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Research Article

Weak stability of ε-isometry Mapping on Real Banach Spaces

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Abstract

The stability of standard ε-isometry mapping in real Banach spaces cannot be determined without using the assumption of surjectivity. However, this mapping remains weakly stable under weak topology. Using this weak stability, there is a bounded linear left-inverse for non-surjective ε-isometry.

Keywords: ε-isometry, Banach space, stability, weak topology, bounded linear left-inverse.

Gerçel Banach Uzaylarındaki ε-izometrinin Zayıf Kararlılığı

Öz

Gerçel Banach uzaylarındaki standart ε-izometrinin kararlılığı, örtenliği varsayımı kullanılmadan belirlenemez. Bununla birlikte, bu dönüşüm, zayıf topoloji altında zayıf bir şekilde kararlı kalır. Bu zayıf kararlılığı kullanarak, örten olmayan ε-izometrisi için sınırlı bir lineer sol-ters vardır.

Anahtar Kelimeler: ε-isometry, Banach uzayı, kararlılık, zayıf topoloji, sınırlı bir lineer sol-ters.

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1. Introduction

Research related to ε-isometry mappings emerged after Mazur and Ulam [14] showed that all isometry mappings are affine. Recall that a function is said to be affine if the function is a translation of a linear mapping. In other words, an isometry mapping $U : X \to Y$ is linear if and only if $U(0) = 0$. Therefore, the concept of an ε-isometry mapping $f : X \to Y$ emerged which is defined as

$$\left| \| f(x) - f(y) \| - \| x - y \| \right| \le \varepsilon \qquad (0.1)$$

for $\varepsilon \ge 0$. If $\varepsilon = 0$, then f is nothing but an isometry mapping. f is said to be standard if $f(0) = 0$. Assuming $y = 0$ in (1.1), then the above condition raises the question, "Is there any isometry mapping $U : X \to Y$ for each given ε-isometry mapping $f : X \to Y$ such that

$$\| f(x) - U(x) \| \le \gamma\varepsilon \qquad (0.2)$$

for some $\gamma > 0$?

On the other hand, Figiel [9] shows that for any isometry mapping U, there exists a bounded linear operator $T : \overline{span} U(X) \to X$ such that $F \circ U = Id_X$. With Figiel's findings, the second question arises, "If given an ε-isometry mapping $f : X \to Y$, does there exist bounded linear operators $F : \overline{span} f(X) \to X$ such that

$$\| Ff(x) - x \| \le \beta\varepsilon \qquad (0.3)$$

for some $\beta > 0$?

The two issues in (1.1) and (1.2) are mainstream research topics related to ε-isometry mapping.

For the first problem, Hyers and Ulam [12] first showed that for any ε -isometry mapping $f : X \to Y$ with $f(0) = 0$, there is an isometry mapping $U : X \to Y$ satisfied (1.2) with $\gamma = 10$ for all $x \in X$, where X and Y are Euclidean spaces. Later Bourgin [2] showed that $\gamma = 12$ where $X = Y = L_p(0,1)$, $1 < p < \infty$. Gruber [11] first generalized to any real Banach spaces and Gevirtz [10] found $\gamma = 5$ that which is reduced by Omladič and Šemrl [17] to $\gamma = 2$. In this first case, the surjectivity assumption cannot be removed.

There are two branches of research for non-surjective ε-isometry cases, namely using the near (almost) surjective concept and Figiel's theorem.

Let $Y_1 \subset Y$ is a closed subspace. A mapping $f : X \to Y$ is said to be *near surjective* if $\forall y \in Y_1$ there exists $x \in X$ such that $\| f(x) - y \| \le \delta$ and $\forall u \in X$ there exists $v \in Y_1$ such that $\| f(u) - v \| \le \delta$ [22]. Dilworth [6] showed that for every δ-surjective ε-isometry mapping $f : X \to Y$ with $f(0) = 0$, where X and Y are Banach spaces, there exists an isometric mapping $U : X \to Y$ such that $\| f(x) - U(x) \| \le 12\varepsilon + 5\delta$. Then Tabor [23] changed this value to $2\varepsilon + 35\delta$ and reduced by Šemrl and Väisälä [22] to $2\varepsilon + 2\delta$. Note that by the definition, a mapping $f : X \to Y$ is said to be near surjective if $\sup_{y \in Y} dist(y, f(X)) < \infty$. Vestfrid [24] showed that the result remains true if the condition of near-surjectivity is relaxed to be $\sup_{y \in Y} \liminf_{|t| \to \infty} dist(ty, f(X))/|t| < \frac{1}{2}$.

