http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 1, Pages 259–285 (2023)
DOI:10.31801/cfsuasmas.1075157
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: February 17, 2022; Accepted: October 6, 2022

NEW TYPES OF CONNECTEDNESS AND INTERMEDIATE VALUE THEOREM IN IDEAL TOPOLOGICAL SPACES

Ferit YALAZ[1] and Aynur KESKIN KAYMAKCI[2]

Department of Mathematics, Selcuk University, Konya, TÜRKİYE

Abstract. The definitions of new type separated subsets are given in ideal topological spaces. By using these definitions, we introduce new types of connectedness. It is shown that these new types of connectedness are more general than some previously defined concepts of connectedness in ideal topological spaces. The new types of connectedness are compared with well-known connectedness in point-set topology. Then, the intermediate value theorem for ideal topological spaces is given. Also, for some special cases, it is shown that the intermediate value theorem in ideal topological spaces and the intermediate value theorem in topological spaces coincide.

1. Introduction

The concept of ideal in topological spaces was first studied by Kuratowski [16] and Vaidyanathswamy [33]. More properties are given for ideal topological spaces in [10]. In [10, 33], it is shown that the local function of a set is a generalization of the concepts of closure point, ω-accumulation point and condensation point of that set. The concept of ideal was applied not only to topology but also to different areas of mathematics. For example, the Cantor-Bendixson Theorem is generalized in [6]. New special spaces such as $\mathcal{I}$-Rothberger [7], $\mathcal{I}$-Baire [17], $\mathcal{I}$-Resolvable and $\mathcal{I}$-Hyperconnected [3],$\mathcal{I}$-Extremally Disconnected [12], $\mathcal{I}$-Alexandroff and $\mathcal{I}_g$-Alexandroff [4] are defined by using ideal. In addition, the concepts of ideal and local function are studied in fuzzy set theory [28], soft set theory [11] and ditopological texture spaces [15].

Connectedness is a topological invariant. So, the concept of connectedness has an important role in general topology. The intermediate value theorem in calculus

2020 *Mathematics Subject Classification.* 54A05, 54A10, 54C05, 54D05.

Keywords. Ideal topological space, local function, local closure function, connected space, separated subset.

[1] ferit.yalaz@selcuk.edu.tr-Corresponding author; 0000-0001-6805-9357

[2] akeskin@selcuk.edu.tr; 0000-0001-5909-8477.

was generalized by means of connectedness in topological spaces [25]. Many types of connectedness are defined by using the local function in [20,31] and these connectedness are stronger connectedness. The generalization of connectedness has been defined in [18,26] More features of connectedness types given in [20] were examined in [14]. In addition, many operators such as local closure function [1], semi-closure local function [9] , weak semi-local function [35,36], semi-local function [13], a-local function [2,21], $\mathcal{M}$-local function [22], c^*-local function [29], Ω-operator [19] and ψ^*-operator [23] are defined in recent years. In this study, we define new types of connectedness by using local functions and local closure functions. In this way, we generalize all connectedness types in [20]. After that, new types of connectedness are compared with well-known connectedness. Also, we define new components with the help of new types of connectedness. In the last section, we give the intermediate value theorem in ideal topological spaces. For the minimal ideal $\mathcal{I} = \{\emptyset\}$, we show that the intermediate value theorem in general topological spaces and the intermediate value theorem in ideal topological spaces coincide.

2. Preliminaries

In any topological space (U, τ), we denote the interior and the closure of the subset M as $Int(M)$ and $Cl(M)$, respectively. The power set of U is denoted by $\mathcal{P}(U)$. Both open and closed subsets are called clopen. The collection of all open neighborhoods of the point x is denoted by $\tau(x)$.

Definition 1. *[16] Let U be nonempty set and $\mathcal{I} \subseteq \mathcal{P}(U)$. If the following conditions are satisfied:*

(1) $\emptyset \in \mathcal{I}$.
(2) *If $M \in \mathcal{I}$ and $K \subseteq M$, then $K \in \mathcal{I}$.*
(3) *If $M, K \in \mathcal{I}$, then $M \cup K \in \mathcal{I}$.*

then the collection $\mathcal{I}$ is called an ideal on U.

The ideal $\mathcal{I} = \{\emptyset\}$ is called minimal ideal and the ideal $\mathcal{I} = \mathcal{P}(U)$ is called maximal ideal. Although the topology is not needed to define an ideal, some collections of sets in the topological spaces form ideals. In any topological space (U, τ), a subset M is called nowhere dense, if $Int(Cl(M)) = \emptyset$. The subset M is called discrete set if $M \cap M^d = \emptyset$ (where M^d is derived set of M). A subset of U is called meager (or set of first category) if it can be written as a countable union of nowhere dense subsets of U. A subset of U is called relatively compact if its closure is compact. The collection of all nowhere dense subsets $\mathcal{I}_{nw} = \{M \subseteq U : M \text{ is nowhere dense}\}$, the collection of all closed-discrete subsets $\mathcal{I}_{cd} = \{M \subseteq U : M \text{ is closed and discrete}\}$, the collection of all meager subsets $\mathcal{I}_{mg} = \{M \subseteq U : M \text{ is meager set}\}$, the collection of all relatively compact subsets $\mathcal{I}_K = \{M \subseteq U : M \text{ is relatively compact}\}$ and $\mathcal{I}_{f \circ g} = \{A \subseteq U : f \circ g(A) = \emptyset\}$, where $f {\sim}^U g$ are ideals on U [16,24,33].

If (U, τ) is a topological space with an ideal $\mathcal{I}$ on U, this space is called an ideal topological space or briefly $\mathcal{I}$-space. Sometimes we denote this case with the triple $(U, \tau, \mathcal{I})$.

Definition 2. *[16] In any $\mathcal{I}$-space (U, τ), a function $(.)^* : \mathcal{P}(U) \to \mathcal{P}(U)$ is defined by*

$$M^*(\mathcal{I}, \tau) = \{x \in U : O \cap M \notin \mathcal{I} \text{ for every } O \in \tau(x)\}$$

is called the local function of a subset M.

Sometimes we write briefly $M^*(\mathcal{I})$ or M^* instead of $M^*(\mathcal{I}, \tau)$. $M \cup M^* = Cl^*(M)$ is a Kuratowski closure operator. So this operator generates a topology on U. This topology is denoted by τ^* and defined as $\tau^* = \{M \subseteq U : Cl^*(U \setminus M) = (U \setminus M)\}$. Moreover $\tau \subseteq \tau^*$ and so $M \subseteq Cl^*(M) \subseteq Cl(M)$. Elements of τ^* are called $*$-open. The complement of a $*$-open subset is called $*$-closed.

Proposition 1. *[10, 16, 33] Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$.*

(1) *If $M \subseteq K$, then $M^* \subseteq K^*$.*
(2) *$M^* = Cl(M^*) \subseteq Cl(M)$. That is, M^* is closed set.*
(3) *$(M \cup K)^* = M^* \cup K^*$.*
(4) *If $\mathcal{I} = \{\emptyset\}$, then $M^*(\{\emptyset\}) = Cl(M)$.*
(5) *If $\mathcal{I} = \mathcal{P}(U)$, then $M^*(\mathcal{P}(U)) = \emptyset$.*

Definition 3. *[1] In any $\mathcal{I}$-space (U, τ), a function $\Gamma(.) : \mathcal{P}(U) \to \mathcal{P}(U)$ defined by*

$$\Gamma(M)(\mathcal{I}, \tau) = \{x \in U : Cl(O) \cap M \notin \mathcal{I} \text{ for every } O \in \tau(x)\}$$

is called the local closure function of the subset M.

Sometimes we write briefly $\Gamma(M)(\mathcal{I})$ or $\Gamma(M)$ instead of $\Gamma(M)(\mathcal{I}, \tau)$.

The θ-closure of any subset M is defined in [34] as $Cl_\theta(M) = \{x \in U : Cl(O) \cap M \neq \emptyset \text{ for every } O \in \tau(x)\}$.

Proposition 2. *[1] Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$.*

(1) *If $M \subseteq K$, then $\Gamma(M) \subseteq \Gamma(K)$.*
(2) *$\Gamma(M) = Cl(\Gamma(M)) \subseteq Cl_\theta(M)$. That is $\Gamma(M)$ is closed set.*
(3) *$\Gamma(M \cup K) = \Gamma(M) \cup \Gamma(K)$*
(4) *If $\mathcal{I} = \{\emptyset\}$, then $\Gamma(M)(\{\emptyset\}) = Cl_\theta(M)$.*
(5) *If $\mathcal{I} = \mathcal{P}(U)$, then $\Gamma(M)(\mathcal{P}(U)) = \emptyset$.*

Lemma 1. *[1] In any $\mathcal{I}$-space (U, τ), $M^*(\mathcal{I}, \tau) \subseteq \Gamma(M)(\mathcal{I}, \tau)$.*

Definition 4. *[30] Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The subset M is called Γ-dense-in-itself if $M \subseteq \Gamma(M)$.*

Definition 5. *[8] Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The subset M is called $*$-dense-in-itself if $M \subseteq M^*$.*

Nonempty subsets M, K of a topological space (U, τ) are called separated if $Cl(M) \cap K = M \cap Cl(K) = \emptyset$. The topological space (U, τ) is called connected if U is not the union of two separated subsets. The subset M in a topological space is connected if and only if M is not the union of separated subsets in the subspace (M, τ_M) or equivalently M is not the union of two separated subsets in (U, τ). There are many expressions equivalent to definition of connectedness in the literature [5, 25, 32]. We say that the subsets M, K are τ-separated if they are separated subsets in (U, τ). We say that the subset M is τ-connected if it is a connected subset in (U, τ). That an $\mathcal{I}$-space (U, τ) is τ-connected means that the topological space (U, τ) is τ-connected.

