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Existence Results for Anti-periodic of a Generalized Fractional Derivative Differential Equations

Kadda Maazouz[1], Devarajan Vivek[2] and El-Sayed Mohamed El-Sayed[3]

[1] Department of Mathematics, University of Tiaret, Algeria.
kadda.maazouz@univ-tiaret.dz
[2] Department of Mathematics,PSG College of Arts & Science, Coimbatore, India.
peppyvivek@gmail.com
[3] Faculty of Science, Mansoura University,Mansoura 35516, Egypt.
emmelsayed@yahoo.com

Abstract. We study in the present work the existence of solutions to antiperiodic boundary value problem for differential equations involving generalized fractional derivative via fixed point methods.

Keywords: Generalized Fractional Derivative; Antiperiodic Boundary Value Problem; Integral Equation; Fixed Point.

1 Introduction

Agarwal and Ahmad studied in [1] existence results of solutions for nonlinear fractional differential equations and inclusions of order $q \in (3,4]$ with anti-periodic boundary conditions establishing both cases convex and nonconvex multivalued maps by means of fixed point. In [3] Ashordia considered the solvability of an antiperiodic boundary value problem for systems of linear generalized differential equations.

In [2] Ahmad and Nieto proved existence of solutions for impulsive differential equations of fractional order $q \in (1,2]$ with anti-periodic boundary conditions in a Banach space, their study was based on the contraction mapping principle and Krasnoselskii fixed point theorem.

Chai in [6] obtained existence and uniqueness of solutions for nonlinear equation via Riemann-Liouville fractional derivative of order $\alpha \in (1,2)$

Wang et. al investigated in [22] the existence of solutions of an anti-periodic fractional boundary value problem for nonlinear fractional differential equations via Green function.

Motivated by some above papers we are interest in this present work with the existence of solutions for the following antiperiodic problem,

$$^{r}\mathrm{D}^{\omega}v(s) = \zeta(s, v(s)) := \zeta_v(s), \quad s \in J := [a,b], \quad 1 < \omega < 2, \quad r > 0, \tag{1}$$

$$v(a) + v(b) = 0, \qquad v'(a) + v'(b) = 0. \tag{2}$$

where $^{r}\mathrm{D}^{\omega}$ is the generalized fractional derivative of order ω, $\zeta : J \times \mathbb{R} \longrightarrow \mathbb{R}$, is given function, and $0 \leq a < b < \infty$.

Our work is organized as follows: In Section 2, we give definitions, lemmas, and some results. Section 3 is devoted to establish our main results. Finally, an illustrative example is given to illustrate our theoritical results.

2 Preliminaries

We now introduce some definitions, preliminary facts about the fractional calculus, notations, and some auxiliary results, which will be used later.

Definition 1. [11] *The generalized left-sided fractional integral of order* $\omega \in \mathbb{C}$, $(Re(\omega) > 0)$ *is defined for* $s > a$ *by*

$$^{r}I^{\omega}f(s) = \frac{r^{1-\omega}}{\Gamma(\omega)} \int_a^s \Big(s^r - x^r\Big)^{\omega-1} x^{r-1} f(x)dx \tag{3}$$

if the integral exists, where $\Gamma(.)$ *is the Gamma function.*

Definition 2. [11] *The generalized left-sided fractional derivative, corresponding to the generalized fractional integral () is defined for* $s > a$ *by*

$$^{r}D^{\omega}f(s) = \frac{r^{\omega-n+1}}{\Gamma(n-\omega)} \Big(s^{1-r}\frac{d}{ds}\Big) \int_a^s \Big(s^r - x^r\Big)^{n-\omega-1} x^{r-1} f(x)dx$$

where $n = [\omega] + 1$, *if the integral exists.*

Lemma 1. *Let $\omega > 0$ and $r > 0$, then the differential equation*

$$^{r}D^{\omega}g(s) = 0$$

has solutions

$$g(s) = a_0 + \sum_{i=1}^{n-1} a_i \Big(\frac{s^r - a^r}{r}\Big)^{\omega - i}, \quad a_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1, \quad n = [\omega] + 1.$$

