1}) + \lambda_2(x_0, x_1)d(x_{2k+1}, x_{2k+2}) + \lambda_4(x_0, x_1)[sd(x_{2k}, x_{2k+1}) + sd(x_{2k+1}, x_{2k+2})],$$

which implies that

$$| d(x_{2k+2}, x_{2k+1}) | \leq \frac{\lambda_1(x_0, x_1) + s\lambda_4(x_0, x_1)}{1 - \lambda_2(x_0, x_1) - s\lambda_4(x_0, x_1)} | d(x_{2k}, x_{2k+1}) |. \tag{3.10}$$

Therefore, set $\mu_2 = \dfrac{\lambda_1(x_0, x_1) + s\lambda_4(x_0, x_1)}{1 - \lambda_2(x_0, x_1) - s\lambda_4(x_0, x_1)}$, where $\mu_2 < 1$. Set $\lambda = \max\{\mu_1, \mu_2\}$, where $\lambda < 1$. Therefore by (3.9) and (3.10), we conclude that $| d(x_{n+1}, x_n) | \leq \lambda | d(x_n, x_{n-1}) |$ for all $n \in N.$ So we get by induction,

$$| d(x_{n+1}, x_n) | \leq \lambda^n | d(x_o, x_1) |, \tag{3.11}$$

for all $n \in N.$ So

$$\lim_{n\to\infty} | d(x_{n+1}, x_n) | = 0. \tag{3.12}$$

Now, we prove $\{x_n\}$ is Cauchy sequence. Without loss of generality, we take $m > n$, and $m, n \in N,$ then by (cvbm-3) and using (3.11), we obtain

$$
\begin{aligned}
&|d(x_n,x_m)|\\
&\le s|d(x_n,x_{n+1})|+s|d(x_{n+1},x_m)|\\
&\le s|d(x_n,x_{n+1})|+s^2|d(x_{n+1},x_{n+2})|+s^2|d(x_{n+2},x_m)|\\
&\le s|d(x_n,x_{n+1})|+s^2|d(x_{n+1},x_{n+2})|+s^3|d(x_{n+2},x_{n+3})|+s^3|d(x_{n+3},x_m)|\\
&\le s|d(x_n,x_{n+1})|+s^2|d(x_{n+1},x_{n+2})|+\ldots+s^{m-n-2}|d(x_{m-3},x_{m-2})|\\
&\quad +s^{m-n-1}|d(x_{m-2},x_{m-1})|+s^{m-n-1}|d(x_{m-1},x_m)|\\
&\le s|d(x_n,x_{n+1})|+s^2|d(x_{n+1},x_{n+2})|+\ldots+s^{m-n-2}|d(x_{m-3},x_{m-2})|\\
&\quad +s^{m-n-1}|d(x_{m-2},x_{m-1})|+s^{m-n}|d(x_{m-1},x_m)|\\
&\le s\lambda^n|d(x_o,x_1)|+s^2\lambda^{n+1}|d(x_o,x_1)|+\ldots+s^{m-n-2}\lambda^{m-3}|d(x_o,x_1)|\\
&\quad +s^{m-n-1}\lambda^{m-2}|d(x_0,x_1)|+s^{m-n}\lambda^{m-1}|d(x_0,x_1)|\\
&\le s^n\lambda^n|d(x_o,x_1)|+s^{n+1}\lambda^{n+1}|d(x_o,x_1)|+\ldots+s^{m-3}\lambda^{m-3}|d(x_o,x_1)|\\
&\quad +s^{m-2}\lambda^{m-2}|d(x_0,x_1)|+s^{m-1}\lambda^{m-1}|d(x_0,x_1)|\\
&\le s^n\lambda^n|d(x_o,x_1)|+s^{n+1}\lambda^{n+1}|d(x_o,x_1)|+\ldots+s^{m-3}\lambda^{m-3}|d(x_o,x_1)|\\
&+s^{m-2}\lambda^{m-2}|d(x_0,x_1)|+s^{m-1}\lambda^{m-1}|d(x_0,x_1)|+s^m\lambda^m|d(x_0,x_1)|+\cdots+\cdots\\
&=\frac{(s\lambda)^n}{1-s\lambda}|d(x_0,x_1)|.
\end{aligned}
$$