Definition 6. *[20] Let (U, τ) be an $\mathcal{I}$-space and M, K be nonempty subsets in this space. These subsets are called $*_*$-separated (resp. $*$-Cl^*-separated, $*$-Cl-separated), if $M^* \cap K = M \cap K^* = M \cap K = \emptyset$ (resp. $M^* \cap Cl^*(K) = Cl^*(M) \cap K^* = M \cap K = \emptyset$, $M^* \cap Cl(K) = Cl(M) \cap K^* = M \cap K = \emptyset$).*

Definition 7. *[20] Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The subset M is called $*_*$-connected (resp. $*$-Cl^*-connected, $*$-Cl-connected) if it is not the union of two $*_*$-separated (resp. $*$-Cl^*-separated, $*$-Cl-separated) subsets.*

From these definitions, the following diagrams are obtained in [20].

$$*\text{-}Cl\text{-separated} \Longrightarrow *\text{-}Cl^*\text{-separated} \Longrightarrow *_*\text{-separated} \Longleftrightarrow \tau^*\text{-separated}$$

Figure 1. Relations among types of separated subsets which are defined via local function

$$\tau^*\text{-connected} \Longleftrightarrow *_*\text{-connected} \Longrightarrow *\text{-}Cl^*\text{-connected} \Longrightarrow *\text{-}Cl\text{-connected}$$

Figure 2. Relations among types of connectedness which are defined via local function

3. New Types of Separated Subsets via Local Closure

Definition 8. *Let (U, τ) be an $\mathcal{I}$-space and M,K be nonempty subsets of U. These subsets are called*

(1) *Γ-Cl-separated if $\Gamma(M) \cap Cl(K) = Cl(M) \cap \Gamma(K) = M \cap K = \emptyset$.*
(2) *Γ-Cl^*-separated if $\Gamma(M) \cap Cl^*(K) = Cl^*(M) \cap \Gamma(K) = M \cap K = \emptyset$.*
(3) *Γ-separated if $\Gamma(M) \cap K = M \cap \Gamma(K) = M \cap K = \emptyset$.*
(4) *Γ-$*$-separated if $\Gamma(M) \cap K^* = M^* \cap \Gamma(K) = M \cap K = \emptyset$.*
(5) *2^*-separated if $M^* \cap K^* = M \cap K = \emptyset$.*

Theorem 1. *Let (U, τ) be an $\mathcal{I}$-space and M,K be nonempty subsets of U.*

(1) *If M,K are Γ-Cl-separated, then they are Γ-Cl^*-separated subsets.*

(2) *If M,K are Γ-Cl-separated, then they are $*$-Cl-separated subsets.*
(3) *If M,K are Γ-Cl^*-separated, then they are Γ-separated subsets.*
(4) *If M,K are Γ-Cl^*-separated, then they are $*$-Cl^*-separated subsets.*
(5) *If M,K are Γ-separated, then they are $*_*$-separated subsets.*
(6) *If M,K are Γ-Cl^*-separated, then they are Γ-$*$-separated subsets.*
(7) *If M,K are Γ-$*$-separated, then they are 2^*-separated subsets.*
(8) *If M,K are $*$-Cl^*-separated, then they are 2^*-separated subsets.*

Proof. Since $M \subseteq Cl^*(M) \subseteq Cl(M)$, $K \subseteq Cl^*(K) \subseteq Cl(K)$ and Definition 8 , (1)-(3)-(6)-(8) are obtained. By using Lemma 1 and Definition 8, (2)-(4)-(5)-(7) are obtained. □

In addition to this theorem, since $\tau \subseteq \tau^*$, τ-separated subsets are τ^*-separated. From Theorem 1 and Figure 1, we obtain the following diagram:

$$\begin{tikzcd}
\Gamma\text{-}Cl\text{-separated} \arrow[r, Rightarrow] \arrow[ddd, Rightarrow] & \Gamma\text{-}Cl^*\text{-separated} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[ddd, Rightarrow, bend left] & \Gamma\text{-separated} \arrow[ddd, Rightarrow] & & \\
& \Gamma\text{-}*\text{-separated} \arrow[d, Rightarrow] & & & \\
& 2^*\text{-separated} & & & \\
*\text{-}Cl\text{-separated} \arrow[r, Rightarrow] & *\text{-}Cl^*\text{-separated} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & *_*\text{-separated} \arrow[r, Leftrightarrow] & \tau^*\text{-separated} & \tau\text{-separated} \arrow[l, Rightarrow]
\end{tikzcd}$$

FIGURE 3. Relations among new types of separated subsets

For this diagram, counterexamples and independent concepts are shown in Example 1 and Example 2.

Example 1. *Let* $\tau = \{\emptyset, U, \{x\}, \{d\}, \{x,y\}, \{x,z\}, \{a,c\}, \{x,d\}, \{x,y,z\}, \{a,c,d\}, \{x,a,c\}, \{x,z,d\}, \{x,y,d\}, \{a,b,c,d\}, \{x,a,c,d\}, \{x,y,a,c\}, \{x,z,a,c\}, \{x,y,z,d\}, \{x,y,z,a,c\}, \{x,a,b,c,d\}, \{x,y,a,c,d\}, \{x,z,a,c,d\}, \{x,z,a,b,c,d\}, \{x,y,a,b,c,d\}, \{x,y,z,a,c,d\}\}$ *be a topology on* $U = \{a,b,c,d,x,y,z\}$ *and let* $\mathcal{I} = \{\emptyset, \{x\}, \{a\}, \{a,x\}\}$ *be an ideal on U. The following table gives information about some subsets of this ideal topological space.*

According to Table:

(1) *C and E are Γ-Cl^*-separated subsets but not Γ-Cl-separated.*
(2) *D and G are $*$-Cl-separated subsets but not Γ-Cl-separated.*
(3) *D and G are $*$-Cl^*-separated but not Γ-Cl^*-separated.*
(4) *C and H are Γ-separated subsets but not Γ-Cl^*-separated.*
(5) *D and G are $*_*$-separated subsets but not Γ-separated.*
(6) *E and F are Γ-$*$-separated subsets but not Γ-Cl^*-separated.*

TABLE 1. Information about some subsets according to the given $\mathcal{I}$-space

$A = \{b\}$	$A^* = \{b\}$	$\Gamma(A) = \{a, b, c, d\}$	$Cl^*(A) = \{b\}$	$Cl(A) = \{b\}$
$B = \{c\}$	$B^* = \{a, b, c\}$	$\Gamma(B) = \{a, b, c\}$	$Cl^*(B) = \{a, b, c\}$	$Cl(B) = \{a, b, c\}$
$C = \{d\}$	$C^* = \{b, d\}$	$\Gamma(C) = \{b, d\}$	$Cl^*(C) = \{b, d\}$	$Cl(C) = \{b, d\}$
$D = \{z\}$	$D^* = \{z\}$	$\Gamma(D) = \{x, y, z\}$	$Cl^*(D) = \{z\}$	$Cl(D) = \{z\}$
$E = \{a, y\}$	$E^* = \{y\}$	$\Gamma(E) = \{x, y, z\}$	$Cl^*(E) = \{a, y\}$	$Cl(E) = \{a, b, c, y\}$
$F = \{b, c\}$	$F^* = \{a, b, c\}$	$\Gamma(F) = \{a, b, c, d\}$	$Cl^*(F) = \{a, b, c\}$	$Cl(F) = \{a, b, c\}$
$G = \{b, y\}$	$G^* = \{b, y\}$	$\Gamma(G) = U$	$Cl^*(G) = \{b, y\}$	$Cl(G) = \{b, y\}$
$H = \{c, y\}$	$H^* = \{a, b, c, y\}$	$\Gamma(H) = \{a, b, c, x, y, z\}$	$Cl^*(H) = \{a, b, c, y\}$	$Cl(H) = \{a, b, c, y\}$
$K = \{d, x\}$	$K^* = \{b, d\}$	$\Gamma(K) = \{b, d\}$	$Cl^*(K) = \{b, d, x\}$	$Cl(K) = \{b, d, x, y, z\}$
$L = \{d, y\}$	$L^* = \{b, d, y\}$	$\Gamma(L) = \{b, d, x, y, z\}$	$Cl^*(L) = \{b, d, y\}$	$Cl(L) = \{b, d, y\}$
$M = \{x, z\}$	$M^* = \{z\}$	$\Gamma(M) = \{x, y, z\}$	$Cl^*(M) = \{x, z\}$	$Cl(M) = \{x, y, z\}$

(7) *G and M are 2^*-separated subsets but not Γ-$*$-separated.*
(8) *E and F are 2^*-separated subsets but not $*$-Cl^*-separated.*
(9) *D and G are $*$-Cl-separated subsets but not Γ-Cl^*-separated. C and E are Γ-Cl^*-separated subsets but not $*$-Cl-separated. That is, the concepts of $*$-Cl-separated and Γ-Cl^*-separated are independent of each other.*
(10) *E and F are Γ-$*$-separated subsets but not $*$-Cl-separated. D and G are $*$-Cl-separated subsets but not Γ-$*$-separated That is, the concepts of $*$-Cl-separated and Γ-$*$-separated are independent of each other.*
(11) *E and F are Γ-$*$-separated subsets but not $*$-Cl^*-separated. D and G are $*$-Cl^*-separated subsets but not Γ-$*$-separated. That is, the concepts of Γ-$*$-separated and $*$-Cl^*-separated are independent of each other.*
(12) *A and E are Γ-$*$-separated subsets but not Γ-separated. C and H are Γ-separated subsets but not Γ-$*$-separated. That is, the concepts of Γ-$*$-separated and Γ-separated are independent of each other.*
(13) *E and F are Γ-$*$-separated subsets but not $*_*$-separated. D and G are $*_*$-separated subsets but not Γ-$*$-separated. That is, the concepts of Γ-$*$-separated and $*_*$-separated are independent of each other.*
(14) *E and F are 2^*-separated subsets but not Γ-separated. C and H are Γ-separated subsets but not 2^*-separated. That is, the concepts of 2^*-separated and Γ-separated are independent of each other.*
(15) *H and K are $*_*$-separated subsets but not 2^*-separated. E and F are 2^*-separated subsets but not $*_*$-separated. That is, the concepts of 2^*-separated and $*_*$-separated are independent of each other.*

(16) *D and G are $*$-Cl-separated subsets but not Γ-separated. C and H are Γ-separated subsets but not $*$-Cl-separated. So, the concepts of $*$-Cl-separated and Γ-separated are independent of each other.*

(17) *D and G are $*$-Cl^*-separated subsets but not Γ-separated. B and L are Γ-separated subsets but not $*$-Cl^*-separated. So, the concepts of $*$-Cl^*-separated and Γ-separated are independent of each other.*

Lemma 2. *Let (U, τ) be $\mathcal{P}(U)$-space and M, K be nonempty subsets of U such that $M \cap K = \emptyset$. Then, the subsets M and K are Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*,2^*, $*_*$)-separated.*

Proof. In this space, since $\Gamma(M) = \Gamma(K) = M^* = K^* = \emptyset$, these subsets are Γ-Cl (Γ-Cl^*,$*$-Cl, $*$-Cl^*, Γ, Γ-$*$,2^*, $*_*$)-separated. □

Example 2. Let $(\mathbb{R}, \tau_L)$ be $\mathcal{P}(\mathbb{R})$-space, where $\mathbb{R}$ is the set of real numbers with left-ray topology τ_L i.e. $\tau_L = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Consider the subsets $M = (-\infty, 3)$ and $K = (3, 5)$. Since $Cl(M) = \mathbb{R}$ and $Cl(K) = [3, +\infty)$, these subsets are not τ-separated. But M and K are Γ-Cl ($*$-Cl, $*$-Cl^*, Γ, Γ-$*$, Γ-Cl^*,2^*)-separated subsets from Lemma 2.