Lemma 2. *Let $\omega > 0$ and $r > 0$, then*

$$^{r}I^{\omega}\Big(^{r}D^{\omega}g(s)\Big) = g(s) + a_0 + \sum_{i=1}^{n-1} a_i \Big(\frac{s^r - a^r}{r}\Big)^{\omega - i},$$

for some

$$a_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1, \quad n = [\omega] + 1.$$

3 Main results

Let us put

$$\mathrm{M}_s^{\omega}(x) = \frac{x^{r-1}}{\Gamma(\omega)}\Big(\frac{s^r - x^r}{r}\Big)^{\omega - 1} \quad \text{and} \quad \psi_s^{\omega} = \Big(\frac{s^r - a^r}{r}\Big)^{\omega - 1}$$

On the other hand It is easy to see that

$$\int_a^s \mathrm{M}_s^{\omega}(x)dx \leqslant \int_a^b \mathrm{M}_b^{\omega}(x)dx = \frac{(b^r - a^r)^{\omega} r^{-\omega}}{\omega\Gamma(\omega)} := \mathrm{M}^{\omega}.$$

$$\int_a^b \mathrm{M}_b^{\omega-1}(x)dx = \frac{(b^r - a^r)^{\omega-1} r^{1-\omega}}{(\omega - 1)\Gamma(\omega)} = \frac{1}{\omega - 1}\mathrm{M}^{\omega-1}.$$

Lemma 3. *Let $1 < \omega < 2$, $r > 0$ and $\sigma \in \mathcal{C}(J, \mathbb{R})$ be a continuous function. Then the following boundary value problem*

$$\begin{cases} ^{r}\mathrm{D}^{\omega}v(s) = \zeta_v(s), \quad t \in J, \\ a_1 v(a) + b_1 v(b) = c_1, \\ a_2 v'(a) + b_2 v'(b) = c_2, \end{cases} \tag{4}$$

has a unique solution given by

$$v(s) = \int_a^s \mathrm{M}_s^{\omega}(x)\zeta_v(x)dx + \chi_{a,b}\psi_s^{\omega} + \frac{1}{a_1 + b_1}\Big(c_1 - b_1\chi_{a,b}\psi_b^{\omega} - b_1\int_a^b \mathrm{M}_b^{\omega}(x)\zeta_v(x)dx\Big). \tag{5}$$

Proof. Let v satisfies () then, by Lemmas and we have

$$v(s) = \int_a^s \mathrm{M}_s^{\omega}(x)\zeta_v(x)dx + \pi_0 + \pi_1\psi_s^{\omega}.$$

Then

$$v'^{r-1}\int_a^s \mathrm{M}_s^{\omega-1}(x)\zeta_v(x)dx + \pi_1(\omega-1)s^{r-1}\psi_s^{\omega-1}.$$

So we have

$$v(a) = \pi_0, \qquad v'(a) = 0,$$

$$v(b) = \int_a^b \mathrm{M}_b^{\omega}(x)\zeta_v(x)dx + \pi_0 + \pi_1\psi_b^{\omega},$$

$$v'^{r-1}\int_a^b \mathrm{M}_b^{\omega-1}(x)\sigma(x)dx + \pi_1(\omega-1)b^{r-1}\psi_b^{\omega-1}.$$

Therefore $a_2v'(a) + b_2v'(b) = c_2$ implies that

$$b_2rb^{r-1}\int_a^b \mathrm{M}_b^{\omega-1}(x)\sigma(x)dx + \pi_1b_2(\omega-1)b^{r-1}\psi_b^{\omega-1} = c_2,$$

so

$$\pi_1 = \frac{b^{1-r}}{b_2(\omega-1)\psi_b^{\omega-1}}\Big(c_2 - rb_2b^{r-1}\int_a^b \mathrm{M}_b^{\omega-1}(x)\zeta_v(x)dx\Big) = \chi_{a,b}.$$