From conditionin inequality above we get $n,m\to\infty$ Therefore, with Tending $s\lambda<1.$ we have (b) such that $z\in X$ is complete, there exists X Since X. is a Cauchy sequence in $\{x_n\}$ Thus, $|d(x_n,x_m)|\to 0,$

$$\lim_{n\to\infty}|d(x_n,z)|=0. \tag{3.13}$$

Now, we show that z is a fixed point S. From (3.3), we have

$$
\begin{aligned}
d(z,Sz)&\preceq sd(z,Tx_{2n+1})+sd(Tx_{2n+1},Sz)=sd(z,x_{2n+2})+sd(Sz,Tx_{2n+1})\\
&\preceq sd(z,x_{2n+2})+s\lambda_1(z,x_{2n+1})d(z,x_{2n+1})\\
&\quad +s\lambda_2(z,x_{2n+1})\frac{d(z,Sz)d(x_{2n+1},Tx_{2n+1})}{1+d(z,x_{2n+1})}\\
&\quad +s\lambda_3(z,x_{2n+1})\frac{d(x_{2n+1},Sz)d(z,Tx_{2n+1})}{1+d(z,x_{2n+1})}\\
&\quad +s\lambda_4(z,x_{2n+1})\frac{d(z,Sz)d(z,Tx_{2n+1})}{1+d(z,x_{2n+1})}\\
&\quad +s\lambda_5(z,x_{2n+1})\frac{d(x_{2n+1},Sz)d(x_{2n+1},Tx_{2n+1})}{1+d(z,x_{2n+1})},
\end{aligned}
$$

which (3.6) imply that

$$
\begin{aligned}
|d(z,Sz)| \leq\ & s|d(z,x_{2n+2})| + s\lambda_1(z,x_1)|d(z,x_{2n+1})| \\
& + s\lambda_2(z,x_1)\frac{|d(z,Sz)d(x_{2n+1},x_{2n+2})|}{|1+d(z,x_{2n+1})|} \\
& + s\lambda_3(z,x_1)\frac{|d(x_{2n+1},Sz)d(z,x_{2n+2})|}{|1+d(z,x_{2n+1})|} \\
& + s\lambda_4(z,x_1)\frac{|d(z,Sz)d(z,x_{2n+2})|}{|1+d(z,x_{2n+1})|} \\
& + s\lambda_5(z,x_1)\frac{|d(x_{2n+1},Sz)d(x_{2n+1},x_{2n+2})|}{|1+d(z,x_{2n+1})|}.
\end{aligned} \tag{3.14}
$$

The condition (cvbm-3) yields $|d(x_{2n+1},z)| \leq s(|d(x_{2n+1},x_{2n})| + |d(x_{2n},z)|)$, so from (3.12) and (3.13)

$$\lim_{n\to\infty}|d(x_{2n+1},z)| = 0. \tag{3.15}$$

On the other hand, from Proposition 3.1 we get

$$\limsup_{n\to\infty}|d(x_{2n+1},x_{2n+2})| \leq s^2|d(z,z)| = 0. \tag{3.16}$$

Taking the upper limit in (3.14) and combining (3.15) and (3.16), we get $|d(z,Sz)|=0$. Thus $d(z,Sz)=0$ and hence $z=Sz$. It follows similarly that $z = Tz$. Therefore, z is a common fixed point of S and T.

Now, we show Common fixed point of S and T is unique. Suppose that there is another common fixed point $z^* \in X$ such that $z^* = Sz^* = Tz^*$. From (3.3), we have

$$
\begin{aligned}
d(z,z^*) = d(Sz,Tz^*) \preceq\ & \lambda_1(z,z^*)d(z,z^*) + \lambda_2(z,z^*)\frac{d(z,Sz)d(z^*,Tz^*)}{1+d(z,z^*)} \\
& + \lambda_3(z,z^*)\frac{d(z^*,Sz)d(z,z^*)}{1+d(z,z^*)} + \lambda_4(z,z^*)\frac{d(z,Sz)d(z,Tz^*)}{1+d(z,z^*)} \\
& + \lambda_5(z,z^*)\frac{d(z^*,Sz)d(z^*,Tz^*)}{1+d(z,z^*)}.
\end{aligned}
$$

Since $|1 + d(z,z^*)| > |d(z,z^*)|$, we get

$$|d(z,z^*)| \leq [\lambda_1(z,z^*) + \lambda_3(z,z^*)]d(z,z^*).$$

This is contradiction because condition (b) imply $\lambda_1(z,z^*) + \lambda_3(z,z^*) < 1$. Therefore, we obtain $|d(z,z^*)| = 0$. Thus $z = z^*$. This completes the proof.

The following examples illustrate of our main result.

Example 3.3. Let $X=[0,1]$. Define the mapping $d: X \times X \to C$ for all $x, y \in X$ by

$$d(x,y) = |x-y|^2 + i|x-y|^2.$$

Then (X,d) by Example 2.3 is a complete complex valued b-metric space with the coefficient $s=2$. Now, define two mappings $T,S:X\to X$ by $S(x)=T(x)=\frac{x}{5}$ for all $x\in X$.