In Example 1, D and G are τ-separated subsets but not Γ-Cl(Γ, Γ-$*$, Γ-Cl^*)-separated. Moreover, B and L are τ-separated subsets but not $*$-Cl ($*$-Cl^*, 2^*) separated.

Consequently, the concepts of Γ-Cl ($*$-Cl, $*$-Cl^*,Γ, Γ-$*$, Γ-Cl^*,2^*)-separated and τ-separated are independent of each other.

Theorem 2. *[27] In any $\mathcal{I}$-space (U, τ), each of the following conditions implies that $M^* = \Gamma(M)$ for any subset M of U:*

(1) *τ has a clopen base.*
(2) *τ is a T_3-space on U.*
(3) *$\mathcal{I} = \mathcal{I}_{cd}$.*
(4) *$\mathcal{I} = \mathcal{I}_K$.*
(5) *$\mathcal{I}_{nw} \subseteq \mathcal{I}$.*
(6) *$\mathcal{I} = \mathcal{I}_{mg}$.*

Corollary 1. *Assume that any of the conditions in Theorem 2 is satisfied and M, K are the subsets in any $\mathcal{I}$-space (U, τ). Then,*

(1) *The subsets M and K are Γ-Cl-separated if and only if they are $*$-Cl-separated.*
(2) *The subsets M and K are Γ-Cl^*-separated if and only if they are $*$-Cl^*-separated.*
(3) *The subsets M and K are Γ-separated if and only if they are $*_*$-separated.*
(4) *The subsets M and K are 2^*-separated if and only if they are Γ-$*$-separated.*

Proof. It is obvious from Definition 8 and Theorem 2. □

Theorem 3. Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. Subsets M and K are both Γ-separated and Γ-$*$-separated if and only if they are Γ-Cl^*-separated.

Proof. Since M and K are both Γ-separated and Γ-$*$-separated,

$$\begin{aligned}\Gamma(M) \cap Cl^*(K) &= \Gamma(M) \cap (K \cup K^*)\\ &= (\Gamma(M) \cap K) \cup (\Gamma(M) \cap K^*)\\ &= \emptyset\end{aligned}$$

$$\begin{aligned}Cl^*(M) \cap \Gamma(K) &= (M \cup M^*) \cap \Gamma(K)\\ &= (M \cap \Gamma(K)) \cup (M^* \cap \Gamma(K))\\ &= \emptyset\end{aligned}$$

and $M \cap K = \emptyset$. So, M and K are Γ-Cl^*-separated subsets.

Conversely, let M and K be Γ-Cl^*-separated subsets. From Figure 3, these subsets are both Γ-separated and Γ-$*$-separated. $\square$

Theorem 4. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. Subsets M and K are both $*_*$-separated and 2^*-separated if and only if these subsets are $*$-Cl^*-separated.*

Proof. Since M and K are both $*_*$-separated and 2^*-separated,

$$\begin{aligned}M^* \cap Cl^*(K) &= M^* \cap (K \cup K^*)\\ &= (M^* \cap K) \cup (M^* \cap K^*)\\ &= \emptyset\end{aligned}$$

$$\begin{aligned}Cl^*(M) \cap K^* &= (M \cup M^*) \cap K^*\\ &= (M \cap K^*) \cup (M^* \cap K^*)\\ &= \emptyset\end{aligned}$$

and $M \cap K = \emptyset$. So, M and K are $*$-Cl^*-separated subsets.

Conversely, let M and K be $*$-Cl^*-separated subsets. From Figure 3, these subsets are both $*_*$-separated and 2^*-separated. $\square$

Theorem 5. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-separated.*
(2) *The subsets M, K are Γ-dense-in-itself.*
(3) $M \cup K \in \tau$.

then $M \in \tau$ and $K \in \tau$.

Proof. Since the subsets M, K are Γ-separated, $M \cap \Gamma(K) = \emptyset$. So, $M \subseteq (U \setminus \Gamma(K))$. From Proposition 2-(2), $U \setminus \Gamma(K)$ is open set and hence $(M \cup K) \cap (U \setminus \Gamma(K)) = M$ is an open subset. Similarly, it can be showed that the subset K is open. $\square$

Corollary 2. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-Cl (Γ-Cl^*)-separated.*
(2) *The subsets M, K are Γ-dense-in-itself.*
(3) $M \cup K \in \tau$.

then $M \in \tau$ and $K \in \tau$.

Proof. From Figure 3 and Theorem 5 , it is obtained. □

Theorem 6. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-separated.*
(2) *The subsets M, K are Γ-dense-in-itself.*
(3) $M \cup K \in \tau^*$.

then $M \in \tau^*$ and $K \in \tau^*$.

Proof. Since the subsets M, K are Γ-separated, $M \cap \Gamma(K) = \emptyset$. So, $M \subseteq (U \setminus \Gamma(K))$. From Proposition 2-(2), $U \setminus \Gamma(K)$ is open set. Since $\tau \subseteq \tau^*$, $U \setminus \Gamma(K) \in \tau^*$ and hence $(M \cup K) \cap (U \setminus \Gamma(K)) = M$ is in τ^*. Similarly, it can be showed that the subset K is in τ^*. □

Corollary 3. *Let (U, τ) be an $\mathcal{I}$-space space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-Cl (Γ-Cl^*)-separated.*
(2) *The subsets M, K are Γ-dense-in-itself.*
(3) $M \cup K \in \tau^*$.

then $M \in \tau^*$ and $K \in \tau^*$.

Proof. It is obtained from Figure 3 and Theorem 6. □

Theorem 7. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-$*$-separated.*
(2) *The subsets M, K are $*$-dense-itself.*
(3) $M \cup \Gamma(K) \in \tau$ *and* $\Gamma(M) \cup K \in \tau$.

then $\Gamma(M)$ and $\Gamma(K)$ are clopen subsets.

Proof. From Proposition 2-(2), $\Gamma(M)$ and $\Gamma(K)$ are closed subsets. We only show that they are open subsets. Since the subsets M, K are Γ-$*$-separated, $\Gamma(M) \cap K^* = \emptyset$. So, $\Gamma(M) \subseteq (U \setminus K^*)$. From Proposition 1-(2), $U \setminus K^*$ is open set and hence $(\Gamma(M) \cup K) \cap (U \setminus K^*) = \Gamma(M)$ is open. Similarly, it can be showed that the subset $\Gamma(K)$ is open. □

Corollary 4. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* Γ-Cl *(* Γ-Cl^**)-separated.*
(2) *The subsets* M, K *are* $*$-*dense-itself.*
(3) $M \cup \Gamma(K) \in \tau$ *and* $\Gamma(M) \cup K \in \tau$.

then $\Gamma(M)$ *and* $\Gamma(K)$ *are clopen subsets.*

Proof. It is obtained from Figure 3 and Theorem 7. □

Theorem 8. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$*. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* Γ-$*$*-separated.*
(2) *The subsets* M, K *are* Γ*-dense-itself.*
(3) $M^* \cup K \in \tau$ *and* $M \cup K^* \in \tau$.

then M^* *and* K^* *are clopen subsets.*

Proof. From Proposition 1-(2), M^* and K^* are closed subsets. We must show that they are open subsets. Since the subsets M, K are Γ-$*$-separated, $M^* \cap \Gamma(K) = \emptyset$. So $M^* \subseteq U \setminus \Gamma(K)$. Since $U \setminus \Gamma(K)$ is open subset, $(M^* \cup K) \cap (U \setminus \Gamma(K)) = M^* \in \tau$. Similarly, it can be showed that the subset K^* is open.

□

Corollary 5. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$*. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* Γ-Cl *(* Γ-Cl^**)-separated.*
(2) *The subsets* M, K *are* Γ*-dense-itself.*
(3) $M^* \cup K \in \tau$ *and* $M \cup K^* \in \tau$.

then M^* *and* K^* *are clopen subsets.*

Proof. It is obtained from Figure 3 and Theorem 8. □

Theorem 9. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$*. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* 2^**-separated.*
(2) *The subsets* M, K *are* $*$-*dense-itself.*
(3) $M^* \cup K \in \tau$ *and* $M \cup K^* \in \tau$.

then M^* *and* K^* *are clopen subsets.*

Proof. From Proposition 1-(2), M^* and K^* are closed subsets. We must show that they are open subsets. Since the subsets M, K are 2^*-separated, $M^* \cap K^* = \emptyset$. So, $M^* \subseteq U \setminus K^*$. Since $U \setminus K^*$ is open, $(M^* \cup K) \cap (U \setminus K^*) = M^* \in \tau$. Similarly, it can be showed that the subset K^* is open. □

Corollary 6. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$*. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* Γ-Cl *(* Γ-Cl^**,* Γ-$*$*)-separated.*
(2) *The subsets* M, K *are* $*$-*dense-itself.*

(3) *$M^* \cup K \in \tau$ and $M \cup K^* \in \tau$.*

then M^* and K^* are clopen subsets.

Proof. From Figure 3 and Theorem 9, it is obtained. □

Theorem 10. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are $*_*$-separated.*
(2) *The subsets M, K are $*$-dense-itself.*
(3) *$M \cup K \in \tau$.*

then M and K are open subsets.

Proof. Since the subsets M, K are $*_*$-separated, $M \cap K^* = \emptyset$. So $M \subseteq U \setminus K^*$. Since $U \setminus K^*$ is open subset, $(M \cup K) \cap (U \setminus K^*) = M$ is in τ. Similarly, it can show that the subset K is open. □

Corollary 7. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-Cl (Γ-Cl^*, Γ, $*$-Cl, $*$-Cl^*, τ)-separated.*
(2) *The subsets M, K are $*$-dense-itself.*
(3) *$M \cup K \in \tau$.*

then M and K are open subsets.