Furthermore, Qian [19] used Figiel theorem to found out the value of β in (1.3). With a counterexample, he showed that the Figiel theorem does not apply in general to ε-isometric mapping. However if $X = Y = L_p$ where $1 < p < \infty$, then for every ε-isometry mapping $f : X \to Y$ there exists a bounded linear operator $F : \overline{span} f(X) \to X$ with $\| F \| = 1$ such that $\| Ff(x) - x \| \le 6\varepsilon$. Furthermore, Šemrl and Väisälä [22] showed that if X is a Banach space and Y is a Hilbert space, then the value of β can be reduced to 2.

From the brief explanation above, it can be seen that research related to ε-isometry is still wide open for non-surjective cases. Recall that the non-surjective condition fails in norm topology. Therefore, we will discuss ε-isometry mapping using a weak topology concept.

2. Material and Method

With the description in the introduction, it can be seen that this research is qualitative with grounded theory method. Books [15] and [8] provide advanced concepts of weak (weak*) topology, Gateaux, and Frechet derivatives while [16] and [21] provide a basic overview of the last two concepts.

If not specifically stated, then X and Y are real Banach spaces. B_X (S_X) is used to denote the unit ball (sphere, resp.) of X, exp(A) ($\overline{co}(A)$) is a set of all exposed points (a closed convex hull, resp.) of $A \subset X$. The authors use the concepts of weak and weak* topology along with symbols that are commonly used.

3. Results and Discussion

As mentioned earlier, non-surjective ε-isometry mapping does not generally apply to any Banach spaces. Therefore, this section will discuss the weaker stability version of an e-isometry mapping.

Theorem 3.1. *Suppose $f : X \to Y$ is a standard ε-isometry, then for any $x^* \in X^*$, there exists $\varphi \in Y^*$ that satisfies $\| \varphi \| = \| x^* \| = r$ such that*

$$\left| \langle \varphi, f(x) \rangle - \langle x^*, x \rangle \right| \le \kappa\varepsilon r, \forall x \in X \qquad (3.1)$$

Using the Hanh-Banach Theorem, do not eliminate generality by assuming r = 1. Cheng, et. al. [5] showed that $\kappa = 4$ in (3.1) and further can be reduced to be 3(see. [3]). Rohman, et. al [2] showed that the weak stability version remains true under Vestfrid condition [24]. The two following lemmas are crucial for the proof of Theorem 3.1.

Lemma 3.2 ([4], Lemma 2.1.) *Let Y be the Banach space,* $g\colon \mathbb{R} \to Y$ *be the standard* ε*-isometric and* $\mathfrak{U}$ *be the free ultrafilter on* $\mathbb{N}$. *For any* $n \in \square$ *, let* $\varphi_n \in S_{Y^*}$ *satisfies*

$$\langle \varphi_n, g(n) - g(-n) \rangle = \| g(n) - g(-n) \|$$

If $\varphi = w^* - lim_{\mathfrak{U}}\, \varphi_n$, *then*

$$\left| \langle \varphi, g(t) \rangle - t \right| \le 3\varepsilon.$$

Lemma 3.3. ([4], Lemma 2.2) *Let* $f : X \to Y$ *be a standard* ε*-isometry,* $z \in S_X$ *be the Gateaux differentiable point of X and recall that its Gateaux derivative is* $d\|z\| = x^*$*, then there exists* $\varphi \in S_{Y^*}$ *such that*

$$\left| \langle \varphi, f(x) \rangle - \langle x^*, x \rangle \right| \le 3\varepsilon, \quad \forall x \in X$$

Proof of Theorem 3.1 for $\kappa = 3$.