On the other hand $a_1v(a) + b_1v(b) = c_1$ implies that

$$b_1\int_a^b \mathrm{M}_b^{\omega}(x)\zeta_v(x)dx + (a_1+b_1)\pi_0 + \pi_1b_1\psi_b^{\omega} = c_1,$$

then

$$\pi_0 = \frac{1}{a_1+b_1}\Big[c_1 - b_1\chi_{a,b}\psi_b^{\omega} - b_1\int_a^b \mathrm{M}_b^{\omega}(x)\zeta_v(x)dx\Big].$$

Finally we get

$$v(s) = \int_a^s \mathrm{M}_s^{\omega}(x)\zeta_v(x)dx + \chi_{a,b}\psi_s^{\omega} + \frac{1}{a_1+b_1}\Big[c_1 - b_1\chi_{a,b}\psi_b^{\omega} - b_1\int_a^b \mathrm{M}_b^{\omega}(x)\zeta_v(x)dx\Big].$$

Lemma 4. *The problem* $() - ()$ *has a unique solution given by*

$$v(s) = \int_a^s \mathrm{M}_s^{\omega}(x)\zeta_v(x)dx - \frac{1}{2}\int_a^b \mathrm{M}_b^{\omega}(x)\zeta_v(x)dx + \big(\psi_s^{\omega} - \frac{1}{2}\psi_b^{\omega}\big)\chi_{a,b}, \qquad (6)$$

with

$$\chi_{a,b} = \frac{r}{(\omega-1)\psi_b^{\omega-1}}\int_a^b \mathrm{M}_b^{\omega-1}(x)\zeta_v(x)dx.$$

Proof.
It sufficient to take $a_1 = a_2 = b_1 = b_2 = 1$ and $c_1 = c_2 = 0$ in () we obtain ().

Our first result is based on Banach fixed point theorem.

Theorem 1. *Assume that*
(C1): ζ *is continuous,*
(C2): $|\zeta_{v_2}(x) - \zeta_{v_1}(x)| < k|v_2 - v_1|$,
in addition if

$$\mu = \frac{3}{2}k\mathrm{M}^{\omega} + \frac{kr\psi_b^{\omega}\mathrm{M}^{\omega-1}}{2(\omega-1)^2\psi_b^{\omega-1}} < 1.$$

Then the problem $() - ()$ *has a unique solution.*

Proof.
We shall show that $\mathcal{F}$ is a contraction.

$$\mathcal{F}v(s) = \int_a^s \mathrm{M}_s^{\omega}(x)\zeta_v(x)dx - \frac{1}{2}\int_a^b \mathrm{M}_b^{\omega}(x)\zeta_v(x)dx + \chi_{a,b}\Big(\psi_s^{\omega} - \frac{1}{2}\psi_b^{\omega}\Big)$$