Proof. From Figure 3 and Theorem 10, it is obtained. □

Theorem 11. *Let (U, τ) be $\{\emptyset\}$-space and $M, K \subseteq U$. Then the following statements are equivalent:*

(1) *The subsets M and K are $*_*$-separated.*
(2) *The subsets M and K are τ-separated.*

Proof. Since $M^*(\{\emptyset\}) = Cl(M)$, $K^*(\{\emptyset\}) = Cl(K)$, these expressions are equivalent. □

Theorem 12. *Let (U, τ) be $\{\emptyset\}$-space and $M, K \subseteq U$. Then the following statements are equivalent:*

(1) *The subsets M and K are 2^*-separated.*
(2) *The subsets M and K are $*$-Cl^*-separated.*
(3) *The subsets M and K are $*$-Cl-separated.*

Proof. Since $M^*(\{\emptyset\}) = Cl^*(M) = Cl(M)$ and $K^*(\{\emptyset\}) = Cl^*(K) = Cl(K)$, these expressions are equivalent. □

Theorem 13. *Let (U, τ) be $\{\emptyset\}$-space and $M, K \subseteq U$. Then the following statements are equivalent:*

(1) *The subsets M and K are Γ-$*$-separated.*
(2) *The subsets M and K are Γ-Cl^*-separated.*

(3) *The subsets M and K are Γ-Cl-separated.*

Proof. Since $M^*(\{\emptyset\}) = Cl^*(M) = Cl(M)$ and $K^*(\{\emptyset\}) = Cl^*(K) = Cl(K)$, these expressions are equivalent. $\square$

4. New Types of Connectedness via Local Closure

Definition 9. *Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The subset M is called Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected if it is not the union of two Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-separated subsets in $\mathcal{I}$-space (U, τ). Otherwise, the subset M is called not Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected . Particularly, if U is Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected, the $\mathcal{I}$-space (U, τ) is called Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected $\mathcal{I}$-space.*

Theorem 14. *In any $\mathcal{I}$-space,*

(1) *Every Γ-Cl^*-connected subset is Γ-Cl-connected.*
(2) *Every $*$-Cl-connected subset is Γ-Cl-connected.*
(3) *Every Γ-connected subset is Γ-Cl^*-connected.*
(4) *Every $*$-Cl^*-connected subset is Γ-Cl^*-connected.*
(5) *Every $*_*$-connected subset is Γ-connected.*
(6) *Every Γ-$*$-connected subset is Γ-Cl^*-connected.*
(7) *Every 2^*-connected subset is Γ-$*$-connected.*
(8) *Every 2^*-connected subset is $*$-Cl^*-connected.*

Proof. (1) Let M be Γ-Cl^*-connected subset. Suppose that it is not Γ-Cl-connected. So, there are subsets K, S which are Γ-Cl-separated and $K \cup S = M$. From Theorem 1-(1), the subsets K and S are Γ-Cl^*-separated. Hence, the subset M is not Γ-Cl^*-connected. This is a contradiction. Consequently, the subset M is Γ-Cl-connected.

By using Theorem 1 (or Figure 3), other proofs are obtained similarly. $\square$

The following diagram is obtained by Theorem 14 and Figure 2.

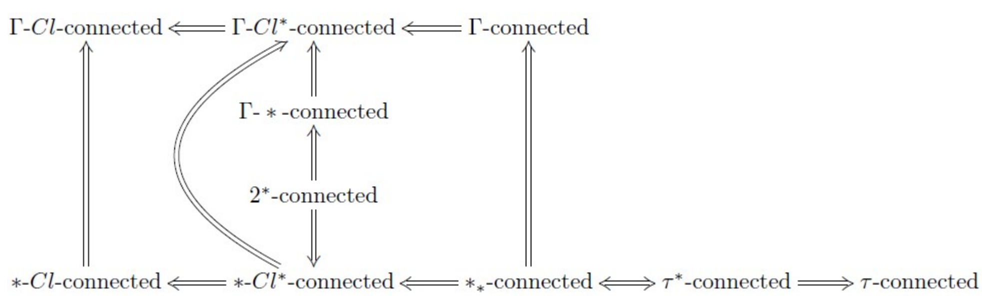

Figure 4. Relations among new types of connectedness

For this diagram, counterexamples and independent concepts are shown in Example 3 and Example 4.

Example 3. *Consider the $\mathcal{I}$-space in Example 1.*

(1) *The subset $P = \{y, z\}$ is Γ (resp. Γ-Cl^*, Γ-Cl, Γ-$*$)-connected but not $*_*$ (resp. $*$-Cl^*, $*$-Cl, 2^*)-connected.*
(2) *The subset $R = \{a, d\}$ is Γ-Cl-connected but not Γ-Cl^*-connected.*
(3) *The subset $S = \{c, d\}$ is Γ-Cl^*-connected but not Γ-connected.*
(4) *The subset $T = \{a, c\}$ is $*$-Cl^* (resp. Γ-Cl^*)-connected but not 2^* (resp. Γ-$*$)-connected.*
(5) *The subset $P = \{y, z\}$ is Γ-Cl^*-connected but not $*$-Cl-connected. The subset $R = \{a, d\}$ is $*$-Cl-connected but not Γ-Cl^*-connected. That is, the concepts of $*$-Cl-connected and Γ-Cl^*-connected are independent of each other.*
(6) *The subset $P = \{y, z\}$ is Γ-connected but not $*$-Cl-connected. The subset $R = \{a, d\}$ is $*$-Cl-connected but not Γ-connected. That is, the concepts of Γ-connected and $*$-Cl-connected are independent of each other.*
(7) *The subset $P = \{y, z\}$ is Γ-$*$-connected but not $*$-Cl-connected. The subset $R = \{a, d\}$ is $*$-Cl-connected but not Γ-$*$-connected. That is, the concepts of Γ-$*$-connected and $*$-Cl-connected are independent of each other.*
(8) *The subset $P = \{y, z\}$ is Γ-$*$-connected but not $*$-Cl^*-connected. The subset $T = \{a, c\}$ is $*$-Cl^*-connected but not Γ-$*$-connected. That is, the concepts of Γ-$*$-connected and $*$-Cl^*-connected are independent of each other.*
(9) *The subset $P = \{y, z\}$ is Γ-connected but not 2^*-connected. The subset $S = \{c, d\}$ is 2^*-connected but not Γ-connected. That is, the concepts of Γ-connected and 2^*-connected are independent of each other.*
(10) *The subset $P = \{y, z\}$ is Γ-connected but not $*$-Cl^*-connected. The subset $S = \{c, d\}$ is $*$-Cl^*-connected but not Γ-connected. That is, the concepts of Γ-connected and $*$-Cl^*-connected are independent of each other.*
(11) *The subset $S = \{c, d\}$ is Γ-$*$-connected but not Γ-connected. The subset $T = \{a, c\}$ is Γ-connected but not Γ-$*$-connected. That is, the concepts of Γ-$*$-connected and Γ-connected are independent of each other.*
(12) *The subset $S = \{c, d\}$ is Γ-$*$-connected but not $*_*$-connected. The subset $T = \{a, c\}$ is $*_*$-connected but not Γ-$*$-connected. That is, the concepts of Γ-$*$-connected and $*_*$-connected are independent of each other.*
(13) *The subset $S = \{c, d\}$ is 2^*-connected but not $*_*$-connected. The subset $T = \{a, c\}$ is $*_*$-connected but not 2^*-connected. That is, the concepts of 2^*-connected and $*_*$-connected are independent of each other.*

Lemma 3. *Let (U, τ) be $\mathcal{P}(U)$-space and M be a subset of U. If the subset M has more than one element, it is not Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*,2^*,$*_*$)-connected.*

Proof. Let K,S be nonempty subsets such that $M = K \cup S$ and $K \cap S = \emptyset$. From Lemma 2, the subsets K and S are Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*,2^*, $*_*$)-separated. So, M is not Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*,2^*, $*_*$)-connected. □

Example 4. *Consider the $\mathcal{P}(\mathbb{R})$-space in Example 2. The subset $M = (-\infty, 3)$ is τ_L-connected but not Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected from Lemma 3.*

According to the $\mathcal{I}$-space given in Example 1, $S = \{c, d\}$ is Γ-Cl ($*$-Cl^*, $*$-Cl, Γ-$*$, Γ-Cl^*, 2^*)-connected but not τ-connected. Moreover, the subset $P = \{y, z\}$ is Γ-connected but not τ-connected.

Consequently, the concepts of Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected and τ-connected are independent of each other.

Lemma 4. *[1] Let (U, τ) be a topological space and $M \subseteq U$. If the subset M is open, $Cl(M) = Cl_\theta(M)$.*

Lemma 5. *If the subset M is clopen in any $\mathcal{I}$-space,*

$$M^* \subseteq \Gamma(M) \subseteq M = Cl(M) = Cl_\theta(M).$$

Proof. It is obtained by Lemma 4, Lemma 1 and Proposition 2-(2). □

Theorem 15. *If any $\mathcal{I}$-space (U, τ) is Γ-Cl-connected, then it is τ-connected. That is, if the set U is Γ-Cl-connected, then U is τ-connected.*

Proof. Suppose that U is Γ-Cl-connected but not τ-connected. So, there is a clopen proper subset M in this space. From Lemma 5,

$$\Gamma(M) \cap Cl(U \setminus M) \subseteq M \cap (U \setminus M) = \emptyset$$
$$Cl(M) \cap \Gamma(U \setminus M) \subseteq M \cap (U \setminus M) = \emptyset$$

and $M \cap (U \setminus M) = \emptyset$. So, the subsets M and $(U \setminus M)$ are Γ-Cl-separated. Since $M \cup (U \setminus M) = U$, U is not Γ-Cl-connected. This is a contradiction. As a result, U is τ-connected. □

Theorem 16. *If any $\mathcal{I}$-space (U, τ) is Γ-Cl^* (Γ, Γ-$*$, 2^*, $*$-Cl, $*$-Cl^*, $*_*$)-connected, then it is τ-connected.*

Proof. The proof is obtained by Figure 4 and Theorem 15. □

Corollary 8. *Suppose that any of the conditions in Theorem 2 is satisfied and let M be subsets in any $\mathcal{I}$-space (U, τ). Then,*

(1) *The subset M is Γ-Cl-connected if and only if it is $*$-Cl-connected.*
(2) *The subset M is Γ-Cl^*-connected if and only if it is $*$-Cl^*-connected.*
(3) *The subset M is Γ-connected if and only if it is $*_*$-connected.*
(4) *The subset M is 2^*-connected if and only if it is Γ-$*$-connected.*

Proof. It is obvious from Definition 9 and Theorem 2. □

Corollary 9. *Let (U, τ) be an $\mathcal{I}$-space and M, K be subsets of U .*

(1) *If the subsets M,K are both Γ-separated, Γ-$*$-separated subsets and $S = M \cup K$, then S is not Γ-Cl^*-connected subset.*
(2) *If the subset S is not Γ-Cl^*-connected, there are both Γ-separated and Γ-$*$-separated subsets M,K such that $M \cup K = S$.*
(3) *If the subsets M,K are both 2^*-separated, $*_*$-separated subsets and $S = M \cup K$, then S is not $*$-Cl^*-connected subset.*
(4) *If the subset S is not $*$-Cl^*-connected, there are both 2^*-separated and $*_*$-separated subsets M,K such that $M \cup K = S$.*

Proof. It is obtained from Theorem 3 and Theorem 4. □

The following corollaries are obtained from Theorem 11, Theorem 12 and Theorem 13, respectively.