Let $f : X \to Y$ be a standard ε-isometry. We denote $\mathfrak{F}$ be a family of all finite-dimensional subspaces of X. Then for any $F \in \mathfrak{F}$, $f_F : F \to Y$ (f is restricted to F) is still a standard ε-isometry. Since F is a Gateaux differentiability space ([18], Proposition 6.5), according to ([18], Proposition 6.9. and Theorem 6.2), the unit ball B_{F^*} of $F^* = X^*/F^{\perp}$ is w^*-closed convex hull of its w^*-exposed point, that is by the definition of GDS, the convex hull of w^*- exposed point of B_{F^*} (w^*-exp(B_{F^*})) is w^*-dense in B_{F^*} (since F is a finite-dimensional space, it is dense in the sense of norm topology). For any $x_F^* \in w^*$-exp(B_{F^*}), from ([18], Proposition 6.9.), we know that there is $z \in S_F$ such that $d\|z\|_F = x_F^*$. By Lemma 3.3, we know that there is $\varphi_F = \varphi \in S_{Y^*}$ such that

$$\left| \langle \varphi_F, f(x) \rangle - \langle x_F^*, x \rangle \right| \le 3\varepsilon, \quad \forall x \in F \tag{3.2}$$

For any $z^* \in S_{F^*}$, from ([18], Theorem 6.2.), there is a family of subsets $\{F_\alpha : \alpha \in I\}$ (where $F_\alpha \subset \mathbb{N}$ is a finite subset), $\left(x_{\alpha,n}^*\right)_{n \in F_\alpha} \subset w^* - \exp\left(B_{F^*}\right)$, $\left(\lambda_{\alpha,n}\right)_{n \in F_\alpha} \subset \square^+$ satisfies $\sum_{n \in F_\alpha} \lambda_{\alpha,n} = 1$ such that

$$\begin{aligned} & w^* - \lim_\alpha z_\alpha^* = z^*, \\ & z_\alpha^* \equiv \sum_{n \in F_\alpha} \lambda_{\alpha,n} x_{\alpha,n}^*, \ \text{for } \alpha \in \mathrm{I} \end{aligned} \tag{3.3}$$

From (3.2) we get

$$\left| \langle \varphi_\alpha, f(x) \rangle - \langle z_\alpha^*, x \rangle \right| \le 3\varepsilon, \quad \forall x \in F \quad \alpha \in I \tag{3.4}$$

where $\varphi_\alpha = \sum_{n \in F_\alpha} \lambda_{\alpha,n} \varphi_{\alpha,n}$, and $\varphi_{\alpha,n}$ satisfies

$$\left| \langle \varphi_{\alpha,n}, f(x) \rangle - \langle x_{\alpha,n}^*, x \rangle \right| \le 3\varepsilon, \quad \forall x \in F \tag{3.5}$$

For (3.3) both ends of the w^*- limit are respectively taken to obtain φ ∈ B_{Y^*} such that

$$\left| \langle \varphi, f(x) \rangle - \langle z^*, x \rangle \right| \le 3\varepsilon, \quad \forall x \in F \tag{3.6}$$

Take $u \in S_F$ such that $\langle z^*, u \rangle = 1$, substitute $x = nu$ into the above inequality and divide by n, and then set $n \to \infty$ we have

$$\lim_{n \to \infty} \left\langle \varphi, \frac{f(nu)}{n} \right\rangle = \langle z^*, u \rangle = 1$$

This shows that $\| \varphi \| \ge 1$. Furthermore, $\| \varphi \| = 1$. In this way, we have proved that for any $z^* \in S_{F^*}$, there exists φ ∈ S_{Y^*} such that (3.1) is true. By the absolute homogeneity of this inequality, it is obtained that for any $z^* \in F^*$, there exists φ ∈ Y^* that satisfies $\|\varphi\| = \|x^*\| = r$, such that

$$\left| \langle \varphi, f(x) \rangle - \langle z^*, x \rangle \right| \le 3\varepsilon r, \quad \forall x \in F \tag{3.7}$$