Let v_1, $v_2 \in \mathcal{C}(J, \mathbb{R})$. For $s \in J$ we have

$$\begin{aligned}
\Big|\mathcal{F}v_2(s) - \mathcal{F}v_1(s)\Big| &\leqslant \int_a^s \mathrm{M}_s^{\omega}(x)|\zeta_{v_2}(x) - \zeta_{v_1}(x)|dx + \frac{1}{2}\int_a^b \mathrm{M}_b^{\omega}(x)|\zeta_{v_2}(x) - \zeta_{v_1}(x)|dx \\
&\quad + \frac{r(\psi_s^{\omega} - \frac{1}{2}\psi_b^{\omega})}{(\omega-1)\psi_b^{\omega-1}}\int_a^b \mathrm{M}_b^{\omega-1}(x)|\zeta_{v_2}(x) - \zeta_{v_1}(x)|dx \\
&\leqslant k\Big[\int_a^s \mathrm{M}_s^{\omega}(x)dx + \frac{1}{2}\int_a^b \mathrm{M}_b^{\omega}(x)dx \\
&\quad + \frac{r(\psi_s^{\omega} - \frac{1}{2}\psi_b^{\omega})}{(\omega-1)\psi_b^{\omega-1}}\int_a^b \mathrm{M}_b^{\omega-1}(x)dx\Big]|v_2(x) - v_1(x)| \\
&\leqslant k\Big[\int_a^b \mathrm{M}_b^{\omega}(x)dx + \frac{1}{2}\int_a^b \mathrm{M}_b^{\omega}(x)dx \\
&\quad + \frac{r(\psi_s^{\omega} - \frac{1}{2}\psi_b^{\omega})}{(\omega-1)\psi_b^{\omega-1}}\int_a^b \mathrm{M}_b^{\omega-1}(x)dx\Big]|v_2(x) - v_1(x)| \\
&\leqslant k\Big[\frac{3}{2}\int_a^b \mathrm{M}_b^{\omega}(x)dx + \frac{r\psi_b^{\omega}}{2(\omega-1)\psi_b^{\omega-1}}\int_a^b \mathrm{M}_b^{\omega-1}(x)dx\Big]|v_2(x) - v_1(x)|
\end{aligned}$$

Thus

$$\Big|\mathcal{F}v_2(s) - \mathcal{F}v_1(s)\Big| \leq k\Big[\frac{3}{2}\mathrm{M}^{\omega} + \frac{r\psi_b^{\omega}\mathrm{M}^{\omega-1}}{2(\omega-1)^2\psi_b^{\omega-1}}\Big]|v_2(x) - v_1(x)|$$

Therefore

$$\|\mathcal{F}v_2 - \mathcal{F}v_1\|_{\infty} \leq \mu\|v_2 - v_1\|_{\infty}.$$

Since $\mu < 1$, the operator $\mathcal{F}$ is a contraction. Then by Banach's fixed point theorem, the problem $() - ()$ has a unique solution.
The second results is based on Schaefer's fixed point theorem.

Theorem 2. *Assume that (C1)and (C2) hold. Then the problem () − () has at least one solution.*

Step 1: $\mathcal{F}$ is continuous.

$$\mathcal{F}v(s) = \int_a^s \mathrm{M}_s^\omega(x)\zeta_v(x)dx - \frac{1}{2}\int_a^b \mathrm{M}_b^\omega(x)\zeta_v(x)dx + \chi_{a,b}\Big(\psi_s^\omega - \frac{1}{2}\psi_b^\omega\Big)$$

Let $\{v_n\}$, $v_n \longrightarrow v \in \mathcal{C}(J,\mathbb{R})$. For $s \in J$ we have

$$\begin{aligned}
\Big|\mathcal{F}v_n(s) - \mathcal{F}v(s)\Big| &\leqslant \int_a^s \mathrm{M}_s^\omega(x)|\zeta_{v_n}(x) - \zeta_v(x)|dx + \frac{1}{2}\int_a^b \mathrm{M}_b^\omega(x)|\zeta_{v_n}(x) - \zeta_v(x)|dx \\
&\quad + \frac{r(\psi_s^\omega - \frac{1}{2}\psi_b^\omega)}{(\omega-1)\psi_b^{\omega-1}}\int_a^b \mathrm{M}_b^{\omega-1}(x)|\zeta_{v_n}(x) - \zeta_v(x)|dx \\
&\leqslant k\Big[\int_a^s \mathrm{M}_s^\omega(x)dx + \frac{1}{2}\int_a^b \mathrm{M}_b^\omega(x)dx \\
&\quad + \frac{r(\psi_s^\omega - \frac{1}{2}\psi_b^\omega)}{(\omega-1)\psi_b^{\omega-1}}\int_a^b \mathrm{M}_b^{\omega-1}(x)dx\Big]|v_n(x) - v(x)| \\
&\leqslant k\Big[\int_a^b \mathrm{M}_b^\omega(x)dx + \frac{1}{2}\int_a^b \mathrm{M}_b^\omega(x)dx \\
&\quad + \frac{r(\psi_s^\omega - \frac{1}{2}\psi_b^\omega)}{(\omega-1)\psi_b^{\omega-1}}\int_a^b \mathrm{M}_b^{\omega-1}(x)dx\Big]|v_n(x) - v(x)| \\
&\leqslant k\Big[\frac{3}{2}\int_a^b \mathrm{M}_b^\omega(x)dx + \frac{r\psi_b^\omega}{2(\omega-1)\psi_b^{\omega-1}}\int_a^b \mathrm{M}_b^{\omega-1}(x)dx\Big]|v_n(x) - v(x)|.
\end{aligned}$$