Corollary 10. *Let (U, τ) be $\{\emptyset\}$-space and $M \subseteq U$. Then the following statements are equivalent:*

(1) *The subset M is $*_*$-connected.*
(2) *The subset M is τ-connected.*

Corollary 11. *Let (U, τ) be $\{\emptyset\}$-space and $M \subseteq U$. Then the following statements are equivalent:*

(1) *The subset M is 2^*-connected.*
(2) *The subset M is $*$-Cl^*-connected.*
(3) *The subset M is $*$-Cl-connected.*

Corollary 12. *Let (U, τ) be $\{\emptyset\}$-space and $M \subseteq U$. Then the following statements are equivalent:*

(1) *The subset M is Γ-$*$-connected.*
(2) *The subset M is Γ-Cl^*-connected.*
(3) *The subset M is Γ-Cl-connected.*

Theorem 17. *Let (U, τ) be $\{\emptyset\}$-space and $M \subseteq U$. If the subset M is τ-connected, then it is Γ (Γ-Cl^*, Γ-Cl, $*$-Cl, $*$-Cl^*, Γ-$*$, 2^*)-connected.*

Proof. Let the subset M be τ-connected. From Corollary 10, M is $*_*$-connected. So, it is Γ (Γ-Cl^*, Γ-Cl, $*$-Cl, $*$-Cl^*)-connected by Figure 4. Moreover M is 2^*-connected and Γ-$*$-connected by Corollary 11 and Corollary 12, respectively. □

Considering $\{\emptyset\}$-space (U, τ) given in Theorem 17, it is seen that Γ (Γ-Cl^*, Γ-Cl, $*$-Cl, $*$-Cl^*, Γ-$*$, 2^*)-connectedness is more general concept than the well-known τ-connectedness. Moreover, in this space, $*_*$-connectedness and τ-connectedness are coincident concepts from Corollary 10. However, in any $\mathcal{I}$-space (U, τ), when τ-connectedness of only the set U is considered in Theorem 15 and Theorem 16, it is seen that the concept of τ-connectedness is more general than the concept of Γ (Γ-Cl^*, Γ-Cl, $*$-Cl, $*$-Cl^*, Γ-$*$, 2^*)-connectedness. So the following result is easily obtained.

Corollary 13. *Let (U, τ) be $\{\emptyset\}$-space. The following statements are equivalent:*

(1) *The set U is Γ-Cl-connected.*
(2) *The set U is Γ-Cl^*-connected.*
(3) *The set U is Γ-$*$-connected.*
(4) *The set U is 2^*-connected.*
(5) *The set U is $*$-Cl-connected.*
(6) *The set U is $*$-Cl^*-connected.*
(7) *The set U is $*_*$-connected.*
(8) *The set U is τ-connected.*
(9) *The set U is Γ-connected.*

Proof. It is obtained by Theorem 15, Theorem 16 and Theorem 17. $\square$

5. Theorems on New Types of Connectedness via Local Closure

Theorem 18. *Let (U, τ) be an $\mathcal{I}$-space. If M is Γ-Cl-connected subset of U and S, T are Γ-Cl-separated subsets such that $M \subseteq S \cup T$, then either $M \subseteq S$ or $M \subseteq T$.*

Proof. Since $M = (M \cap S) \cup (M \cap T)$ and the subsets S, T are Γ-Cl-separated,

$$\Gamma(M \cap S) \cap Cl(M \cap T) \subseteq \Gamma(S) \cap Cl(T) = \emptyset$$
$$Cl(M \cap S) \cap \Gamma(M \cap T) \subseteq Cl(S) \cap \Gamma(T) = \emptyset$$

and $(M \cap S) \cap (M \cap T) \subseteq S \cap T = \emptyset$. If $(M \cap S)$ and $(M \cap T)$ are nonempty subsets, the subset M is not Γ-Cl-connected. This is a contradiction. So, either $(M \cap S) = \emptyset$ or $(M \cap T) = \emptyset$. Since $M \subseteq S \cup T$, either $M \subseteq S$ or $M \subseteq T$. $\square$

Theorem 19. *Let (U, τ) be an $\mathcal{I}$-space. If M is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset of U and S, T are Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-separated subsets such that $M \subseteq S \cup T$, then either $M \subseteq S$ or $M \subseteq T$.*

Proof. It is obtained similar to the proof of Theorem 18. $\square$

Theorem 20. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If M is Γ-Cl-connected subset and $M \subseteq K \subseteq \Gamma(M)$, then K is Γ-Cl-connected subset.*

Proof. Suppose that the subset K is not Γ-Cl-connected. Then, there exist Γ-Cl-separated nonempty subsets T, S such that $T \cup S = K$. Since the subsets S and T are Γ-Cl-separated and $M \subseteq K = S \cup T$, by using Theorem 18, we have $M \subseteq S$ or $M \subseteq T$. Suppose that $M \subseteq S$. Then, from Proposition 2-(1), $\Gamma(M) \subseteq \Gamma(S)$. From the hypothesis, $T \subseteq K \subseteq \Gamma(M) \subseteq \Gamma(S)$. Since $\Gamma(M)$, $\Gamma(S)$ are closed subsets by Proposition 2-(2), $Cl(T) \subseteq \Gamma(M) \subseteq \Gamma(S)$, and since the subsets S and T are Γ-Cl-separated, $Cl(T) = Cl(T) \cap \Gamma(M) \subseteq Cl(T) \cap \Gamma(S) = \emptyset$. That is, $T = \emptyset$. This is a contradiction. Similarly, a contradiction is obtained if $M \subseteq T$. Consequently, the subset K is Γ-Cl-connected. □

Theorem 21. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If M is Γ-Cl^*(resp. Γ)-connected subset of U and $M \subseteq K \subseteq \Gamma(M)$, then K is Γ-Cl^*(resp. Γ)-connected subset.*

Proof. It is obtained similar to the proof of Theorem 20. □

Corollary 14. *Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$.*

(1) *If M is both $*$-dense-in-itself and Γ-Cl-connected subset, then M^* is Γ-Cl-connected.*
(2) *If M is both $*$-dense-in-itself and Γ-Cl^*(resp. Γ)-connected subset, then M^* is Γ-Cl^*(resp. Γ)-connected.*
(3) *If M is both Γ-dense-in-itself and Γ-Cl-connected subset, then $\Gamma(M)$ is Γ-Cl-connected.*
(4) *If M is both Γ-dense-in-itself and Γ-Cl^*(resp. Γ)-connected subset, then $\Gamma(M)$ is Γ-Cl^*(resp. Γ)-connected.*
(5) *If M is both Γ-dense-in-itself and Γ-Cl-connected subset, then $Cl(M)$ is Γ-Cl-connected.*
(6) *If M is both Γ-dense-in-itself and Γ-Cl^*(resp. Γ)-connected subset, then $Cl(M)$ is Γ-Cl^*(resp. Γ)-connected.*

Proof. (1) Since M is $*$-dense-in-itself and by Lemma 1, $M \subseteq M^* \subseteq \Gamma(M)$. From Theorem 20 , M^* is Γ-Cl-connected subset.

(2) By using Theorem 21, it is obtained similar to the proof of (1).
(3) Since M is Γ-dense-in-itself, we have $M \subseteq \Gamma(M) \subseteq \Gamma(M)$. From Theorem 20, $\Gamma(M)$ is Γ-Cl-connected subset.
(4) By using Theorem 21, it is obtained similar to the proof of (3).
(5) Since M is Γ-dense-in-itself, $M \subseteq \Gamma(M)$ and so $M \subseteq Cl(M) \subseteq Cl(\Gamma(M))$. Since $\Gamma(M)$ is closed subset from Proposition 2-(2), $M \subseteq Cl(M) \subseteq Cl(\Gamma(M)) = \Gamma(M)$. That is, $M \subseteq Cl(M) \subseteq \Gamma(M)$ and M is Γ-Cl-connected from the hypothesis. Using Theorem 20, we obtain that $Cl(M)$ is Γ-Cl-connected subset.
(6) Since M is Γ-dense-in-itself, $M \subseteq Cl(M) \subseteq \Gamma(M)$ is obtained as in the proof of (5). M is Γ-Cl^*(resp. Γ)-connected from the hypothesis. By using Theorem 21, we obtain that $Cl(M)$ is Γ-Cl^*(resp. Γ)-connected subset.

□

Theorem 22. *Let (U, τ) be an $\mathcal{I}$-space and $\{N_k : k \in \Delta\}$ be a nonempty collection of Γ-Cl-connected subsets of U (where Δ is arbitrary index set). If $\bigcap_{k\in\Delta} N_k \neq \emptyset$, then $\bigcup_{k\in\Delta} N_k$ is Γ-Cl-connected.*

Proof. Suppose that $\bigcup_{k\in\Delta} N_k$ is not Γ-Cl-connected. Then, there exist Γ-Cl-separated nonempty subsets T, S such that $T \cup S = \bigcup_{k\in\Delta} N_k$. Since $\bigcap_{k\in\Delta} N_k \neq \emptyset$, there exists a point $x \in N_k$ for every $k \in \Delta$. Since T, S are Γ-Cl-separated and $x \in \bigcup_{k\in\Delta} N_k$, we have $x \in T$ or $x \in S$. Suppose now that $x \in S$. So, $N_k \cap S \neq \emptyset$ for every $k \in \Delta$. Then, by Theorem 18, $N_k \subseteq S$ for every $k \in \Delta$. Therefore, we obtain $\bigcup_{k\in\Delta} N_k \subseteq S$. That is, $T = \emptyset$. This is a contradiction. Similarly, a contradiction is also obtained if we suppose that $x \in T$. Consequently, $\bigcup_{k\in\Delta} N_k$ is Γ-Cl-connected. □

Theorem 23. *Let (U, τ) be an $\mathcal{I}$-space and $\{N_k : k \in \Delta\}$ be a nonempty collection of Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subsets of U. If $\bigcap_{k\in\Delta} N_k \neq \emptyset$, then $\bigcup_{k\in\Delta} N_k$ is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected.*

Proof. By using Theorem 19, it is obtained similar to the proof of Theorem 22. □

Theorem 24. *Let (U, τ) be an $\mathcal{I}$-space, $\{N_k : k \in \Delta\}$ be a nonempty collection of Γ-Cl-connected subsets and M be Γ-Cl-connected subset. If $M \cap N_k \neq \emptyset$ for every $k \in \Delta$, then $M \cup (\bigcup_{k\in\Delta} N_k)$ is a Γ-Cl-connected subset.*

Proof. For every $k \in \Delta$, since N_k and M are Γ-Cl-connected subsets such that $M \cap N_k \neq \emptyset$, by using Theorem 22, we obtain that the subset $M \cup N_k$ are Γ-Cl-connected for every $k \in \Delta$. Since $M \subseteq M \cup N_k$ for every $k \in \Delta$, $M \subseteq \bigcap_{k\in\Delta}(M \cup N_k) \neq \emptyset$. From Theorem 22, $\bigcup_{k\in\Delta}(M \cup N_k) = M \cup (\bigcup_{k\in\Delta} N_k)$ is a Γ-Cl-connected subset. □

Theorem 25. *Let (U, τ) be an $\mathcal{I}$-space, $\{N_k : k \in \Delta\}$ be a nonempty collection of Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subsets and M be Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset. If $M \cap N_k \neq \emptyset$ for every $k \in \Delta$, then $M \cup (\bigcup_{k\in\Delta} N_k)$ is a Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset.*