The following proves that for any norm attaining functional $x^* \in X^*$, there exists φ ∈ Y^* that satisfies $\|\varphi\| = \|x^*\| = r$, such that

$$\left| \langle \varphi, f(x) \rangle - \langle x^*, x \rangle \right| \le 3\varepsilon r, \quad \forall x \in X \tag{3.8}$$

Let $x_0 \in S_X$ such that $\langle x^*, x_0 \rangle = \|x^*\| = r$. We denote the set of all finite-dimensional subspaces containing x_0 as $\mathfrak{F}_0$, then for any $F \in \mathfrak{F}_0$ there is $\varphi_F \in rS_{Y^*}$ such that

$$\left| \langle \varphi_F, f(x) \rangle - \langle x^*, x \rangle \right| \le 3\varepsilon r, \quad \forall x \in F \tag{3.9}$$

We denote the set of all φ_F satisfying (3.9) and $\|\varphi_F\| = \|x^*\| = r$ as K_F for the above x^*. It is not difficult to verify, $\forall F \in \mathfrak{F}_0$, K_F is a non-empty $w*$-compact convex subset in rS_{Y^*}. Let $\mathfrak{K} = \{K_F : F \in \mathfrak{F}_0\}$, then this is a collection of closed $w*$-compact convex subset. $\forall E, F \in \mathfrak{F}_0$,

$$\varnothing \ne K_G \subset K_E \cap K_F$$

where $G = span(E \cup F)$.

This shows that $\mathfrak{K}$ has a finite intersection property, and then

$$K_0 \equiv \bigcap \{K_F : F \in F_0\} \neq \varnothing$$

If $\varphi \in K_0$ is chosen, it is easy to show that $\varphi \in rS_{Y^*}$ and gives

$$\left|\langle \varphi, f(x)\rangle - \langle x^*, x\rangle\right| \leq 3\varepsilon r, \quad \forall x \in X$$

Finally, we will prove that for any $x^* \in X^*$, there exists $\varphi \in Y^*$ that satisfies $\|\varphi\| = \|x^*\| = r$, such that

$$\left|\langle \varphi, f(x)\rangle - \langle x^*, x\rangle\right| \leq 3\varepsilon r, \quad \forall x \in X$$

In fact, according to the Bishop-Phelps theorem that every Banach space is subreflexive [26], according to ([15], Theorem 2.11.13 and [8], Theorem 7.41) there exists a sequence of norm-attaining functional $(x_n^*) \subset rS_{X^*}$ such that $x_n^* \to x^*$, $x^* \in rS_{X^*}$. Let $\varphi_n \in rS_{Y^*}$ such that

$$\left|\langle \varphi_n, f(x)\rangle - \langle x_n^*, x\rangle\right| \leq 3\varepsilon r, \quad \forall x \in X$$

then for any (φ_n) there exists w^*-convergence point φ such that $\| \varphi \| \leq r$, and

$$\left|\langle \varphi, f(x)\rangle - \langle x^*, x\rangle\right| \leq 3\varepsilon r, \quad \forall x \in X$$

by the above inequality, we get $\| \varphi \| \geq r$. Therefore, the theorem is proved. ■

By using Theorem 3.1. for $\kappa = 4$, Cheng, et. al. [5] gave the generalization of Figiel's Theorem from isometry to ε-isometry for specific spaces.

Theorem 3.4. *Let $f: X \to Y$ be a standard ε-isometry and $E \subset Y$ be the annihilator of $F \subset Y^*$ consisting of all bounded functional on $\overline{co}\left(f(x), -f(x)\right)$. If E is α-complemented in Y, then there is a bounded linear operator with $\|T\| \leq \alpha$ such that*

$$\|Tf(x) - x\| \leq \beta\varepsilon, \quad \forall x \in X \tag{3.10}$$

If X and Y are Banach spaces with Y reflexive, then $\beta = 4$ in (3.10). If $Y = \overline{co}\left(f(x), -f(x)\right)$ or Y is reflexive, Gateaux smooth and strictly convex Banach space with Kadec-Klee property, then $\beta = 2$.

4. Conclusions and Recommendations

When we cannot know the stability of non-surjective ε-isometry mappings on real Banach spaces under norm topology, such mappings remain stable under weak topology. Besides the result still supports Figiel theorem for such mapping.

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