Let $\lambda_i : X\times X \to [0,1)$ where $i=1,2,3,4,5$ for all $x,y\in X$ be defined as follows:

$$\lambda_1(x,y)=\frac{xy}{10}+\frac{1}{24} \qquad \lambda_2(x,y)=\frac{xy}{24} \qquad \lambda_3(x,y)=\frac{x^2y^2}{24}$$

$$\lambda_4(x,y)=\frac{x^3y^3}{24} \qquad \lambda_5(x,y)=\frac{x^4y^4}{24}.$$

Clearly $2\lambda_1(x,y)+\lambda_2(x,y)+\lambda_3(x,y)+6(\lambda_4(x,y)+\lambda_5(x,y))<1$ and also,

$$\lambda_1(TSx,y)=\lambda_1(T(\frac{x}{5}),y)=\lambda_1(\frac{x}{25},y)=\frac{xy}{250}+\frac{1}{24}\le\frac{xy}{10}+\frac{1}{24}=\lambda_1(x,y)$$

$$\lambda_1(x,STy)=\lambda_1(x,S(\frac{y}{5}))=\lambda_1(x,\frac{y}{25})=\frac{xy}{250}+\frac{1}{24}\le\frac{xy}{10}+\frac{1}{24}=\lambda_1(x,y),$$

$$\lambda_2(TSx,y)=\lambda_2(T(\frac{x}{5}),y)=\lambda_2(\frac{x}{25},y)=\frac{xy}{600}\le\frac{xy}{24}=\lambda_2(x,y)$$

$$\lambda_2(x,STy)=\lambda_2(x,S(\frac{y}{5}))=\lambda_2(x,\frac{y}{25})=\frac{xy}{600}\le\frac{xy}{24}=\lambda_2(x,y).$$

Similarly for $i=3,4,5$ we will have

$$\lambda_i(TSx,y)\le\lambda_i(x,y) \qquad \text{and} \qquad \lambda_i(x,STy)\le\lambda_i(x,y).$$

We next verify inequality (3.3) of Theorem 3.2. For all $x,y\in X$

$$0\preceq d(x,y),\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)},\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)},\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)},\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)}.$$

Thus,

$$
\begin{aligned}
d(Sx,Ty) &= \frac{1}{25}\Big[|x-y|^2 + i|x-y|^2\Big] \preceq \frac{1}{24}[|x-y|^2 + i|x-y|^2] \\
&\preceq \left(\frac{xy}{10}+\frac{1}{24}\right)\Big[|x-y|^2 + i|x-y|^2\Big] = \lambda_1(x,y)d(x,y) \\
&\preceq \lambda_1(x,y)d(x,y) + \lambda_2(x,y)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} \\
&\preceq \lambda_3(x,y)\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)} + \lambda_4(x,y)\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)} \\
&\preceq \lambda_5(x,y)\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)}.
\end{aligned}
$$

So we get inequality (3.3). Hence the required condition of Theorem 3.2 are satisfied and there exists a unique common fixed point 0 of the mappings S and T .

Example 3.4. Let $X=\{(1,2),(2,3),(3,4),(4,5),(5,6)\}$. Define the mapping $d: X\times X \to C$ for all $z_1, z_2 \in X$, by

$$d(x,y) = |x-y|^2 + i|x-y|^2,$$

where $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$. Then (*X; d*) is a complete complex valued b-metric space with the coefficient $s=2$. Now, define two mappings $T,S: X \to X$ by $S(z)=T(z)=|x-y|+2i|x-y|$, for all $z = x+iy \in X$.

Let $\lambda_i : X\times X \to [0,1)$ where $i=1,2,3,4,5$ for all $z_1, z_2 \in X$ be defined as follows:

$$\lambda_1(z_1,z_2) = \frac{|x_1-y_1||x_2-y_2|}{10} + \frac{1}{24}, \qquad \lambda_i(z_1,z_2) = \frac{|x_1-y_1||x_2-y_2|}{24} \qquad \forall i = 2,3,4,5.$$