Proof. By using Theorem 23, it is obtained similar to the proof of Theorem 24. □

Theorem 26. *Let (U, τ) be an $\mathcal{I}$-space and $\{N_k : k \in \mathbb{N}\}$ be a nonempty collection of Γ-Cl-connected subsets such that $N_k \cap N_{k+1} \neq \emptyset$ for every $k \in \mathbb{N}$. Then $\bigcup_{k\in\mathbb{N}} N_k$ is a Γ-Cl-connected subset.*

Proof. We can use induction method. Firstly, N_1 is Γ-Cl-connected. Now assume that the theorem is true for $k-1$. That is, $N_1 \cup N_2 \cup ... \cup N_{k-1}$ is Γ-Cl-connected. From Theorem 22, $M_k = N_1 \cup N_2 \cup ... \cup N_k$ is Γ-Cl-connected and $\bigcap_{k\in\mathbb{N}} M_k = N_1 \neq \emptyset$. Again from Theorem 22, $\bigcup_{k\in\mathbb{N}} M_k = \bigcup_{k\in\mathbb{N}} N_k$ is a Γ-Cl-connected subset. □

Theorem 27. *Let (U, τ) be an $\mathcal{I}$-space and $\{N_k : k \in \mathbb{N}\}$ be a nonempty collection of Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subsets such that $N_k \cap N_{k+1} \neq \emptyset$ for every $k \in \mathbb{N}$. Then $\bigcup_{k\in\mathbb{N}} N_k$ is a Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset.*

Proof. By using Theorem 23, it is obtained similar to the proof of Theorem 26. □

Theorem 28. *Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. If for each distinct pair of points $a, b \in M$ there is a Γ-Cl-connected subset E such that $a, b \in E \subseteq M$, then M is Γ-Cl-connected subset.*

Proof. Suppose that the subset M is not Γ-Cl-connected. Then there are Γ-Cl-separated nonempty subsets S, K such that $S \cup K = M$. Let $a \in S$ and $b \in K$. By hypothesis, there is Γ-Cl-connected subset E such that $a, b \in E \subseteq M$. Since $E \subseteq S \cup K$, $E \subseteq S$ or $E \subseteq K$ by Theorem 18. Suppose that $E \subseteq S$. So, $b \in S \cap K \neq \emptyset$. This is a contradiction. Similarly, a contradiction is obtained if we suppose that $E \subseteq K$. □

Theorem 29. *Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. If for each distinct pair of points $a, b \in M$ there is a Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset E such that $a, b \in E \subseteq M$, then M is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset.*

Proof. By using Theorem 19, it is obtained similar to the proof of Theorem 28. □

Theorem 30. *Let (U, τ) be Γ-Cl-connected $\mathcal{I}$-space, M be Γ-Cl-connected subset and K, C be Γ-Cl-separated subsets. If $U \setminus M = K \cup C$, then both $M \cup K$ and $M \cup C$ are Γ-Cl-connected subsets.*

Proof. Suppose that $M \cup K$ is not Γ-Cl-connected. There are Γ-Cl-separated nonempty subsets S, T such that $S \cup T = M \cup K$. Since $M \subseteq S \cup T = M \cup K$ and M is a Γ-Cl-connected subset, $M \subseteq S$ or $M \subseteq T$, by Theorem 18. Suppose that $M \subseteq T$. Then, $S \cup T = M \cup K \subseteq T \cup K$, and so $S \subseteq K$. Since K and C are Γ-Cl-separated subsets, S and C are Γ-Cl-separated subsets. So,

$$\Gamma(S) \cap Cl(T \cup C) = [\Gamma(S) \cap Cl(T)] \cup [\Gamma(S) \cap Cl(C)] = \emptyset$$
$$Cl(S) \cap \Gamma(T \cup C) = [Cl(S) \cap \Gamma(T)] \cup [Cl(S) \cap \Gamma(C)] = \emptyset$$

and $S \cap (T \cup C) = (S \cap T) \cup (S \cap C) = \emptyset$. As a result, S and $T \cup C$ are Γ-Cl-separated subsets. Since $U \setminus M = K \cup C$, we have $U = M \cup (K \cup C) = S \cup (T \cup C)$. This contradicts with the fact that (U, τ) is an Γ-Cl-connected $\mathcal{I}$-space. Consequently, the subset $M \cup K$ is Γ-Cl-connected.

If $M \subseteq S$, a contradiction can be obtained again in this way. Similarly, it can be proved that $M \cup C$ is Γ-Cl-connected subset. □

Theorem 31. *Let (U, τ) be Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected $\mathcal{I}$-space, M be a Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset and K, C be Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-separated subsets. If $U \setminus M = K \cup C$, then $M \cup K$ and $M \cup C$ are Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subsets.*

Proof. By using Theorem 19, it is obtained similar to the proof of Theorem 30. □

Theorem 32. *Let (U, τ) be an $\mathcal{I}$-space and M, K be Γ-Cl-connected subsets of U. If these subsets are not Γ-Cl-separated, then $M \cup K$ is Γ-Cl-connected subset.*

Proof. Suppose that $M \cup K$ is not Γ-Cl-connected subset. So, there are Γ-Cl-separated nonempty subsets S, T such that $S \cup T = M \cup K$. Then, we have $M \subseteq S \cup T$ and $K \subseteq S \cup T$. From Theorem 18, there are four cases to be considered:

(1) $M \subseteq S$ and $K \subseteq S$

(2) $M \subseteq S$ and $K \subseteq T$
(3) $M \subseteq T$ and $K \subseteq T$
(4) $M \subseteq T$ and $K \subseteq S$

If case (1) or case (3) is satisfied, then $T = \emptyset$ or $S = \emptyset$, respectively. Both are contradiction.

Suppose that case (2) is satisfied. If $M = S$ and $K = T$, then the subsets M and K are Γ-Cl-separated. This is a contradiction. If $M \subsetneq S$, then $T \subsetneq K$ due to $S \cup T = M \cup K$. Similarly, if $K \subsetneq T$, then $S \subsetneq M$. These contradict with case (2). Additionally, for case (4), we obtain similar contradictions. Consequently, $M \cup K$ is Γ-Cl-connected subset. □

Theorem 33. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If these subsets are not Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-separated, then $M \cup K$ is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset.*

Proof. By using Theorem 19, it is obtained similar to the proof of Theorem 32. □

Lemma 6. *Let (U, τ) be an $\mathcal{I}$-space and M, K be subsets of U. Then*

$$\Gamma(M \cap K) \subseteq \Gamma(M) \cap \Gamma(K).$$

Proof. Let $x \in \Gamma(M \cap K)$. Then, $[Cl(O) \cap (M \cap K)] \notin \mathcal{I}$ for every $O \in \tau(x)$. Because of the definition of ideal, $Cl(O) \cap M \notin \mathcal{I}$ and $Cl(O) \cap K \notin \mathcal{I}$. So, $x \in \Gamma(M)$ and $x \in \Gamma(K)$. That is, $x \in \Gamma(M) \cap \Gamma(K)$. □

In the following example, we show that the inclusion $\Gamma(M \cap K) \subseteq \Gamma(M) \cap \Gamma(K)$ strictly hold.

Example 5. *Consider the $\mathcal{I}$-space in Example 1. In Table 1, $\Gamma(A \cap B) = \emptyset \subsetneq \{a, b, c\} = \Gamma(A) \cap \Gamma(B)$.*

Theorem 34. *Let (U, τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

(1) *The subset K is both Γ-Cl-connected and closed.*
(2) *$\Gamma(M) \subseteq Cl(M)$ and $\Gamma(U \setminus M) \subseteq Cl(U \setminus M)$.*
(3) *$K \cap M \neq \emptyset$ and $K \cap (U \setminus M) \neq \emptyset$.*

then $K \cap Bd(M) \neq \emptyset$ where $Bd(M)$ is boundary of the subset M.

Proof. Suppose that $K \cap Bd(M) = \emptyset$. So, $K \cap (Cl(M) \cap Cl(U \setminus M)) = \emptyset$. The subset K can be expressed as $K = U \cap K = (M \cup (U \setminus M)) \cap K = (M \cap K) \cup ((U \setminus M) \cap K)$. Then, by using Lemma 6,

$$\begin{aligned}\Gamma(M \cap K) \cap Cl((U \setminus M) \cap K) &\subseteq \Gamma(M) \cap \Gamma(K) \cap [Cl(U \setminus M) \cap Cl(K)] \\ &\subseteq Cl(M) \cap \Gamma(K) \cap Cl(U \setminus M) \cap K = \emptyset\end{aligned}$$

$$\begin{aligned}Cl(M \cap K) \cap \Gamma((U \setminus M) \cap K) &\subseteq Cl(M) \cap Cl(K) \cap [\Gamma(U \setminus M) \cap \Gamma(K)] \\ &\subseteq Cl(M) \cap K \cap Cl(U \setminus M) \cap \Gamma(K) = \emptyset\end{aligned}$$

and $(M \cap K) \cap ((U \setminus M) \cap K) = \emptyset$. Therefore, the subset K is not Γ-Cl-connected. This is a contradiction. Consequently, $K \cap Bd(M) \neq \emptyset$. □

Theorem 35. *Let (U, τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

(1) *The subset K is Γ-connected.*
(2) $\Gamma(M) \subseteq Cl(M)$ *and* $\Gamma(U \setminus M) \subseteq Cl(U \setminus M)$.
(3) $K \cap M \neq \emptyset$ *and* $K \cap (U \setminus M) \neq \emptyset$.

then $K \cap Bd(M) \neq \emptyset$.

Proof. Suppose that $K \cap Bd(M) = \emptyset$. So, $K \cap (Cl(M) \cap Cl(U \setminus M)) = \emptyset$. The subset K can be expressed as $K = U \cap K = (M \cup (U \setminus M)) \cap K = (M \cap K) \cup ((U \setminus M) \cap K)$. Then, by using Lemma 6,

$$\begin{aligned}\Gamma(M \cap K) \cap ((U \setminus M) \cap K) &\subseteq \Gamma(M) \cap \Gamma(K) \cap (U \setminus M) \cap K \\ &\subseteq Cl(M) \cap \Gamma(K) \cap Cl(U \setminus M) \cap K = \emptyset\end{aligned}$$

$$\begin{aligned}(M \cap K) \cap \Gamma((U \setminus M) \cap K) &\subseteq M \cap K \cap \Gamma(U \setminus M) \cap \Gamma(K) \\ &\subseteq Cl(M) \cap K \cap Cl(U \setminus M) \cap \Gamma(K) = \emptyset\end{aligned}$$

and $(M \cap K) \cap ((U \setminus M) \cap K) = \emptyset$. Therefore, the subset K is not Γ-connected. This is a contradiction. Consequently, $K \cap Bd(M) \neq \emptyset$. □

Theorem 36. *Let (U, τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

(1) *The subset K is both Γ-Cl^*-connected and $*$-closed.*
(2) $\Gamma(M) \subseteq Cl^*(M)$ *and* $\Gamma(U \setminus M) \subseteq Cl^*(U \setminus M)$.
(3) $K \cap M \neq \emptyset$ *and* $K \cap (U \setminus M) \neq \emptyset$.

then $K \cap Bd^*(M) \neq \emptyset$ where $Bd^*(M)$ is boundary of the subset M with respect to τ^*.