Therefore

$$\big\|\mathcal{F}v_n - \mathcal{F}v\big\|_\infty \leqslant k\Big[\frac{3}{2}\mathrm{M}^\omega + \frac{r\psi_b^\omega}{2(\omega-1)\psi_b^{\omega-1}}\mathrm{M}^{\omega-1}\Big]\|v_n - v\|_\infty.$$

As v_n converges to v, $\big\|\mathcal{F}v_n - \mathcal{F}v\big\|_\infty \longrightarrow 0$ so $\mathcal{F}$ is continuous.

Step 2: $\mathcal{F}$ maps bounded sets into bounded sets in $\mathcal{C}(J,\mathbb{R})$.

$$\begin{aligned}
|\mathcal{F}v(s)| &\leqslant \int_a^s \mathrm{M}_s^\omega(x)|\zeta_v(x)|dx + \frac{1}{2}\int_a^b \mathrm{M}_b^\omega(x)|\zeta_v(x)|dx + \Big|\chi_{a,b}\big(\psi_s^\omega - \frac{1}{2}\psi_b^\omega\big)\Big| \\
&\leqslant \frac{3}{2}\int_a^b \mathrm{M}_b^\omega(x)|\zeta_v(x)|dx + \frac{1}{2}\psi_b^\omega\big|\chi_{a,b}\big|
\end{aligned}$$

But we have

$$\begin{aligned}
|\zeta_v(x)| &\leqslant |\zeta_v(x) - \zeta_0(x)| + |\zeta_0(x)| \\
&\leqslant k|v| + |\zeta_0(x)|
\end{aligned}$$

and

$$
\begin{aligned}
|\chi_{a,b}| &\leqslant \frac{r}{(\omega-1)\psi_b^{\omega-1}} \int_a^b \mathrm{M}_b^{\omega-1}(x)\big|\zeta_v(x)\big|dx \\
&\leqslant \frac{r(k\|v\| + \max_{x\in J}|\zeta_0(x)|)}{(\omega-1)\psi_b^{\omega-1}} \int_a^b \mathrm{M}_b^{\omega-1}(x)dx \\
&\leqslant \frac{r(k\|v\| + \max_{x\in J}|\zeta_0(x)|)}{(\omega-1)^2\psi_b^{\omega-1}} \mathrm{M}^{\omega-1}
\end{aligned}
$$

Then we get

$$
\begin{aligned}
|\mathcal{F}v(s)| &\leqslant \frac{3}{2}\int_a^b \mathrm{M}_b^{\omega}(x)dx + \frac{r(k\|v\| + \max_{x\in J}|\zeta_0(x)|)\psi_b^{\omega}}{2(\omega-1)^2\psi_b^{\omega-1}}\mathrm{M}^{\omega-1} \\
&\leqslant \frac{3}{2}(k\|v\| + \max_{x\in J}|\zeta_0(x)|)\int_a^b \mathrm{M}_b^{\omega}(x)dx + \frac{r(k\|v\| + \max_{x\in J}|\zeta_0(x)|)\psi_b^{\omega}}{2(\omega-1)^2\psi_b^{\omega-1}}\mathrm{M}^{\omega-1} \\
&\leqslant (k\|v\| + \max_{x\in J}|\zeta_0(x)|)\Big(\frac{3}{2}\mathrm{M}^{\omega} + \frac{r\psi_b^{\omega}\mathrm{M}^{\omega-1}}{2(\omega-1)^2\psi_b^{\omega-1}}\Big) := \ell,
\end{aligned}
$$