Clearly $2\lambda_1(z_1,z_2)+\lambda_2(z_1,z_2)+\lambda_3(z_1,z_2)+6(\lambda_4(z_1,z_2)+\lambda_5(z_1,z_2))<1$ and also,

$$
\begin{aligned}
&\lambda_1(TSz_1,z_2) = \lambda_1(T(|x_1 - y_1| + 2i|x_1 - y_1|), z_2) \\
&= \lambda_1(||x_1 - y_1| - 2|x_1 - y_1|| + 2i||x_1 - y_1| - 2|x_1 - y_1||, z_2) \\
&= \lambda_1(|x_1 - y_1| + 2i|x_1 - y_1|, z_2) = \frac{||x_1 - y_1| - 2|x_1 - y_1|||x_2 - y_2|}{10} + \frac{1}{24} \\
&= \frac{|x_1 - y_1||x_2 - y_2|}{10} + \frac{1}{24} = \lambda_1(z_1,z_2),
\end{aligned}
$$

$$\begin{aligned}\lambda_1(z_1, STz_2) &= \lambda_1(z_1, S(|x_2-y_2|+2i|x_2-y_2|))\\ &= \lambda_1(z_1, ||x_2-y_2|-2|x_2-y_2||+2i||x_2-y_2|-2|x_2-y_2||)\\ &= \lambda_1(z_1, |x_2-y_2|+2i|x_2-y_2|) = \frac{||x_2-y_2|-2|x_2-y_2|||x_1-y_1|}{10}+\frac{1}{24}\\ &= \frac{|x_1-y_1||x_2-y_2|}{10}+\frac{1}{24} = \lambda_1(z_1,z_2).\end{aligned}$$

Similarly for i=2,3,4,5 we will have

$$\lambda_i(TSz_1, z_2) \le \lambda_i(z_1,z_2) \quad \text{and} \quad \lambda_i(z_1, STz_2) \le \lambda_i(z_1,z_2).$$

For all $z_1, z_2 \in X$ we have 0=d(Sz_1, Tz_2), therefore the inequality (3.3) is trivial. Hence the required condition of Theorem 3.2 are satisfied also the point $(1,2) \in X$ is a unique common fixed point of S and T.

If we let S=T in Theorem 3.2, then we get the following corollary.

Corollary 3.5. Let (X,d) be a complete complex valued b -metric space with the coefficient $s \ge 1$ and $T: X \to X$. Suppose there exist mappings $\lambda_i : X \times X \to [0,1)$ where i=1,2,3,4,5 such that

$$(a)\ \lambda_i(T^2x, y) \le \lambda_i(x,y) \ and \ \lambda_i(x, T^2y) \le \lambda_i(x,y),$$

$$\text{(b)}\ s\lambda_1(x,y)+\lambda_2(x,y)+\lambda_3(x,y)+(s^2+s)(\lambda_4(x,y)+\lambda_5(x,y))<1,$$

$$\begin{aligned}\text{(c)}\ d(Tx,Ty) \preceq{}& \lambda_1(x,y)d(x,y)+\lambda_2(x,y)\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\\ &+\lambda_3(x,y)\frac{d(y,Tx)d(x,Ty)}{1+d(x,y)}+\lambda_4(x,y)\frac{d(x,Tx)d(x,Ty)}{1+d(x,y)}\\ &+\lambda_5(x,y)\frac{d(y,Tx)d(y,Ty)}{1+d(x,y)},\end{aligned}$$

for all $x, y \in X$. Then there exists a unique point $u \in X$ such that u = Tu .

Remark 3.6. For $\lambda_i(x,y)=0$ for i=4,5 and s=1 in Theorem 3.2, we get result obtained by Sitthikul and Saejung in [13, Theorems 2.4].

Remark 3.7. By definition $\lambda_1(x,y)=\Lambda(x), \lambda_2(x,y)=\Xi(x)$ and $\lambda_i(x,y)=0$ for i=3,4,5 and s=1 in Theorem 3.2, we get result obtained by Sintunavarat and Kumam in [14, Theorems 3.1].

Remark 3.8. By choosing $\lambda_i(x,y)=\lambda_i$ for i=1, 2, $\lambda_i(x,y)=0$ for i=3,4,5 and s=1 in Theorem 3.2, we deduce Theorem 2.8.

Remark 3.9. By choosing $\lambda_i(x,y)=\lambda_i$ for i=1, 2 and $\lambda_i(x,y)=0$ for i=3,4,5 in Theorem 3.2, we deduce Theorem 2.9.

Remark 3.10. By letting $\lambda_i(x,y)=\lambda_i$ for i=1, 2,3, $\lambda_i(x,y)=0$ for i=4,5 and s=1 in Theorem 3.2, we get Theorem 2.10.

Remark 3.11. By letting $\lambda_i(x,y)=\lambda_i$ for i=1, 2, 3, 4, 5 and s=1 in Theorem 3.2, we get Theorem 2.11.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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