Proof. Suppose that $K \cap Bd^*(M) = \emptyset$. So, $K \cap (Cl^*(M) \cap Cl^*(U \setminus M)) = \emptyset$. The subset K can be expressed as $K = U \cap K = (M \cup (U \setminus M)) \cap K = (M \cap K) \cup ((U \setminus M) \cap K)$. Then,

$$\begin{aligned}\Gamma(M \cap K) \cap Cl^*((U \setminus M) \cap K) &\subseteq \Gamma(M) \cap \Gamma(K) \cap [Cl^*(U \setminus M) \cap Cl^*(K)] \\ &\subseteq Cl^*(M) \cap \Gamma(K) \cap Cl^*(U \setminus M) \cap K = \emptyset\end{aligned}$$

$$\begin{aligned}Cl^*(M \cap K) \cap \Gamma((U \setminus M) \cap K) &\subseteq Cl^*(M) \cap Cl^*(K) \cap [\Gamma(U \setminus M) \cap \Gamma(K)] \\ &\subseteq Cl^*(M) \cap K \cap Cl^*(U \setminus M) \cap \Gamma(K) = \emptyset\end{aligned}$$

and $(M \cap K) \cap (K \cap (U \setminus M)) = \emptyset$. Therefore, the subset K is not Γ-Cl^*-connected. This is a contradiction. So, $K \cap Bd^*(M) \neq \emptyset$.

□

Corollary 15. *Let (U,τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

1. *The subset K is Γ-$*$ (2^*)-connected and $*$-closed.*
2. $\Gamma(M) \subseteq Cl^*(M)$ *and* $\Gamma(U \setminus M) \subseteq Cl^*(U \setminus M)$.
3. $K \cap M \neq \emptyset$ *and* $K \cap (U \setminus M) \neq \emptyset$.

then $K \cap Bd^*(M) \neq \emptyset$.

Proof. It is obvious from Figure 4 and Theorem 36. □

Theorem 37. *Let (U,τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

1. *The subset K is Γ-connected.*
2. $\Gamma(M) \subseteq Cl^*(M)$ *and* $\Gamma(U \setminus M) \subseteq Cl^*(U \setminus M)$.
3. $K \cap M \neq \emptyset$ *and* $K \cap (U \setminus M) \neq \emptyset$.

then $K \cap Bd^*(M) \neq \emptyset$.

Proof. Suppose that $K \cap Bd^*(M) = \emptyset$. So, $K \cap (Cl^*(M) \cap Cl^*(U \setminus M)) = \emptyset$. The subset K can be expressed as $K = U \cap K = (M \cup (U \setminus M)) \cap K = (M \cap K) \cup ((U \setminus M) \cap K)$. Then

$$\begin{aligned} \Gamma(M \cap K) \cap ((U \setminus M) \cap K) &\subseteq \Gamma(M) \cap \Gamma(K) \cap (U \setminus M) \cap K \\ &\subseteq Cl^*(M) \cap \Gamma(K) \cap Cl^*(U \setminus M) \cap K = \emptyset \end{aligned}$$

$$\begin{aligned} (M \cap K) \cap \Gamma((U \setminus M) \cap K) &\subseteq M \cap K \cap [\Gamma(U \setminus M) \cap \Gamma(K)] \\ &\subseteq Cl^*(M) \cap K \cap Cl^*(U \setminus M) \cap \Gamma(K) = \emptyset \end{aligned}$$

and $(M \cap K) \cap (K \cap (U \setminus M)) = \emptyset$. Therefore, the subset K is not Γ-Cl^*-connected. This is a contradiction. Finally, $K \cap Bd^*(M) \neq \emptyset$.

□

6. New Type Components via Local Closure

Definition 10. *Let (U,τ) be an $\mathcal{I}$-space and x be a point of U. The union of all Γ-Cl(resp. Γ-Cl^*, Γ, Γ-$*$, 2^*)-connected subsets that contain the point x is called Γ-Cl(resp. Γ-Cl^*, Γ, Γ-$*$, 2^*)-component of U containing x. That is, we define a Γ-Cl(resp. Γ-Cl^*, Γ, Γ-$*$, 2^*)-component of the point x as follows:*

1. *The subset $\mathcal{C}_{\Gamma\text{-}Cl}(x) = \bigcup\{M \subseteq U : M$ is Γ-Cl-connected and $x \in M\}$ is called Γ-Cl-component of the point x.*
2. *The subset $\mathcal{C}_{\Gamma\text{-}Cl^*}(x) = \bigcup\{M \subseteq U : M$ is Γ-Cl^*-connected and $x \in M\}$ is called Γ-Cl^*-component of the point x.*
3. *The subset $\mathcal{C}_{\Gamma}(x) = \bigcup\{M \subseteq U : M$ is Γ-connected and $x \in M\}$ is called Γ-component of the point x.*
4. *The subset $\mathcal{C}_{\Gamma\text{-}*}(x) = \bigcup\{M \subseteq U : M$ is Γ-$*$-connected and $x \in M\}$ is called Γ-$*$-component of the point x.*

(5) *The subset* $\mathcal{C}_{2^*}(x) = \bigcup\{M \subseteq U : M$ *is* 2^**-connected and* $x \in M\}$ *is called* 2^**-component of the point* x.

Theorem 38. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* x *be a point of* U.

(1) *The subset* $\mathcal{C}_{\Gamma\text{-}Cl}(x)$ *is* Γ-Cl*-connected subset which contains* x.
(2) *The subset* $\mathcal{C}_{\Gamma\text{-}Cl}(x)$ *is maximal* Γ-Cl*-connected subset which contains* x .

Proof. (1) Since $x \in \bigcap\{M \subseteq U : M$ is Γ-Cl-connected and $x \in M\} \neq \emptyset$, $\mathcal{C}_{\Gamma\text{-}Cl}(x) = \bigcup\{M \subseteq U : M$ is Γ-Cl-connected and $x \in M\}$ is Γ-Cl-connected by Theorem 22.
(2) It is obvious from Definition 10 and (1).

□

Theorem 39. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* x *be a point of* U.

(1) *The subset* $\mathcal{C}_{\Gamma\text{-}Cl^*}(x)$*(resp.* $\mathcal{C}_{\Gamma}(x)$, $\mathcal{C}_{\Gamma\text{-}*}(x)$, $\mathcal{C}_{2^*}(x)$*) is* Γ-Cl^**(resp.* Γ, Γ-$*$, 2^**)-connected subset which contains* x.
(2) *The subset* $\mathcal{C}_{\Gamma\text{-}Cl^*}(x)$*(resp.* $\mathcal{C}_{\Gamma}(x)$, $\mathcal{C}_{\Gamma\text{-}*}(x)$, $\mathcal{C}_{2^*}(x)$*) is maximal* Γ-Cl^**(resp.* Γ, Γ-$*$, 2^**)-connected subset which contains* x .

Proof. By using Theorem 23 and Definition 10 , it is obtained similar to the proof of Theorem 38 . □

Theorem 40. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $x, y \in U$. *Then*

(1) $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cap \mathcal{C}_{\Gamma\text{-}Cl}(y) = \emptyset$ *or* $\mathcal{C}_{\Gamma\text{-}Cl}(x) = \mathcal{C}_{\Gamma\text{-}Cl}(y)$.
(2) *The set of all distinct* Γ-Cl*-components forms a partition of* U.

Proof. (1) Let $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cap \mathcal{C}_{\Gamma\text{-}Cl}(y) \neq \emptyset$. From Theorem 38-(1) and Theorem 22, $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y)$ is Γ-Cl-connected. We have $\mathcal{C}_{\Gamma\text{-}Cl}(x) \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y)$ and $\mathcal{C}_{\Gamma\text{-}Cl}(y) \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y)$. From Theorem 38 -(2), $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y) \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(x)$ and $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y) \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(y)$. So, $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y) = \mathcal{C}_{\Gamma\text{-}Cl}(x) = \mathcal{C}_{\Gamma\text{-}Cl}(y)$.
(2) Since $\bigcup_{x \in U} \mathcal{C}_{\Gamma\text{-}Cl}(x) = U$, it is obvious from (1).

□

Theorem 41. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $x, y \in U$. *Then,*

(1) $\mathcal{C}_{\Gamma\text{-}Cl^*}(x) \cap \mathcal{C}_{\Gamma\text{-}Cl^*}(y) = \emptyset$ *or* $\mathcal{C}_{\Gamma\text{-}Cl^*}(x) = \mathcal{C}_{\Gamma\text{-}Cl^*}(y)$.
(2) $\mathcal{C}_{\Gamma}(x) \cap \mathcal{C}_{\Gamma}(y) = \emptyset$ *or* $\mathcal{C}_{\Gamma}(x) = \mathcal{C}_{\Gamma}(y)$.
(3) $\mathcal{C}_{\Gamma\text{-}*}(x) \cap \mathcal{C}_{\Gamma\text{-}*}(y) = \emptyset$ *or* $\mathcal{C}_{\Gamma\text{-}*}(x) = \mathcal{C}_{\Gamma\text{-}*}(y)$.
(4) $\mathcal{C}_{2^*}(x) \cap \mathcal{C}_{2^*}(y) = \emptyset$ *or* $\mathcal{C}_{2^*}(x) = \mathcal{C}_{2^*}(y)$.
(5) *The set of all distinct* $\mathcal{C}_{\Gamma\text{-}Cl^*}(x)$*(resp.* $\mathcal{C}_{\Gamma}(x)$, $\mathcal{C}_{\Gamma\text{-}*}(x)$, $\mathcal{C}_{2^*}(x)$*)-components forms a partition of* U.