so we conclude that

$$\|\mathcal{F}v\|_\infty \leq \ell.$$

Step 3: $\mathcal{F}$ maps bonded sets into equicontinuous set of $\mathcal{C}(J,\mathbb{R})$.
Let $v \in \mathbb{M}_\delta$. for each $s_1,\ s_2 \in J$ with $s_1 < s_2$ we obtain

$$
\begin{aligned}
\Big|\mathcal{F}v(s_2) - \mathcal{F}v(s_1)\Big| &\leqslant \Big|\int_a^{s_2} \mathrm{M}_{s_2}^{\omega}(x)\zeta_v(x)dx - \int_a^{s_1} \mathrm{M}_{s_1}^{\omega}(x)\zeta_v(x)dx\Big| + (\psi_{s_2}^{\omega} - \psi_{s_1}^{\omega})|\chi_{a,b}| \\
&\leqslant \Big|\int_a^{s_1} (\mathrm{M}_{s_2}^{\omega} - \mathrm{M}_{s_1}^{\omega})(x)\zeta_v(x)dx + \int_{s_1}^{s_2} \mathrm{M}_{s_1}^{\omega}(x)\zeta_v(x)dx\Big| + (\psi_{s_2}^{\omega} - \psi_{s_1}^{\omega})|\chi_{a,b}| \\
&\leqslant (k\|v\| + \max_{x\in J}|\zeta_0(x)|)\int_a^{s_1} \Big|(\mathrm{M}_{s_2}^{\omega} - \mathrm{M}_{s_1}^{\omega})(x)\Big|dx + \int_{s_1}^{s_2} \mathrm{M}_{s_1}^{\omega}(x)dx \\
&\quad +(\psi_{s_2}^{\omega} - \psi_{s_1}^{\omega})|\chi_{a,b}| \\
&\leq (k\|v\| + \max_{x\in J}|\zeta_0(x)|)\Big[2\big(s_2^r - s_1^r\big)^\beta + (s_1^r - a^r)^\beta - (s_2^r - a^r)^\beta\Big] \\
&\quad +(\psi_{s_2}^{\omega} - \psi_{s_1}^{\omega})|\chi_{a,b}|.
\end{aligned}
$$

If $s_1 \longrightarrow s_2$ the second hand side of above inequality tends to zero, it follows that $\mathcal{F}$ is continuous and completely continuous.
Step 4: A priori bounds.
Let $v \in \mathfrak{N} = \{v \in \mathcal{C}(J,\mathbb{R}) : v = \sigma\mathcal{F}(v);\, \sigma \in (0,1)\}$, then $v = \sigma\mathcal{F}(v)$ with

$\sigma \in (0,1)$, so, for each $s \in J$, we have

$$\begin{aligned}
|\mathcal{F}v(s)| &\leq \int_a^s \mathrm{M}_s^{\omega}(x)|\zeta_v(x)|dx + \frac{1}{2}\int_a^b \mathrm{M}_b^{\omega}(x)|\zeta_v(x)|dx + \Big|\chi_{a,b}\big(\psi_s^{\omega} - \frac{1}{2}\psi_b^{\omega}\big)\Big| \\
&\leq \frac{3}{2}\int_a^b \mathrm{M}_b^{\omega}(x)|\zeta_v(x)|dx + \frac{1}{2}\psi_b^{\omega}\big|\chi_{a,b}\big| \\
&\leq (k\|v\| + \max_{x\in J}|\zeta_0(x)|)\Big(\frac{3}{2}\mathrm{M}^{\omega} + \frac{r\psi_b^{\omega}\mathrm{M}^{\omega-1}}{2(\omega-1)^2\psi_b^{\omega-1}}\Big)
\end{aligned}$$

Thus

$$\|\mathcal{F}(v)\|_{\infty} < \infty.$$

This shows that $\mathfrak{N}$ is bounded. As a sequence of Schaefer's fixed point theorem, we deduce that $\mathcal{F}$ has a fixed point which is a solution of the problem $() - ()$.

4 Example

Example 1. Let us consider the following anti-periodic problem

$$^{2}\mathrm{D}^{\frac{3}{2}}v(s) = \frac{1}{25}v(s) + \frac{s+1}{v(s)+5}, \quad s \in J := [0,1], \tag{7}$$

$$v(0) + v(1) = 0, \qquad v'(0) + v'(1) = 0. \tag{8}$$

Let us set

$$\zeta(s,v) = \frac{1}{25}v + \frac{s+1}{v+5}, \quad s \in J, \ \textit{and}\ v_1, v_2 \in \mathbb{R}_+.$$

It is clear that ζ is continuous. For $s \in J$, and v_1, $v_2 \in \mathbb{R}_+$ we have

$$\begin{aligned}
\Big|\zeta(s,v_2) - \zeta(s,v_1)\Big| &= \Big|\frac{1}{25}(v_2 - v_1) + \frac{s+1}{v_2+5} - \frac{s+1}{v_1+5}\Big| \\
&\leq \frac{1}{25}\Big|v_2 - v_1\Big| + \frac{\Big|(s+1)(v_1-v_2)\Big|}{(v_1+5)(v_2+5)} \\
&\leq \frac{1}{25}|v_2 - v_1| + \frac{s+1}{25}|v_2 - v_1| \\
&\leq \frac{3}{25}|v_2 - v_1|,
\end{aligned}$$

so the assumption $(C2)$ holds for $k = \frac{3}{25}$. On the other hand we have

$$\mu_{r,\omega} = \frac{3}{2}k\mathrm{M}^{\omega} + \frac{kr\psi_b^{\omega}\mathrm{M}^{\omega-1}}{2(\omega-1)^2\psi_b^{\omega-1}} < 1.$$

So for $r = 2$ and $\omega = \frac{3}{2}$ we have

$$\begin{aligned}\mu_{2,\frac{3}{2}} &= \frac{3}{2} \times \frac{3}{25}\mathrm{M}^{\frac{3}{2}} + \frac{\frac{3}{25}2\psi_1^{\frac{3}{2}}\mathrm{M}^{\frac{1}{2}}}{2(\frac{1}{2})^2\psi_1^{\frac{1}{2}}} \\ &= \frac{9}{50}\mathrm{M}^{\frac{3}{2}} + \frac{12\psi_1^{\frac{3}{2}}\mathrm{M}^{\frac{1}{2}}}{25\psi_1^{\frac{1}{2}}} \\ &= \frac{9}{50}\mathrm{M}^{\frac{3}{2}} + \frac{12}{25}\psi_1^1\mathrm{M}^{\frac{1}{2}}.\end{aligned}$$

In addition we have

$$\mathrm{M}^{\frac{3}{2}} = \frac{2^{-\frac{3}{2}}}{\frac{3}{2}\Gamma(\frac{3}{2})} = \frac{\sqrt{2}}{3\sqrt{\pi}}, \qquad \mathrm{M}^{\frac{1}{2}} = \frac{2^{-\frac{1}{2}}}{(\frac{1}{2})\Gamma(\frac{3}{2})} = \frac{2\sqrt{2}}{\sqrt{\pi}}, \quad \textit{and} \quad \psi_1^1 = 1.$$

So

$$\mu_{2,\frac{3}{2}} = \frac{3\sqrt{2}}{50\sqrt{\pi}} + \frac{24\sqrt{2}}{25\sqrt{\pi}} < 1.$$

Then by Theorem the problem ()-() has a unique solution.

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