Proof. By using Theorem 39 and Theorem 23, all statements above are obtained similar to the proof of Theorem 40. □

Theorem 42. *Let* (U, τ) *be an* $\mathcal{I}$*-space. If* M *is* Γ-Cl*-connected and nonempty clopen subset of* U, *then* M *is* Γ-Cl*-component.*

Proof. Let $\mathcal{C}_{\Gamma\text{-}Cl}(x)$ be Γ-Cl-component of the point $x \in M$. From Theorem 38-(2), $M \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(x)$. Suppose that $M \subsetneqq \mathcal{C}_{\Gamma\text{-}Cl}(x)$. Then, $(M \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) \cap [(U \setminus M) \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)] = \emptyset$ and $(M \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) \cup [(U \setminus M) \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)] = \mathcal{C}_{\Gamma\text{-}Cl}(x)$. From Lemma 5,

$$\Gamma(M) \cap Cl(U \setminus M) \subseteq Cl(M) \cap (U \setminus M) = M \cap (U \setminus M) = \emptyset$$
$$Cl(M) \cap \Gamma(U \setminus M) \subseteq M \cap Cl(U \setminus M) = M \cap (U \setminus M) = \emptyset$$

These imply that

$$\Gamma(M \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) \cap Cl((U \setminus M) \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) = \emptyset$$
$$Cl(M \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) \cap \Gamma((U \setminus M) \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) = \emptyset$$

So, $\mathcal{C}_{\Gamma\text{-}Cl}(x)$ is not Γ-Cl-connected. This is a contradiction. Consequently, $M = \mathcal{C}_{\Gamma\text{-}Cl}(x)$. That is, M is Γ-Cl-component. □

Theorem 43. *Let (U, τ) be an $\mathcal{I}$-space. If M is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected and nonempty clopen subset of U, then M is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-component.*

Proof. By using Lemma 5, it is obtained similar to the proof of Theorem 42. □

7. The Image of New Types of Connectedness Under a Continuous Map in Ideal Topological Spaces

$f : (U, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ is continuous map means that $f : (U, \tau_1) \to (Y, \tau_2)$ is continuous.

Theorem 44. *Let (U, τ_1) be Γ-Cl-connected $\mathcal{I}$-space and (Y, τ_2) be any topological space. If $f : (U, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ is a continuous map, then $f(U)$ is τ_2-connected.*

Proof. From Theorem 15 , the set U is τ_1-connected. Since the image of a connected space under a continuous map is connected, $f(U)$ is τ_2-connected. □

Corollary 16. *Let (U, τ_1) be Γ-Cl^* (Γ, Γ-$*$, 2^*, $*$-Cl, $*$-Cl^*, $*_*$)-connected $\mathcal{I}$-space and (Y, τ_2) be any topological space. If $f : (U, \tau_1, I) \to (Y, \tau_2)$ is a continuous map, then $f(U)$ is τ_2-connected.*

Proof. It is obvious from Theorem 44 and Figure 4. □

Corollary 17. *Let $f : (U, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ be continuous and surjective function. If U is Γ-Cl (Γ-Cl^*, Γ, Γ-$*$, 2^*)-connected, then Y is τ-connected.*

Proof. It is obvious from Theorem 44 and Corollary 16. □

It is shown in [14] that Corollary 17 is also satisfied for $*$-Cl ($*$-Cl^*, $*_*$)-connectedness. This is clear from Theorem 44 and Corollary 16. Because Γ-Cl-connectedness is more general than $*$-Cl ($*$-Cl^*, $*_*$)-connectedness.

Theorem 45. *[25](Intermediate Value Theorem) Let $f:(U,\tau_1)\to(Y,\tau_2)$ be continuous map, where (U,τ_1) is a τ_1-connected topological space, Y is an ordered set with " $<$ " and τ_2 is order topology on Y. If $a,b\in U$ and $f(a)<r<f(b)$, then there exists a point $c\in U$ such that $f(c)=r$.*

Now, we give the intermediate value theorem for the ideal topological spaces.

Theorem 46. *Let $f:(U,\tau_1,\mathcal{I})\to(Y,\tau_2)$ be continuous map, where (U,τ_1) is a Γ-Cl (Γ-Cl^*, Γ, Γ-$*$, 2^*, $*$-Cl, $*$-Cl^*, $*_*$)-connected $\mathcal{I}$-space, Y is an ordered set with " $<$ " and τ_2 is order topology on Y. If $a,b\in U$ and $f(a)<r<f(b)$, then there exists a point $c\in U$ such that $f(c)=r$.*

Proof. From Theorem 15 (and Corollary 16), the set U is τ_1-connected. That is, (U,τ_1) is connected space. Then, the claim is obtained by Theorem 45. $\square$

Specially, if we choose the minimal ideal $\mathcal{I}=\{\emptyset\}$ in Theorem 46, by using Corollary 13, we obtain the intermediate value theorem. That is, a special case of Theorem 46 gives the intermediate value theorem.

Author Contribution Statements All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

Acknowledgements Authors would like to thank the referees and the editor for their constructive comments and suggestions.
The first author would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during his doctorate studies.

References

[1] Al-Omari, A., Noiri, T., Local closure functions in ideal topological spaces, *Novi Sad J. Math.*, 43(2) (2013), 139-149.
[2] Al-Omeri, W., Noorani, M., Al-Omari, A., a-local function and its properties in ideal topological spaces, *Fasc. Math.*, 53 (2014), 5-15.
[3] Dontchev, J., Ganster, M., Rose D. A., Ideal resolvability, *Topology Appl.*, 93(1) (1999), 1-16. https://doi.org/10.1016/s0166-8641(97)00257-5
[4] Ekici, E., On I-Alexandroff and I_g-Alexandroff ideal topological spaces, *Filomat*, 25(4) (2011), 99-108. https://doi.org/10.2298/fil1104099e
[5] Engelking, R., General Topology, Heldermann Verlag, Berlin, 1989.
[6] Freud, G., Ein beitrag zu dem satze von Cantor und Bendixson, *Acta Math. Hungar.*, 9 (1958), 333-336. https://doi.org/10.1007/bf02020262
[7] Güldürdek, A., Ideal Rothberger spaces, *Hacet. J. Math. Stat.*, 47(1) (2018), 69–75. 1015672/HJMS.2017.445
[8] Hayashi, E., Topologies defined by local properties, *Math. Ann.*, 156 (1964), 205-215. https://doi.org/10.1007/bf01363287

[9] Islam, M. M., Modak, S., Second approximation of local functions in ideal topological spaces, *Acta Comment. Univ. Tartu. Math.*, 22(2) (2018), 245-255. https://doi.org/10.12697/acutm.2018.22.20
[10] Janković, D., Hamlett T. R., New topologies from old via ideals, *Amer. Math. Monthly*, 97(4) (1992), 295-310. https://doi.org/10.2307/2324512
[11] Kandil, A., Tantawy O. A. E., El-Sheikh, S. A., Abd El-latif, A. M., Soft ideal theory, soft local function and generated soft topological spaces, *Appl. Math. Inf. Sci.*, 8(4) (2014), 1595-1603. https://doi.org/10.12785/amis/080413
[12] Keskin, A., Yüksel, Ş., Noiri, T., On I-extremally disconnected spaces, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 56(1) (2007), 33-40. https://doi.org/10.1501/commua1_0000000195
[13] Khan, M., Noiri, T., Semi-local functions in ideal topological spaces, *J. Adv. Res. Pure Math.*, 2(1) (2010), 36-42. 10.5373/jarpm.237.100909
[14] Kilinc, S., More on $*_*$-Connected, *Matematichki Bilten*, 43(2) (2019), 53-60.
[15] Kule, M., Dost, Ş., Texture spaces with ideal, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 68(2) (2019), 1596-1610. https://doi.org/10.31801/cfsuasmas.416238
[16] Kuratowski, K., Topology Volume I, Academic Press, New York-London, 1966.
[17] Li, Z., Lin, F., On I-Baire spaces, *Filomat*, 27(2) (2013), 301-310. https://doi.org/10.2298/FIL1302301L
[18] Modak, S., Noiri, T., A weaker form of connectedness,*Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 65(1) (2016), 49-52. https://doi.org/10.1501/Commua1_0000000743
[19] Modak, S., Grill-filter space, *Journal of the Indian Math. Soc.*, 80(3-4) (2013), 313-320.
[20] Modak, S., Noiri, T., Connectedness of ideal topological spaces, *Filomat*, 29(4) (2015), 661-665. https://doi.org/10.2298/FIL1504661M
[21] Modak, S., Islam, M. M., More on α-topological spaces, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 66(2) (2017), 323–331. https://doi.org/10.1501/Commua_10000000822
[22] Modak, S., Minimal spaces with a mathematical structure, *J. Assoc. Arab Univ. Basic Appl. Sci.*, 22(1) (2018), 98-101. https://doi.org/10.1016/j.jaubas.2016.05.005
[23] Modak, S., Islam M. M., On $*$ and ψ operators in topological spaces with ideals, *Trans. A. Razmadze Math. Inst.*, 172 (2018), 491-497. https://doi.org/10.1016/j.trmi.2018.07.002
[24] Modak, S., Selim, S., Set operators and associated functions,*Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 70(1) (2021), 456-467. https://doi.org/10.31801/cfsuasmas.644689
[25] Munkres, J. R., Topology A First Course, Prentice-Hall, Englewood Cliffs N. J., 1974.
[26] Pachón Rubiano, N. R., A generalization of connectedness via ideals, *Armen.J.Math.*, 14(7) (2022), 1-18. https://doi.org/10.52737/18291163-2022.14.7-1-18
[27] Pavlović, A., Local function versus local closure function in ideal topological spaces, *Filomat*, 30(14) (2016), 3725-3731. https://doi.org/10.2298/fil1614725p
[28] Sarker, D., Fuzzy ideal theory, Fuzzy local function and generated fuzzy topology, *Fuzzy Sets and Systems*, 87(1) (1997), 117-123. https://doi.org/10.1016/S0165-0114(96)00032-2
[29] Selim, S., Islam, M. M., Modak, S., Common properties and approximations of local function and set operator ψ, *Cumhuriyet Sci. J.*, 41(2) (2020), 360-368. https://doi.org/10.17776/csj.644158
[30] Tunc, A. N., Yıldırım, Ö. S., New sets obtained by local closure functions, *Ann. Pure and Appl. Math. Sci.*, 1(1) (2021), 50-59. http://doi.org/10.26637/aopams11/006
[31] Tyagı, B. K., Bhardwaj, M., Sıngh, S., Cl^*-connectedness and Cl-Cl^*-connectedness in ideal topological space, *Matematichki Bilten*, 42(2) (2018), 91–100.
[32] Willard, S., General Topology, Addison-Wesley Publishing Company, Reading, 1970.
[33] Vaidyanathswamy, R., The localisation theory in set topology, *Proc. Indian Acad. Sci.*, 20 (1945), 51-61. https://doi.org/10.1007/bf03048958
[34] Veličko, N. V., H-closed topological spaces, *Amer. Math. Soc.*, 78(2) (1968), 103-118. https://doi.org/10.1090/trans2/078/05

[35] Yalaz, F., Kaymakcı, A. K., Weak semi-local functions in ideal topological spaces, *Turk. J. Math. Comput. Sci.*, 11 (2019), 137-140.

[36] Yalaz, F., Kaymakcı, A. K., New topologies from obtained operators via weak semi-local function and some comparisons, *Filomat*, 15(35) (2021), 5073-5081. https://doi.org/10.2298/FIL2115073Y